Announcements:

- Exercise 6 now bonus.
- Office hours this week Tuesday 8:30pm–10:00pm (last ones!).
- Final exam May 3rd, 10:30am
  Details are on website.
- Problem rewrites due before midterm starts.
- Please fill out course feedback form!

Theorem If G is a finite group, $V_1, \dots, V_k$ are representatives for the isomorphism classes of irreps of G over $\mathbb{C}$ then the characters χ_{V_i} are an orthonormal basis for $C(G)$

$C(G)$ = functions on G which are constant on conjugacy classes.

$$\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{h(g)}.$$

Last time: · Deduced some nice consequences
· Proved orthonormal

Need to prove χ_{V_i} span $C(G)$.

Suffices to show that if $f \in C(G)$ s.t. $\langle \chi_{V_i}, f \rangle = 0$ for $i = 1, \dots, k$ then $f = 0$. (I.e, show that $\langle \chi_{V_1}, \chi_{V_2}, \dots, \chi_{V_k} \rangle^{\perp} = 0$).

Observe: If $0 = \langle \chi_{V_i}, f \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{V_i}(g) \overline{f(g)}.$

$$= \sum_{g \in G} \overline{f(g)} \chi_{V_i}(g)$$

$$= \sum_{g \in G} \overline{f(g)} \operatorname{Tr} \rho_{V_i}(g)$$

$$0 = \mathrm{Tr}\Big(\sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(g)\Big)$$

$$\text{End}(V_i) \quad (\cong M_{n\times n}(\mathbb{C}) \text{ after choosing a basis})$$

<u>Claim:</u> $\sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(g) = 0$ in $\text{End}(V_i)$.

Suffices to show $= \lambda\, \text{Id}$, $\lambda \in \mathbb{C}$, because then

$$0 = \mathrm{Tr}(\_\_) = (\dim V_i)\lambda$$

$$\Rightarrow \lambda = 0.$$

Want to apply Schur's Lemma, which says this is true if $\sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(g) \in \text{End}_G(V_i) = \text{End}(V_i)^G$.

Need to check this:

Let $h \in G$.

$$\rho_{\text{End}(V_i)}(h)\Big(\sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(g)\Big)$$

$$= \sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(h)\rho_{V_i}(g)\rho_{V_i}(h)^{-1}$$

$$= \sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(hgh^{-1}) \longleftarrow \text{because } f \in C(G)$$

$$= \sum_{g\in G} \overline{f(hgh^{-1})}\, \rho_{V_i}(hgh^{-1})$$

$$= \sum_{g\in G} \overline{f(g)}\, \rho_{V_i}(g). \quad \checkmark$$

$g \in G$

Consequence: For <u>any</u> representation V,

$$\sum_{g \in G} \overline{F(g)}\, \rho_V(g) = 0 \quad (\text{in } \mathrm{End}(V)).$$

Because $V = V_1^{m_1} \oplus V_2^{m_2} \oplus \dots \oplus V_k^{m_k}$

$$\rho_V(g) = (\underbrace{\rho_{V_1}(g), \rho_{V_1}(g), \dots, \rho_{V_1}(g)}_{m_1}, \underbrace{\rho_{V_2}(g), \dots}_{m_2}, \dots)$$

Conclude by taking $V = \mathbb{C}[G]$.

$$\left(\sum_{g \in G} \overline{F}(g)\, \rho_V(g)\right) e$$

$$= \sum_{g \in G} \overline{F}(g)\, g = 0$$

$$\Rightarrow \overline{F}(g) = 0 \quad \forall g \in G.$$

$\square$

Irreps of S_3:

Character table

	e 1	(12) 3	(123) 2
triv	1	1	1
sgn	1	-1	1
std	2	0	-1

$\text{std} = \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right]$ $x+y+z=0$ in $\mathbb{C}^3$

$= \text{triv}^{\perp}$

$\langle \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right] \rangle$

$\mathbb{C}^3 = \text{triv} \oplus \text{std}$

$\chi_{\mathbb{C}^3} = \chi_{\text{triv}} + \chi_{\text{std}}$

$\chi_{\text{std}} = \chi_{\mathbb{C}^3} - \chi_{\text{triv}}$

$(\chi_{V_i}, \chi_{V_i}) = 1$

$$\longleftrightarrow \overset{\text{triv}}{1} + 3\cdot 1 + 2\cdot 1 = 6 = |S_3|$$

$= \chi_{std}(g) = $ # of fixed pts of g $- 1$.

sgn

$$1 + 3\cdot 1 + 2\cdot 1 = 6$$

$(-1)\overline{(-1)} = |-1|^2$.

std

$$1\cdot \underset{= 2\cdot\overline{2}}{4} + 3\cdot 0 + 2\cdot \underset{= (-1)\overline{(-1)}}{1} = 6.$$

$\langle \chi_{V_i}, \chi_{V_j} \rangle = 0 \longleftrightarrow$ rows of character table are orthogonal. (sorted by conj-class)

$i \neq j$

$$1^2 + 1^2 + 2^2 = 6 = |S_3|.$$

	e	(12)	(123)
	1	3	2
triv	1	1	1
sgn	1	-1	1
std	2	0	-1

$$1^2 + (-1)^2 + 0^2 = 2 = \frac{6}{3}$$

$$1^2 + 1^2 + (-1)^2 = 3 = \frac{6}{2}$$

$\dfrac{|S_3|}{|\mathcal{C}|}$ ← conjugacy class

Columns are also orthogonal!

Theorem: (2nd orthogonality).

Distinct columns are orthogonal and the column c corresponding to a conj class $\mathcal{C}$

satisfies $\left(\begin{matrix} c_i \bar{c} \\ \shortparallel \\ \langle c, c \rangle \end{matrix}\right) = \frac{|G|}{|C|}$

Proof: Character table is a $\overset{\text{the transpose}}{\text{change}}$ of basis matrix from $\mathbb{1}_C$ ← indicator functions for the conj. classes to χ_{V_i}.

Almost orthonormal basis. $(\mathbb{1}_C, \mathbb{1}_C) = \frac{|C|}{|G|}$.

Orthonormal basis

The matrix is basically unitary . .

$A\,(\overline{A^t}) =$ diagonal matrix.

and

$\overline{A}^t A =$ diagonal matrix.

one is first ortho. other is 2nd ortho.

Unitary $A^* = \overline{A^t}$

$A^* = A^{-1}$

$\Rightarrow A^* A = Id$

$A A^* = Id$

Example: character table of S_4. $|S_4| = 24$

	e 1	(12) 6	(123) 8	(1234) 6	$(12)(34)$ 3
triv	1	1	1	1	1
sgn	1	−1	1	−1	1
std	3	1	0	−1	−1
sgn ⊗ std.	3	−1	0	1	−1
?	2	0	−1	0	2

We never constructed this!!!

Remarks: · Nice basic application to Galois theory in HW.

· Other approach based on group algebras.

$\mathbb{C}[G]$ ← natural $\mathbb{C}$-algebra

finite dim'l

semisimple (complete reducibility)