Last time: nilpotent & solvable groups

Today: A few examples

Example: There is no simple group of order 3675.

Suppose $|G| = 3675$. $= 3 \cdot 5^2 \cdot 7^2$

$n_7 \equiv 1 \mod 7$ $\quad | \, 3 \cdot 5^2$

if $n_7 = 1$ done ✓.

otherwise we get $n_7 = 15$.

Let Q be a Sylow-7 subgroup

$N_G(Q) \leq G$. $\quad \dfrac{|G|}{|N_G(Q)|} = 15 \leftarrow n_7$.

$\underline{|N_G(Q)|} = 5 \cdot 7^2$.

$N_G(Q)$ has a unique Sylow 5-subgroup. (by congruence and 1 + ...).

Call it P, $\quad P \trianglelefteq N_G(Q)$.

i.e. $N_G(Q) \leq \boxed{N_G(P)}$.

P is a 5-group in G. $\quad (|P| = 5)$

so contained in P^* a Sylow 5-subgroup of G

$|P^*| = 5^2$.

We've seen a group of order 5^2 is abelian.

so in particular P^* normalizes P.

i.e. $P^* \leq N_G(P)$.

$N_G(Q) \leq N_G(P)$ $\quad \Rightarrow \quad 5 \cdot 7^2 \mid |N_G(P)|$

$P^* \leq N_G(P)$. $\quad\quad 5^2 \mid |N_G(P)|$.

$5^2 \cdot 7^2 \mid |N_G(P)|$.

$[G : N_G(P)] = 3$ or 1.

If it's $1 \Rightarrow N_G(P) = G$

so P is a normal subgroup

otherwise $G \xrightarrow{\text{nontrivial}} S_3$

ker is a nontrivial normal subgroup

because $|G| \geq |S_3| = 6$.

by action on cosets $(\leftarrow \ldots)$.

(i.e. action of P on conjugation).

$\boxed{}$

One more example of constructing a simple group

<u>Example:</u> $GL_3(\mathbb{F}_2)$ is simple.

(Remark: $GL_n(\mathbb{F}_2) = PGL_n(\mathbb{F}_2) = PSL_n(\mathbb{F}_2)$

For p odd $GL_n(\mathbb{F}_p) \twoheadrightarrow PGL_n(\mathbb{F}_p) \trianglerighteq PSL_n(\mathbb{F}_p)$

(index depending on p, n).)

To generalize this example should look at $PSL_n(\mathbb{F}_p)$.

$|GL_3(\mathbb{F}_p)| = (p^3-1)(p^3-p)(p^3-p^2)$.

For a 3×3 matrix to have

$\det \neq 0 \Leftarrow\Rightarrow$

columns are lin. independent.

First column ~ $p^3 - 1$ ← 0 vector

$\uparrow$ all vectors

Second column ~ $p^3 - p$ ← scalar multiples of first column

Third column ~ $p^3 - p^2$ ← linear combinations of first 2 vectors

$$|GL_3(\mathbb{F}_2)| = (2^3-1)(2^3-2)(2^3-2^2)$$
$$= 7 \cdot 6 \cdot 4 = 2^3 \cdot 3 \cdot 7$$
$$= 168.$$

<u>By definition</u> $GL_3(\mathbb{F}_2)$ has a faithful action on $\mathbb{F}_2^3$

$$GL_3(\mathbb{F}_2) = \mathrm{Aut}(\mathbb{F}_2^3).$$

$\mathbb{F}_2$-vector space

$GL_3(\mathbb{F}_2)$ = 3x3 matrices w/non-zero det.

$\mathbb{F}_2^3$ = column-vectors of height 3 w/entries in $\mathbb{F}_2$.

$A\vec{v}$ given by multiplication

A: 3x3, $\vec{v}$: 3x1 matrix.

Q: What are the orbits of $GL_3(\mathbb{F}_2)$ acting on $\mathbb{F}_2^3$?

$A\vec{0} = \vec{0}$ so one orbit is $\vec{0}$

$\mathbb{F}_2^3 \setminus \{\vec{0}\}$ = Non-zero vectors.

$A\begin{bmatrix}1\\0\\0\end{bmatrix}$ = First column of A.

First column of A can be any non-zero

vector.

<u>Claim:</u> $GL_3(\mathbb{F}_2) \circlearrowright \mathbb{F}_2^3 \setminus \{\vec{0}\}$ is faithful.

(If A in kernel then $Ae_1 = e_1$, $Ae_2 = e_2$, $Ae_3 = e_3$, so $A = I_d$.)

i.e. $\displaystyle\bigcap_{x \in \mathbb{F}_2^3 \setminus \{0\}} \operatorname{stab}(x) = \{e\}$

For any $\operatorname{stab}(x)$ has index $\underset{\left|\mathbb{F}_2^3 \setminus \{\vec{0}\}\right|}{\overset{}{7}}$ in $GL_3(\mathbb{F}_2)$.

size $24 = 2^3 \cdot 3$.

$$\operatorname{stab}(e_1) = \operatorname{stab}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

in GL_3 if the 2 columns are lin ind,

so there are $(2^3-2)(2^3-2^2) = 6 \cdot 4 = 24$.

Suppose $H \trianglelefteq GL_3(\mathbb{F}_2)$.

If $H \leq \operatorname{stab}(\vec{e}_1)$.

then $H = \{e\}$.

then $H < \operatorname{stab}(\vec{v})$ for all non-zero so $H \leq \ker$ action on $\mathbb{F}_2^3 \setminus \{0\}$

$H =$ trivial.

So $H \not\leq \mathrm{Stab}(\vec{e_1})$

so $H\,\mathrm{Stab}(\vec{e_1}) = GL_3(\mathbb{F}_2)$.

(Because $[G : \mathrm{Stab}(\vec{e_1})] = 7$).

$7 \mid |H|$.

Sylow 7-subgroups of H

Assume $H \neq G$.

$$|H| = 7\cdot 2^2 \cdot 3$$
$$7 \cdot 2^3$$
$$7 \cdot 2 \cdot 3$$
$$7 \cdot 3$$
$$7 \cdot 2$$

n_7 for H

$= 2^3 = 8$ or 1

Then there is a unique Sylow 7 subgroup in G also because $H \trianglelefteq G$.
(Edited for clarity after lecture).

Exercise: Construct Sylow 7-subgroups of $GL_3(\mathbb{F}_2)$.

By constructing a degree 3 field extension of $\mathbb{F}_2$ $\quad \mathbb{F}_8$
and look at $\mathbb{F}_8^\times \curvearrowright \mathbb{F}_8$

3-dimensional $\mathbb{F}_2$-vector space.

Conclude $|H| = 7 \cdot 2^3$.

<u>To finish</u>: Look at Sylow 2-subgroups. (Exercise).

[Expanded after class: use a counting argument to show H has a unique Sylow 2-subgroup. $H \trianglelefteq GL_3(\mathbb{F}_2)$ + same power of 2 dividing order $\Rightarrow$ $GL_3(\mathbb{F}_2)$ has unique Sylow 2-subgroup. Then get a contradiction by exhibiting 2 distinct Sylow 2-subgroups in $GL_3(\mathbb{F}_2)$.]