If $L/K$ is a field extension
\[ \alpha_1, \ldots, \alpha_n \in L \]
$K(\alpha_1, \ldots, \alpha_n)$ is the smallest subfield of $L$ containing $K, \alpha_1, \ldots, \alpha_n$.

Note: $K(\alpha_1, \alpha_2, \ldots, \alpha_n) = K(\alpha_i)(\alpha_{2i-1}, \ldots, \alpha_{2n})$
\[ = (K(\alpha_1))(\alpha_2)(\alpha_3)(\alpha_4, \ldots, \alpha_n). \]

In a set $\alpha_i \in L \; i \in I$.
Then $K(\alpha_i \mid i \in I)$.

If $L = K(\alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in L$, then $L$ is finitely generated over $K$.

If $[L:K] < \infty$, $L$ is a finite extension of $K$.

Example $K(x)/K$ is not finite.
But it is finitely generated.
Field of rational functions in the variable $x$ over $K$.

$L/K$ is algebraic if every $\alpha \in L$
\[ \text{is algebraic over } K. \]
(Transcendental otherwise).

Theorem $L/K$ is finite if $[L:K] < \infty$. 

if and only if $L/K$ is finitely generated and algebraic.

**Main lemma:** If $L/K$ is finite, then $L/K$ is algebraic.

**Other main lemma:** If $M/L/K$ then $[M:K] = [M:L][L:K]$.

**Proof** Let $\alpha \in L$ and $d = [L:K]$.

$$1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^d$$

$d+1$ vectors in $d$-dimensional space.

There is a non-trivial linear dependence:

$$a_0 \cdot 1 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_d \alpha^d = 0$$

$a_i \in K$.

$$f(\alpha) = a_0 + a_1 \alpha + a_2 \alpha^2 + \cdots + a_d \alpha^d$$

This is a non-zero polynomial w.r.t. $\alpha$ in $K$.

$$f(\alpha) = 0$$

So $\alpha$ is algebraic.

**Theorem:** If $K$ is a field and $f$ is a non-constant polynomial over $K$
Then \( \exists \) an extension \( L/K \) s.t. \( f \) has a root in \( L \).

**Proof:** \( K[y] \) is a PID

Take an irreducible factor \( g(x) \) of \( f(x) \).

\[ L = \frac{K[y]}{g(y)} \] is a field

**Claim:** \( \bar{y} \) for the image of \( y \) in \( L \), then \( \bar{y} \) is a root of \( f(x) \) and \( LC(y) \)

Need to check \( f(\bar{y}) = 0 \) in \( L \)

\( f(x) = g(x)h(x) \).

\[ f(\bar{y}) = g(\bar{y})h(\bar{y}) \]

\( h \) suffice to show \( g(\bar{y}) \) is zero.

\[ g(\bar{y}) \in L = \frac{K[y]}{g(y)} \]

\[ a(n) \overset{df}{=} \lambda(n) \text{ and } \mu(n) \]
Uniqueness statement: If $L_1/K$, $L_2/K$ are both roots of an irreducible $f(x) \in K[x]$, then there is an isomorphism of extensions of $K$

$$L_1 \cong L_2$$

The unique isomorphism sends $a$ to $b$.

Example $\phi(e^{2\pi i/3}) = e^{2\pi i/3}$. 

This isomorphism is not typically unique, but here it is explicitly term of the extra data.
Remark: If \( F \in K[x] \) and \( L_1/K \), \( L_2/K \) are extensions. Then any \( \psi : L_1 \to L_2 \) s.t. \( \psi|_K = \text{Id} \), sends roots of \( f \) in \( L_1 \) to roots of \( f \) in \( L_2 \).

If \( \alpha \in L_1 \) is a root of \( f \):

\[
f = a_0 + a_1x + \ldots + a_dx^d
\]

\( a_i \in K \).

\[
\psi(f(\alpha)) = \psi(a_0 + a_1\alpha + \ldots + a_d\alpha^d)
\]

\[
= \psi(a_0) + \psi(a_1)\psi(\alpha) + \ldots + \psi(a_d)\psi^d(\alpha)
\]

\[
= a_0 + a_1\psi(\alpha) + \ldots + a_d\psi^d(\alpha)
\]

\[
= f(\psi(\alpha))
\]

\( \implies f(\alpha) = 0 \quad f(\psi(\alpha)) = \psi(f(\alpha)) = \psi(0) = 0 \).
**Bootstrap:** Splitting field.

If $F \subseteq K(x)$ then an extension $L/K$ is a splitting field for $F$ if

1. $F$ factors into degree 1 polynomials in $L(X)$.

2. If $\alpha_1, \ldots, \alpha_s$ are the roots of $F$, $\text{deg}(F)$.

Then $L = K(\alpha_1, \ldots, \alpha_s)$.

**Theorem:** Splitting fields exist and any 2 are isomorphic as extensions of $K$.

**Proof:** For existence - suffices to construct an extension satisfying (1) then take subext. generated by roots.

Do by induction on degree.

**Note:** If $L/K$ is a splitting field for $F$ then $[L:K] \leq \text{deg}(F)$.
Given a field $K$, is there an extension $L/K$ s.t. any polynomial splits into linear factors in $L$?

Theorem/Defn: There exist such an extension which is unique, up to isomorphism, such an $L$ is called an algebraic closure of $K$.

Proof
1) Construct an extension where all polynomials split.
2) Show that the subfield of elements algebraic over $K$ is a field.
3) Build up uniquely from splitting field uniqueness.

Part 2: Claim if $L/K$ then
\[ L^{\alpha,B} \leq L \]
\[ \{ \alpha \in L \mid \alpha \text{ is algebraic} \} \]
i.e. a subfield.

Need to check, e.g., if \( \alpha, \beta \) are algebraic then so is \( \alpha + \beta \leq^{-1} \alpha \beta = \alpha. \)

**Note** if \( f(\alpha) = 0 \) \( g(\beta) = 0 \)

It's not clear if \( h \) \( h(\alpha + \beta) = 0. \)

\[ \alpha + \beta \in K(\alpha, \beta) \]

\[ K \leq K(\alpha) \leq K(\alpha, \beta) \]

\[ \text{finite extn.} \quad \text{finite extn.} \]

so \( K(\alpha, \beta)/K \) is finite

\[ (K(\alpha, \beta):K) \leq (K(\alpha):K) (K(\alpha, \beta):K(\alpha)) \]

Finite \( \Rightarrow \) algebraic.
Example: Every polynomial in \( \mathbb{C}[X] \) splits by Liouville's Theorem.

- \( \mathbb{C}/\mathbb{Q} \) is a field extension \( \overline{\mathbb{Q}} := \mathbb{C} \) is an algebraic closure of \( \mathbb{Q} \).