Announcements: ① Fixed the online whiteboard.
② Notes will be available after future class sessions.

G a group, K a field

(ρ, V) a representation of G on a K vector-space is irreducible if the sub representations are $\{0\}$ and V.

Lemma: If V & W are 2 irreducible representations of G and $\phi: V \to W$ is a homomorphism of G-reps then ϕ is either an isomorphism or the zero map.

Proof: Hint: $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$ are both subrepresentations.

Theorem (Complete reducibility): If V is a representation of a finite group G, char K, $|G|$ are coprime, then V is a direct sum of irreducibles (unique up to isomorphism).

Proof: Last time: Averaged a projection operator.

Theorem (Schur's lemma): If V is irreducible, K alg. closed then $\operatorname{End}_G(V,V) = K$.

Proof: Last time: use existence of an eigenvalue.

Remark: Lemma above $\Rightarrow$ (with no hypothesis on K) that $\operatorname{End}_G(V,V)$ is a division algebra for V irreducible.

Proposition If G is a finite group char K, $|G|$ coprime then there are only finitely many isomorphism classes of irreducible G-representations.

Proof: Observation 1 If V is irreducible then

There is a surjection of G-representations

$$K[G] \twoheadrightarrow V$$

Regular representation. (G is viewed as a G-set via left multiplication).

<u>Why?</u> Take any nonzero $v \in V$

Then take the map

$$\sum_{g \in G} a_g g \to \sum_{g \in G} a_g (g \cdot v)$$

Map of rep's - easy to check.

Image nonzero because $v = e \cdot v$ the image of e.

so because V irreducible it is surjective.

<u>Conclusion:</u> V has to appear in the decomposition of $K[G]$ into irreducible

<u>Why?</u> Write $K[G] = \bigoplus_{i=1}^{m} W_i$ where W_i is irred.

We've seen $\mathrm{Hom}_G(K[G], V) \neq 0$.

$$\mathrm{Hom}_G(K[G], V) = \bigoplus_{i=1}^{m} \mathrm{Hom}_G(W_i, V)$$

$\mathrm{Hom}_G(W_i, V) \neq 0$
$\iff W_i \cong V$.

Thus one of the W_i is isomorphic to V (as a G-rep.)

<u>Corollary (of proof)</u> Every irrep of G is a summand of $K[G]$.

Definition/ Lemma: If W is irreducible and V is any rep. and K alg. closed!

then the multiplicity of W in V

is $\dim_K \mathrm{Hom}_G(W, V)$

$=$ # of irred. subrep isomorphic to W in the decomp. of V into irrep.

Proof: $V = \bigoplus_{i=1}^{m} V_i$ irreducibles.

$$\mathrm{Hom}_G(V, W) = \bigoplus_{i=1}^{m} \mathrm{Hom}_G(V_i, W)$$

$$= \bigoplus_{\substack{i \text{ s.t.} \\ V_i \cong W}} \mathrm{Hom}_G(V_i, W)$$

$$= \bigoplus_{\substack{i \text{ st} \\ V_i \cong W}} K \quad \leftarrow \dim = \# \text{ of } V_i \cong W$$

G finite, K a.c., char K $|G|$ coprime.

Remark/Example: Fix $W_1, \dots, W_k$ representatives for the isomorphism classes of irreps of G

If V is any rep.

$$V \cong W_1^{m_1} \oplus W_2^{m_2} \oplus \dots \oplus W_k^{m_k}$$

where m_i is the mult. of W_i in V.

This like choosing a basis for a vector space.

If $V' = W_1^{n_1} \oplus W_2^{n_2} \oplus \dots \oplus W_k^{n_k}$

$$\mathrm{Hom}_G(V, V') = M_{n_1 \times m_1}(K) \times M_{n_2 \times m_2}(K) \times \dots \times M_{n_k \times m_k}(K)$$

Q: 1. Given a group G, how do we find its irreps?
2. Given a representation V, how do we decompose it into irreducibles?

A: Characters!

Definition: If (ρ, V) is a representation of G in a K vector space, the character of ρ = (ρ, V) is the K-valued function on G

$$\chi_\rho(g) = \underline{\mathrm{Tr}\,\rho(g)}.$$

$\rho(g) \in GL(V)$.
$\wr\wr\wr$ fix a basis
$GL_n(K)$
take the trace of corresp. matrix.

AMAZING FACT:

(Note: Trace of a matrix doesn't determine matrix up to conjugacy!)

χ_ρ determines ρ up to isomorphism (really in a computable way).

Basic properties: If (ρ, V) is a rep.

- $\chi_\rho(e) = \mathrm{Tr}\,\rho(e) = \mathrm{Tr}\,\mathrm{Id}_V$
  $= \dim V.$

- $\chi_\rho(ghg^{-1}) = \chi_\rho(h)$
  (Trace is invariant under conjugation)
  $\rightarrow$ So χ_ρ is a <u>class function</u>
  i.e., constant on conj. classes.

- $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$
- $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

$\Big\}$ Fix bases to compute.