Theorem: Let G be a finite group. Let $V_1, \dots, V_m$ be representatives for the irreps of G (over $\mathbb{C}$). Then $\chi_{V_1}, \dots, \chi_{V_m}$ are an orthonormal base for the space of $\underbrace{\text{conjugation-invariant functions on } G}_{\text{class functions}}$ with inner product $\langle f, h \rangle = \frac{1}{|G|} \sum_{g \in G} f(g) \overline{h(g)}$.

Hermitian inner product.

$X = G$ w/ conjugation action

$\mathbb{C}[X]^G$

Corollary: $\dim_{\mathbb{C}} C(G) = \#$ of conjugacy classes.

class functions on G

So $\#$ of irreps of $G = \#$ of conjugacy classes

Corollary: The multiplicity of V_i in any rep W is $\langle \chi_W, \chi_{V_i} \rangle$.

Pf: $W \cong V_1^{K_1} \oplus V_2^{K_2} \oplus \dots \oplus V_m^{K_m}$

$\chi_W = K_1 \chi_{V_1} + K_2 \chi_{V_2} \dots + K_m \chi_{V_m}$

$\langle \chi_W, \chi_{V_i} \rangle = K_i$. (by orthonormality).

Corollary: Multiplicity of V_i in $\mathbb{C}[G]$ is $\dim V_i$. Thus, $\sum_{i=1}^{m} (\dim V_i)^2 = |G|$. In particular, G is abelian $\Leftrightarrow$ all irreps are 1-dim'l.

Example Irreps of S_3: triv. (1-dim'l) sgn (1-dim'l) std $\subseteq \mathbb{C}^3 = \mathbb{C}[\{1,2,3\}]$

$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x+y+z=0 \right\}$

2 dim'l.

$1^2 + 1^2 + 2^2 = 6$

Proof: $\chi_{\mathbb{C}[G]}(g) = \#$ of fixed points of g acting on G by left multiplication

$$= \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases}$$

Mult. of V_i in $\mathbb{C}[G]$

$$= \langle \chi_{\mathbb{C}[G]}, \chi_{V_i} \rangle$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}[G]}(g) \overline{\chi_{V_i}(g)}$$

$$= \cancel{\frac{1}{|G|} |G|} \overline{\chi_{V_i}(e)}$$

$\overline{\chi_{V_i}(e)}$ = Trace of Id matrix on a $\dim V_i$-dim'l space

$$= \dim V_i.$$

$$\mathbb{C}[G] = \bigoplus_{i=1}^{m} V_i^{\dim V_i}$$

$$\underbrace{|G|}_{\dim} = \underbrace{\sum_{i=1}^{m} (\dim V_i)^2}_{= \dim}. \quad \blacksquare$$

Proof of orthonormality for characters:

$$\mathrm{Hom}_G(V_i, V_j) = \begin{matrix} \mathbb{C} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{matrix}$$

$$\|$$

$$\mathrm{Hom}(V_i, V_j)^G$$

$\mathrm{Hom}(V_i, V_j)$ $\qquad \rho_i : G \to GL(V_i)$

$$\rho_{\mathrm{Hom}(V_i,V_j)}(g)\phi = \rho_j(g)\circ\phi\circ\rho_i(g)^{-1}.$$

$$\mathrm{Hom}(V_i,V_j) = V_i^* \otimes V_j$$
as G-representations.

$\left\{\begin{array}{l} \mathrm{Hom}(A,B) = A^* \otimes B \\ a \mapsto a^*(a)b \leftarrow a^* \otimes b \end{array}\right.$

Have a projection operator onto G-invariants for any rep V:

$$\frac{1}{|G|}\sum_{g\in G}\rho(g) \in \mathrm{End}(V)$$

Apply this to $V = \mathrm{Hom}(V_i,V_j)$.

Tr of projection operator = dimension of its image.

$$\dim V^G = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G}\rho(g)\right) = \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}(\rho(g))$$
$$= \frac{1}{|G|}\sum_{g\in G}\chi_\rho(g)$$

$$V = \mathrm{Hom}(V_i,V_j) = V_i^* \otimes V_j$$

$$\dim \mathrm{Hom}_G(V_i,V_j) = \frac{1}{|G|}\sum \chi_{V_i^*}(g)\chi_{V_j}(g)$$
$$= \frac{1}{|G|}\sum \chi_{V_j}(g)\overline{\chi_{V_i}(g)}$$
$$= \langle \chi_{V_j}, \chi_{V_i}\rangle \quad \boxed{\Xi}$$

$$\dim \mathrm{Hom}_G(V_i,V_j) = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$$

Remains to show the χ_{V_i} span $C(G)$.
Next time.