Representation theory of finite groups

Group action / G-sets ⇝ Theory of groups acting on sets

Set X, $G \to \mathrm{Aut}(X)$.

Structure theorem $X = \bigsqcup$ orbits

orbit $\cong G/\mathrm{stab}(x)$

for any x in the orbit.

Group representation:

$$G \xrightarrow{\rho} \mathrm{Aut}(V)$$
$$(= GL(V)).$$

V is a vector space over a field k

Sometimes write ρ, sometimes V, sometimes (ρ, V), ...

A map of reps $\phi : (\rho_1, V_1) \to (\rho_2, V_2)$

is a map $\phi : V_1 \to V_2$ of vector spaces

s.t. $\phi(\rho_1(g)v) = \rho_2(g)\phi(v)$

$\forall g, v$

$\iff \rho_2(g) \circ \phi \circ \rho_1(g)^{-1} = \phi$.

G-invariants

$$\iff \phi \in \mathrm{Hom}(V_1, V_2)^G$$
$$\parallel$$
$$(V_1^* \otimes_k V_2)^G$$

Definition: A rep. V is irreducible if its only subrep's are $\{0\}$ and itself (V)

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Subspace preserved by group action.

Example If X is a finite G-set then

$K[X]$ = K - vector space with basis X

$$\sum_{x \in X} a_x x$$

$$g \sum_{x \in X} a_x x = \sum_{x \in X} a_x \, g \cdot x.$$

Claim: If $X = \{*\}$ then $K[X]$ is not irreducible.

$\langle \sum_{x \in X} x \rangle$ is a subrepresentation. (action of G is trivial on this line).

$$\| $$

$K[X]^G$ ← if X is a transitive G-set.

Example: 1-dim'l representations are characters (in the sense of week 6).

The meaning of this word will soon change.

If V is 1-dimensional $GL(V) \cong K^\times$

$$\rho : G \to K^\times$$

and they are irreducible.

For non-abelian groups there are higher dim'l irreps.

Example $S_3 \circlearrowright \{1,2,3\}$.

$$\rightsquigarrow$$

$S_3 \circlearrowright \mathbb{C}^3$ by permutation of coordinates

Not irreducible:
saw $\langle e_1 + e_2 + e_3 \rangle$
is a subrep

$\{ae_1 + be_2 + ce_3 \mid a+b+c=0\}$
is a 2-dim'l irreducible subspace.

Theorem: If G is a finite group and char K is coprime to $|G|$, then any finite dim'l representation of G on a K vector space V can be written as a direct sum of irreducible sub-representations $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$.

Schur's Lemma: If G is a finite group and K is algebraically closed and V is an irrep of G

then $\mathrm{End}_G(V)$ $\left(= \mathrm{Hom}(V,V)^G\right)$.
$= K$.

Proof of Schur's Lemma:

Suppose $\phi \in End_G(V)$.

Then ϕ has an eigenvalue $\lambda \in K$.
(because K is alg. closed).

$Ker(\phi - \lambda Id)$ is a nonzero G-invariant subspace.

V irred $\Rightarrow Ker(\phi - \lambda Id) = V$

$\phi - \lambda Id = 0$

$\phi = \lambda Id.$

Proof of complete reducibility:

$K = \mathbb{C}$: suffices to show if

V f.d. rep. $W \subseteq V$ sub rep,

$\exists$ $W' \subseteq V$ s.t. $W \oplus W' = V$
and W' is a subrep.

Suppose V had a Hermitian inner product

$\langle \, , \rangle$ s.t. $\langle \rho(g) v, \rho(g) w \rangle = \langle v, w \rangle$

Then claim $W^{\perp}$ is preserved by G.

To find such an inner product:

Let $(\, ,)$ be an arbitrary Hermitian product,

define

$$\langle v, w \rangle = \frac{1}{|G|} \sum_{g \in G} (\rho(g) v, \rho(g) w)$$

$g \in G$

Proof over arbitrary K:

W invariant

Take W' a complementary subspace

$$\pi : V = W \oplus W' \to W$$
$$(w, w') \to w.$$

$$\overline{\pi} := \frac{1}{|G|} \sum_{g \in G} g \cdot \pi = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi \circ \rho(g)^{-1}$$

Check $\overline{\pi}(w) = w$ for $w \in W$

$\operatorname{Im} \overline{\pi} \subseteq W$. $\overline{\pi}$ G-invariant projection operator

$\operatorname{Ker} \overline{\pi}$ is a G-invariant complementary subspace