Lecture: More on irreducibles, multiplicities, decomposition into irreducibles

**Recall** If $W$ is irreducible and $k$ alg closed char 0,

$$\dim_k \mathrm{Hom}_G(W, V) = \text{multiplicity of } W \text{ in } V.$$

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**Recall**: If $G$ is a finite group, then there are a finite # of irreducible representations of $G$.
(In lecture $\leq |G|$, all appear in $k[G]$).

Next lecture: In fact (if $k = \mathbb{C}$) # of irreps = # of conjugacy classes

Q: How do we actually compute decompositions into irreps / find irred. rep'ns. ?

If $(\rho, V)$ is a representation then the character of $(\rho, V)$ is the function on $G$

$$\chi_\rho(g) = \operatorname{tr} \rho(g).$$

$$(\chi_V)$$

<u>Properties:</u>

$$\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}.$$

(because of the corresp property for trace)

$$\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}.$$

$(V_1, \rho_1)$ $(V_2, \rho_2)$ are reps

then $V_1 \otimes V_2, \rho_1 \otimes \rho_2$ is a rep.

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v$$

**Exercise:** $G$ finite group, $X$ finite $G$-set, Compute the character of $\mathbb{C}[X]$.

Give a formula for $\chi_{\mathbb{C}[X]}(g)$ in terms of simple invariant of $g$ acting in $X$.

Answer: $\chi_{\mathbb{C}[X]}(g) = \#$ of fixed points of $g$.

(i.e. $\{\# x \in X$ s.t. $gx = x\}$.

**Example** $G = S_3$ $X = \{1\ 2\ 3\}$

$(1\ 2\ 3)$

$$\operatorname{tr}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = 0$$

**Exercise:** Compute $\sum_{g \in G} \chi_{\mathbb{C}[X]}(g)$.

$= \#$ ~~of~~

$g \in G$ in $g$.

**Answer:** $|G| \cdot (\#$ of orbits of $X)$

Outline: ① If $X$ is transitive then use orbit-stabilizer

② $X = \bigsqcup_{i=1}^{n} \mathcal{O}_i$

$$\mathbb{C}[X] = \bigoplus_{i=1}^{n} \mathbb{C}[\mathcal{O}_i]$$

$$\chi_{\mathbb{C}[X]} = \sum_{i=1}^{n} \chi_{\mathbb{C}[\mathcal{O}_i]}$$

Then sum values over $g \in G$

triv: $\mathbb{C}$,

Rewrite this:

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}[X]}(g) = (\# \text{ of orbits in } X)$$

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}[X]}(g) \cdot 1 = \# \text{ of orbits in } X$$

$$\frac{1}{|G|} \sum_{g \in G} \chi_{\mathbb{C}[X]}(g) \cdot \overline{\chi_{\text{triv}}(g)} = \# \text{ of orbits in } X$$

$$= \boxed{\text{multiplicity of triv. rep in } \mathbb{C}[X]}$$

Generalizes: If $W$ is an irrep and $V$ is any rep.

multiplicity of $W$ in $V = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_{W}(g)}$

(In example above verified by hand for $V = \mathbb{C}[x]$ ; $W = \text{triv}$).

Exercise / question:

If $G$ a finite group
$V$ f.d. $\mathbb{C}$-vector space
$\rho: G \to GL(V)$.

Show that for any $g \in G$ the eigenvalues of $\rho(g)$ are roots of unity.

Corollary: $\chi_{V^*} = \overline{\chi_V}$