Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}$, and let $\mathfrak{p}$ be a prime ideal of $\mathbb{K}$ lying above $\mathfrak{p}$. Consider the ring $\mathbb{Z}[\mathfrak{p}]$, which is a complete local ring.

Moreover, this extension is wild and not admissible. The reason is that the action of $\mathbb{Z}$ on $\mathbb{Z}[\mathfrak{p}]$ is not compatible with the action of $\mathbb{Q}$ on $\mathbb{Z}[\mathfrak{p}]$. As a result, we cannot define a well-behaved $\mathbb{Z}$-module structure on $\mathbb{Z}[\mathfrak{p}]$. This is in contrast to the admissible case, where $\mathbb{Z}$ acts properly on $\mathbb{Z}[\mathfrak{p}]$.

In summary, the situation is quite different from the admissible case, where $\mathbb{Z}$ acts properly on $\mathbb{Z}[\mathfrak{p}]$. This highlights the importance of understanding the compatibility between the actions of $\mathbb{Z}$ and $\mathbb{Q}$ on $\mathbb{Z}[\mathfrak{p}]$.
**Remark:** One more piece we can describe explicitly:

\[ G_K = R_K^x \times \hat{\mathbb{Z}} \]

Via classical field theory, \(K = \mathbb{Q}_p\), \(\mathbb{Z}_p^x = R_K^x\)

\( \mathbb{Q}_p^{ab} = \varphi_p(\mathcal{O}_\mathbb{Q}) \) (includes your field)

\[ \mathbb{Z}_p^x \cong \frac{1 + p\mathbb{Z}_p}{\text{wild inertia}} \]

**Representations:**

Understand groups via their actions.

Galois groups are given with an action.

\[ G_{\mathbb{Q}_p} \rightarrow \mathbb{Q}_p \]

\( \times \text{roots of } f \text{ for } f \in \mathbb{Q}_p \text{ Ext} \)

\( \times \text{set, vector space, manifold} \)

Obvious things \( G_{\mathbb{Q}_p} \) acts on.

\( \mathbb{M} \text{ is a manifold. } \text{Diff}(\mathbb{M}) \circ \mathbb{M} \)

\( \text{Diff}(\mathbb{M}) \circ H^i(\mathbb{M}, \mathbb{Z}) \)

\( \text{Diff}(\mathbb{M}) \circ \pi_1(\mathbb{M}) \leftarrow \text{fundamental groupoid} \)

If \( K \) is not (field, \( K \) is an algebraic closure

\[ G_u \circ K = C_{G_K} = \text{Aut}(\mathbb{F}_K/K) \)]
Finite Galois extensions of \( F \in K \). 

\[ \text{Gal}(F) \cong \text{Aut}(F) \]

Any algebraic variety over \( K \), \( \sim \) can act \( \mathbb{Q}_l \) vector spaces with

\( \sim \) (continuous) action of \( \text{Gal}(K) \) via étale cohomology.

Important problem in NT: understand these representations.

Need to understand continuous representations of \( \text{Gal}(K) \) on \( \mathbb{Q}_l \) vector spaces.

\( K \) = finite extension of \( \mathbb{Q}_p \).

\[ l = p \rightarrow \text{etale \ H^i} \]

\[ l \neq p \rightarrow \text{pretty same} \]

\( \mathbb{Q}_l \) is a profinite group

\( \text{topological group} \).

\[ \text{Gal}(K) \cong \lim_{\leftarrow} \text{Gal} \left( K \right) \]

\( \text{topological subgroups} \).

\[ \text{Gal}(K) \cong \lim_{\leftarrow} G_i \]

\( \text{considered limit for} \).

\[ \mathbb{Z}_l \in \mathbb{Q}_l \]

\[ \text{topologized using} \ l \]

\[ \mathbb{Z}_l \subseteq \mathbb{Q}_l \]

\[ \text{profinite group} \]

\[ \mathbb{Z}_l \cong \lim_{\leftarrow} \mathbb{Z}/n \mathbb{Z} \].
Definition: A profinite group \( G \) is pro-\( p \) for a prime \( p \) if

- For any quotient by a normal subgroup is a \( p \)-group,

\[
\Rightarrow G \cong \lim_{\rightarrow i} G_i
\]

Exercise: If \( G_1 \) is pro-\( p \) and \( G_2 \) is pro-\( l \) for \( l \neq p \), then

\[
\text{Hom}_{\text{pro-}p}(G_1, G_2) = 0.
\]

Sketch: Suppose \( f \) is such a map.

\[
f : G_1 \to G_2
\]

is uniquely determined by

\[
G_1 \to G_2 \to G_2/N
\]

For all \( N \) a normal, \( N \) in \( G_2 \), (definition of \( N \)),

\[
\ker f_N = f^{-1}(N)
\]

which is open by continuity

and they are open normal subgroups of \( G_1 \).
$f_N : G_1 \to G_1/\ker_f \to G_2/\ker_f$

finite p-group, finite l-group

so $f_N = 0 \Rightarrow f = 0$.

Lemma Let $\varphi : G_K \to \text{GL}_N(\mathbb{Q}_l)$ be a continuous homomorphism.

then there is a finite extension $L/K$

s.t. $\varphi|_{G_L}$ factors through $G^\text{tor} = \pi_L \backslash G_L$.

Proof: Step 1 - Assume $\varphi : G_K \to \text{GL}_N(\mathbb{Z}_l)$.

(exercise to check this).

Tua $1 \to 1 + l\mathbb{M}_n(\mathbb{Z}_l) \to \text{GL}_n(\mathbb{Z}_l) \to \text{GL}_n(\mathbb{F}_l) \to 1$

$\longrightarrow$

$1 \to 1 + l\mathbb{M}_n(\mathbb{Z}_l) \to \mathbb{Z}_l^\times \to \mathbb{F}_l^\times \to 1$

open pro-$l$ subgroup.

$1 + l\mathbb{M}_n(\mathbb{Z}_l)$

is an open subgroup $U$ of $\text{GL}_n(\mathbb{Z}_l)$.
so \( U = G \) for finite extension of \( K \).

\[ \phi|_{G_{\mathfrak{m}}} \rightarrow 1 + \ell M_n(\mathbb{Z}_\ell) \]

Need to show \( \phi|_{\mathfrak{p}_L} \) is trivial.

\[ \phi|_{\mathfrak{p}_L} : \mathfrak{p}_L \rightarrow 1 + \ell M_n(\mathbb{Z}_\ell) \]

By exercise we are done.