Announcements:

- HW for week 6 due March 16.
- Rewrites for problems due March 4th @ 11:59pm. ON GRADESCOPE
- Exam on March 4th. - In class on Zoom. (contact if not possible).
- On March 2nd: Review / Q & A.

This week: Bridge between the first & second halves of the class

Characters:

Let G be a group and L a field.
(eg $G = \mathbb{Z}/n\mathbb{Z}$ $L = \mathbb{C}$).

A character of G (with values in L) is a homomorphism
$$G \to L^\times = GL_1(L) = L \setminus \{0\}.$$

Exercise What are the characters of $\mathbb{Z}/n\mathbb{Z}$ with values in $\mathbb{C}$?
$$\mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$$

Need: $1 \in \mathbb{Z}/n\mathbb{Z}$ maps to an nth root of unity.

Why? $n \cdot 1 = 0$

identity in

$\mathbb{Z}/n\mathbb{Z}$

$$\chi: \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$$

$$\chi(0) = 1$$

$$\chi(\overset{\shortparallel}{n\cdot 1}) = 1$$

$$\chi(1)^n = 1$$

So $\chi(1)$ is an nth root of unity.

Conversely: For any nth root of unity ζ

$\exists!$ $\chi_\zeta: \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$

$\chi_\zeta(1) = \zeta$.

nth roots of unity in $\mathbb{C}^\times$:

Suppose $\zeta^n = 1$, $\zeta \in \mathbb{C}$

Then $|\zeta|^n = |1| = 1$ so $|\zeta| = 1$.

Thus $\zeta \in S^1$ = complex numbers with absolute value 1.

$$\underset{U(1)}{\overset{\shortparallel}{S^1}} \leq \mathbb{C}^\times$$

Since $\zeta \in S^1$ $\exists!$ $\theta \in [0, 2\pi)$

s.t. $\zeta = \cos\theta + i\sin\theta$

$= e^{i\theta}$ (Euler's identity).

$e^{z_1+z_2} = e^{z_1}e^{z_2}$ for any z_1, z_2 in $\mathbb{C}$.

$$(e^{i\theta})^n = e^{in\theta} = 1$$

$\iff$ $n\theta$ is a multiple of 2π.

So $\zeta^n = 1$ $\iff$ $\zeta = e^{2\pi i k/n}$

for $0 \leq k \leq n-1$

$\downarrow$

$k \in \mathbb{Z}/n\mathbb{Z}$

$X(\mathbb{Z}/n\mathbb{Z}) = \mathbb{C}$-valued characters of $\mathbb{Z}/n\mathbb{Z}$.

This is a group: ($X(G)$ for any group).

$(\chi_1 \chi_2)(x) = \chi_1(x)\chi_2(x)$

$$X(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

$$\chi_k(x) = e^{\frac{2\pi i x k}{n}} \longleftrightarrow k$$

$$= \left(e^{\frac{2\pi i k}{n}}\right)^x$$

<u>Example</u> Continuous characters of $S^1 = \{|z|, |z| \leq 1\}$.

Can show: any $\chi : S^1 \to \mathbb{C}^\times$ factors through S^1 and is of the form $z \mapsto z^n$.

$$X(S^1) \cong \mathbb{Z}.$$

$$z \mapsto z^n \leftrightarrow n$$

<u>Example</u> Continuous characters of $\mathbb{R}$

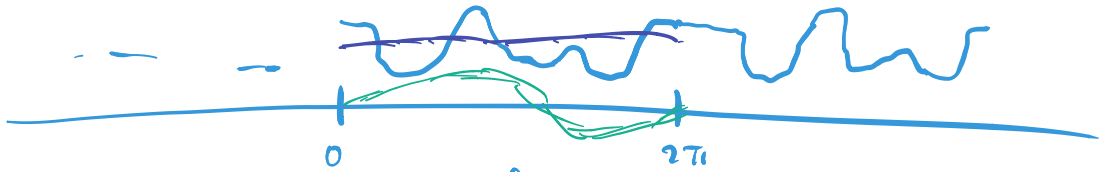
All of the form $t \to e^{\lambda t}$ for $\lambda \in \mathbb{C}$

$$X(\mathbb{R}) = (\mathbb{C}, +).$$

$$e^{\lambda_1 t} e^{\lambda_2 t} = e^{(\lambda_1 + \lambda_2) t}.$$

<u>Fourier Theory:</u>

Periodic function on $\mathbb{R}$



Decompose into its fundamental waves.
(sin & cos waves)

$\lambda \sin(nx)$ $\quad \lambda\cos(mx)$

$\rightsquigarrow$ express a periodic function as

$$\sum_{k\geq 0} \alpha_k \sin(kx) + \beta_k \cos(kx).$$

$\alpha_k, \beta_k \in \mathbb{R}.$

Signal analysis (Harmonic analysis).

What does this have to do with characters?

(1) Periodic function period 2π $\longleftrightarrow$ Function on S^1

$\mathbb{R}/2\pi\mathbb{Z} \cong S^1$

$t \mapsto e^{it}$

Real valued functions $\subseteq$ complex valued functions

$\cos(n\theta) + i\sin(n\theta) = \underline{e^{int}}$

$$\frac{e^{-int} + e^{int}}{2} = \cos(nt)$$

$$\frac{e^{int} - e^{-int}}{2i} = \sin(nt).$$

$t \mapsto e^{int}$ as a function ... on S^1.

$\mathbb{R}/2\pi\mathbb{Z}$ $\longleftrightarrow$ $z \mapsto z^n$

(a character of S^1).

Fourier Theory for S^1:

$L^2(S^1, \mathbb{C})$

measurable complex functions f on S^1 s.t. $\int |f|^2 < \infty$.

Statement: the characters $z \mapsto z^n$ they form an orthonormal basis for $L^2(S^1, \mathbb{C})$.

Infinite dim'l vector space w/ inner product

$$\langle f, g \rangle = \int_{S^1} f(z)\overline{g(z)}$$

Any $f \in L^2(S^1, \mathbb{C})$

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n \qquad a_n \in \mathbb{C}.$$

$z = e^{2\pi i t}$

$$f(t) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n t} \qquad a_n \in \mathbb{C}.$$

$$\sum_{n \in \mathbb{Z}} |a_n|^2 < \infty.$$

A calculus exercise to prove.

A different perspective:

$L^2(S^1, \mathbb{C})$ or $C^\infty(S^1, \mathbb{C})$

$C^0(S^1, \mathbb{C})$

...

Spaces of functions on S^1.

F = some space of functions on S^1.

$S^1 \circlearrowright F$ by right translation

$z \in S^1$, $f \in F$

$$(z \cdot f)(x) = f(xz).$$

(Could replace S^1 with any group).

F is a $\mathbb{C}$-vector space.

$S^1 \circlearrowright F$ is by linear operators

This is a "representation" of S^1.

(on an infinite dimensional vector space)

Hilbert space w/ inner product preserved by S^1.

This representation decomposes as a direct sum of characters.

$$L^2(S^1) = \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C} \cdot (z \mapsto z^n)$$

— 1-dim'l subspace

decomposition into irred. subrepresentation

$t \in S^1$ $\quad t \cdot (z \mapsto z^n) = t^n (z \mapsto z^n)$