SAS: $\text{GL}_2(\mathbb{F}_3) \times \mathbb{F}_3^2 \times \mathbb{F}_3^2$

(i.e. pairs of column vectors $\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}$)

Q: How many orbits are there for this action?


First orbit: $\begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}$ s.t. $\vec{v}_1 \neq \vec{v}_2$

are linearly independent.

If $(\vec{v}_1, \vec{v}_2)$ are lin. ind.,

• $(\vec{w}_1, \vec{w}_2)$ are lin. ind.

Then there exists a linear transformation sending $\vec{v}_1$ to $\vec{w}_1$ and $\vec{v}_2$ to $\vec{w}_2$

and its inverse.

Second orbit: $\{ (0, \vec{v}) \}$

1 of these $\rightarrow (\vec{v}, \lambda \vec{v})$ for $\vec{v} \neq \vec{0}$ and $\lambda \in \mathbb{F}_3$

(1 of these $\rightarrow (0, \vec{v})$ for $\vec{v} \neq \vec{0}$, $\vec{v} \cdot (\vec{v}, \lambda \vec{v}) = (\vec{v}, \lambda \vec{v})$)
How many elements are in the conjugacy class of $E_{1,0}^0$ in $GL_2(\mathbb{F}_5)$?

30.

Linear algebra answer: $M$ conjugate to $E_{1,0}^0$ if

$$M \text{ has eigenvalues } 1, i, -1, \text{ each with multiplicity } 1.$$ 

To give $M$, just need to give the 1-dim subspaces

$$V_1 \subseteq \mathbb{F}_5^2 \quad V_{-1} \subseteq \mathbb{F}_5^2$$

such that $Mv = v$ for $v \in V_1$ and $Mv = -v$ for $v \in V_{-1}$.

$$\Rightarrow 5^{+1} \quad V_1 \neq V_{-1}.$$ 

6 1-dim subspaces of $\mathbb{F}_5^2$

$\Rightarrow$ 6 choices for $V_1 \times 5$ choices for $V_{-1}$

$= 30$. 
Group Theory reason: 1 orbit of $\mathbb{Z}_4^{1.0}$ under conjugation, $|\text{stab} \, [0.1]| = \frac{|\text{stab} \, [1.0]|}{1}$

$\text{stab} \, [0.1] = \left[ \begin{smallmatrix} 0 & 0 \\ 0 & \alpha \end{smallmatrix} \right] \alpha \in \mathbb{F}_5$

has order $\alpha^2 = 16$.

$|\text{GL}_2(\mathbb{F}_5)| = (5^2-1)(5^2-5) = 4 \cdot 6 \cdot 5 \cdot 1$

1. Every group of order $121 = 11^2$ is abelian.
   
   TRUE. (In class we saw this).

2. Every group of order $573.19$ is abelian.
   
   FALSE: $\mathbb{Z}/19\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

   True $\mathbb{Z}/3\mathbb{Z} \rightarrow (\mathbb{Z}/19\mathbb{Z})^x \cong \mathbb{Z}/18\mathbb{Z}$

   non-trivial map

3. Every group of order $91 = 7.13$ is cyclic.
   
   TRUE $\eta_3 = 1$, $\eta_7 = 1$. 

6.5

11

30
4. Every subgroup of $Q_8$ is normal.
   TRUE. (In an exercise).

5. The dihedral group $D_{30}$ is solvable.
   TRUE — rotations are a normal subgroup of index 2.

6. There is a simple group of order 60.
   TRUE: $A_5$.

7. There is a simple group of order 168.
   TRUE: $GL_3(\mathbb{F}_2)$. (In class).

8. There is a simple group of order 88 = 8.11
   FALSE: $A_{11}$.

9. There is a simple group of order 360 = $2^3 \cdot 3^2 \cdot 5$.
   TRUE: $A_6$.

10. There is a simple group of order 256 = $2^8$.
    FALSE: Groups prime power order are solvable.

     SA

1. How many Sylow 7-subgroups in a simple group of order $169 = 2^3 \cdot 7$.
   \[ n_7 = 1 \text{ or } 9 \]
   but can't be 1 because simple.
2. How many conjugacy classes in $S_5$?

- $\{e\}$
- $(12)(34)$
- $2$-cycles, $2 \times 2$-cycles
- $3$-cycles, $A_4 2$-cycle + 3 cycle $(12)(345)$
- $4$-cycles
- $5$-cycles

3. How many elements are in the center of $D_{14}$?

$1$. $\mathbb{Z}(D_{14}) = \{e\}$.

$180^\circ$ rotation isn't a symmetry if a mirror runs when $n$ is odd.