Announcements:
- Exercise 6 now bonus.
- Office hours this week Tuesday 9:30pm-10:00pm
- Final exam May 3rd, 10:30am
- Details are on website.
- Problem sets due before mid-term starts.
- Please fill out course feedback form!

Theorem If $G$ is a finite group, $V_1, \ldots, V_k$ are representations for the isomorphism classes of irreps of $G$ over $F$ then the characters $\chi_{V_i}$ are an orthonormal basis for $C(G)$

function on $G$ which are constant on conjugacy classes

$$\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) h(g).$$

Last time: Deduced some nice consequences

Need to prove $\langle \chi_{V_i}, f \rangle = 0$ for $i=1, \ldots, k$.

Suffices to show that if $f \in C(G)$

such that $\langle \chi_{V_i}, f \rangle = 0$ for $i=1, \ldots, k$ then $f = 0$. (i.e., show that $\langle \chi_{V_1}, \chi_{V_2}, \ldots, \chi_{V_k} \rangle = 0$.)

Observe: $0 = \langle \chi_{V_i}, f \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{V_i}(g) \overline{f(g)} = \sum_{g \in F(g)} \chi_{V_i}(g)$

$= \sum_{g \in F(g)} \overline{f(g)} \chi_{V_i}(g) = \sum_{g \in F(g)} \overline{f(g)} \chi_{V_i}(g)$
\[ 0 = \text{Tr} \left( \sum_{g \in G} \overline{F(g)} P_{V_i}(g) \right) \]

\[ \text{End}(V_i) \left( \cong M_{n_i}(\mathbb{C}) \right) \]

Claim: \[ \sum_{g \in G} \overline{F(g)} P_{V_i}(g) = 0 \] \[ \forall \lambda \in \text{End}(V_i). \]

Suffices to show \[ \lambda \text{Id} \] because then

\[ 0 = \text{Tr}(\lambda \text{Id}) = (\text{dim} V_i) \lambda \]

\[ \implies \lambda = 0. \]

Want to apply Schur's Lemma, which says this is true if \[ \sum_{g \in G} \overline{F(g)} P_{V_i}(g) \in \text{End}_0(V_i). \]

Need to check this:

Let \[ h \in G. \]

\[ \text{End}(V_i)(h) \left( \sum_{g \in G} \overline{F(g)} P_{V_i}(g) \right) \]

\[ = \sum_{g \in G} \overline{F(g)} P_{V_i}(h) P_{V_i}(g) P_{V_i}(h)^{-1} \]

\[ = \sum_{g \in G} \overline{F(g)} P_{V_i}(h g h^{-1}) \]

\[ \left( \text{because } F \in C(G) \right) \]

\[ = \sum_{g \in G} \overline{F(h g h^{-1})} P_{V_i}(h g h^{-1}) \]

\[ = \sum_{g \in G} \overline{F(g)} P_{V_i}(g). \quad \checkmark \]
Consequence: For any representation $V$,
\[ \sum_{g \in G} f(g) \rho_V(g) = 0 \quad \text{(in End}(V)) \].

Because $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$,
\[ \rho_V(g) = \begin{pmatrix} \rho_{V_1}(g) & \rho_{V_2}(g) & \ldots & \rho_{V_m}(g) \end{pmatrix} \in \mathfrak{gl}_m \]

Conclude by taking $V = \mathbb{C}[G]$.
\[ \left( \sum_{g \in G} f(g) \rho_V(g) \right) e \]
\[ = \sum_{g \in G} f(g) g = 0 \]
\[ \Rightarrow f(g) = 0 \quad \forall \, g \in G. \]

Irreps of $S_3$:

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
<th>$(12)$</th>
<th>$(123)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>triv</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>sgn</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>std</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

(\chi_V, \chi_{\bar{W}}) = 1

\[ \chi_{\mathfrak{gl}_m} = x_{\text{triv}} + x_{\text{std}} \]

\[ \chi_{\text{std}} = \chi_{\mathfrak{gl}_m} - x_{\text{triv}} \]
\[ \chi(\text{tri}) = 1 + 3 \cdot 1 + 2 \cdot 1 = 6 = \chi_5(1) = \# \text{ of fixed pts of } 6 \]

\[ \chi(\text{sign}) = 1 + 3 \cdot 1 + 2 \cdot 1 = 6 \]

\[ (-1)(-1) = 1 \rightarrow 1 \quad \frac{1}{2} \quad \frac{1}{2} \]

\[ \langle \chi_{v_i}, \chi_{v_j} \rangle = 0 \quad \text{rows of character table are orthogonal (proved by hand)} \]

\[ 1^2 + 1^2 + 2^2 = 6 = 1 \chi_3 \]

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>(12)</th>
<th>(123)</th>
<th>12</th>
<th>12</th>
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<td>0</td>
<td>-1</td>
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</tr>
</tbody>
</table>

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Columns are also orthogonal!

**Theorem:** (2nd orthogonality)
Distinct columns are orthogonal and the column \( \bar{c} \) corresponding to a conjugate class \( C \)
Proof: Character table is a change of basis matrix from $\mathbf{1}_C$ (indicator function for the c class) to $\mathbf{v}_i$. About orthogonal basis, $\langle \mathbf{1}_C, \mathbf{v}_i \rangle = \frac{|C|}{|G|}$. The matrix is basically unitary, $A^* A = \text{diagonal matrix}$.


$A^* A = \text{Id}$

Example: Character table of $S_4$. $|S_4| = 24$

<table>
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<tr>
<th></th>
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<th>(12)</th>
<th>(123)</th>
<th>(1234)</th>
<th>(12)(34)</th>
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<tbody>
<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>sgn</td>
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<td>-1</td>
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<td>-1</td>
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<tr>
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<td>0</td>
<td>-1</td>
<td>-1</td>
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<tr>
<td>sgn &amp; std.</td>
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<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>?</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
We never constructed this!!!

Remarks:

- Nice basic application to Galois theory in HW.
- Other approach based on group algebra.

\[ \mathbb{C}[G] \cong \text{natural } \mathbb{C} \text{-algebra} \]

finite dim',

semisimple (complete reducibility)