Exercise 1:

G a finite group

(ρ, V) is a rep. of G on a $\mathbb{C}$-vector space. Show $\rho(g)$ is diagonalizeable and its eigenvalues are roots of unity for any $g \in G$.

<u>Proof</u>: $g^n = e$ for some n

$$\rho(g^n) = \rho(e)$$

$$\rho(g)^n = Id$$

$$\rho(g)^n - Id = 0$$

$\rho(g)$ satisfies $x^n - 1 = 0$

Minimal poly of $\rho(g) \mid x^n - 1$

$x^n - 1$ has no repeated roots and splits into linear factors

so $\rho(g)$ diagonalizeable.

Eigenvalues are all roots of $x^n - 1$.

Corollary: $\chi_{V^*} = \chi_V$

Proof: (ρ, V) is a $\mathbb{C}$-rep. of G.

$$V^* = \mathrm{Hom}(V, \mathbb{C})$$

If $\phi \in V^*$

$$\Big(\rho_{V^*}(g)(\phi)\Big)(v) = \phi\left(\rho_V(g)^{-1}(v)\right)$$

i.e. $\rho_{V^*}(g)(\phi) = \phi \circ \rho_V(g)^{-1}$.

Note If $T: V \to W$ is a linear transformation, get

$$T^*: W^* \to V^*$$

$$T^*(\phi) = \phi \circ T$$

If you fix bases $\underbrace{e_1, \dots, e_m}_{V}$ $\underbrace{f_1, \dots, f_n}_{W}$ for

and dual bases $\underbrace{e_1^*, \dots, e_m^*}_{V^*}$ $\underbrace{f_1^*, \dots, f_n^*}_{W^*}$

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

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In the bases $e_\cdot$, $f_\cdot$,

T is given by a matrix

A is an $m \times n$ matrix

In the bases $f^*_\cdot$, $e^*_\cdot$

T^* is given by a matrix

B is an $n \times m$ matrix

$$B = A^t$$

$$\chi_{V^*}(g) = \mathrm{Tr}\, \rho_{V^*}(g)$$

$$\underline{\rho_{V^*}(g)}(\phi) = \phi \circ \rho_V(g)^{-1}$$

$$\rho_{V^*}(g) = \left(\rho_V(g)^{-1}\right)^*$$

(if I fix a basis

$$A = (A^{-1})^T$$

I know $\rho_V(g)$ is diagonalizable with roots of unity on diagonal.

$$A = \begin{pmatrix} \zeta_1 & & & 0 \\ & \zeta_2 & & \\ & & \ddots & \\ 0 & & & \zeta_n \end{pmatrix}$$

ζ_i are roots of unity

$$(A^{-1})^T = A^{-1} = \begin{pmatrix} \zeta_1^{-1} & & & 0 \\ & \zeta_2^{-1} & & \\ & & \ddots & \\ 0 & & & \zeta_n^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\zeta}_1 & & & 0 \\ & \bar{\zeta}_2 & & \\ & & \ddots & \\ 0 & & & \bar{\zeta}_n \end{pmatrix}$$

$$\operatorname{Tr}((A^{-1})^T) = \overline{\operatorname{Tr}(A)}$$

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How to check if V is irreducible using χ_V:

$$V = V_1^{m_1} \oplus V_2^{m_2} \oplus \dots \oplus V_k^{m_k}$$

V_i distinct irreducibles

$$\chi_V = m_1 \chi_{V_1} + \dots + m_k \chi_{V_k}$$

$$\langle \chi_V, \chi_V \rangle = \sum_{i,j} m_i m_j \underbrace{\langle \chi_{V_i}, \chi_{V_j} \rangle}_{\substack{=0 \text{ if } i \neq j \\ =1 \text{ if } i=j}}$$

$$= m_1^2 + m_2^2 + \dots + m_k^2$$

$= 1$ if and only if $m_i = 1$ for some i and 0 for all of the others.

$\iff$

$V = V_i$ is irreducible.

i.e. V is irreducible

$$\Longleftrightarrow$$

$$\langle \chi_V, \chi_V \rangle = 1.$$

(In general $\langle \chi_V, \chi_V \rangle$ = sum of squares of multiplicities)

<u>Exercise</u> Let V_n be the rep'n of S_n given by $(x_1, \dots, x_n) \in \mathbb{C}^n$ s.t. $\sum x_i = 0$

orthogonal to trivial representation $\langle (1, 1, \dots, 1) \rangle$

S_n acts by permutation of the coordinates.

(i.e. $\mathbb{C}^n = \mathbb{C}[\{1, 2, \dots, n\}]$)

(standard rep. of S_n).

$\boxed{\text{Prove } V_n \text{ is irreducible.}}$

Remarks: $\chi_{\mathbb{C}^n} = \chi_{triv} + \chi_{V_n}$

$\chi_{V_n} = \chi_{\mathbb{C}^n} - \chi_{triv}$

$=$ # of fixed points of σ minus 1.

In general: Suffices to show $\langle \chi_{\mathbb{C}^n}, \chi_{\mathbb{C}^n} \rangle = 2$.

$\frac{1}{|S_n|} \sum_{\sigma \in S_n} \chi_{\mathbb{C}^n}(\sigma) \chi_{\mathbb{C}^n}(\sigma)$ ← drop conjugate because integer

$\cancel{\frac{1}{|S_n|} \sum_{\sigma \in S_n} \chi_{\mathbb{C}^n \otimes \mathbb{C}^n}(\sigma)}$

$\mathbb{C}^n = \mathbb{C}[\{1,2,\dots,n\}]$

$\mathbb{C}[X] \otimes \mathbb{C}[Y] = \mathbb{C}[X \times Y]$

$\mathbb{C}^n \otimes \mathbb{C}^n = \mathbb{C}[\{1,2,\dots,n\}^2]$

The set of ordered pairs (a,b) $1 \le a, b \le n$.

Computes the mult. of triv rep. $=$ # of orbits

So just need to check there are 2 orbits