This week is the last week on finite group theory.

Next week: character theory of abelian groups

Not on exam

+ Homework will only have exercises.

Rewrites of problems on HW 1-5 sent via email by 11:59pm on March 4th.

all in one pdf in one email

Today: groups built up from abelian groups.

Cyclic groups ⊆ Abelian groups ⊆ nilpotent groups ⊆ solvable groups ⊆ all groups

Nice structure theorem

Jordan-Holder factors are cyclic (for finite solvable groups)

Def'n: the commutator subgroup of G (written $[G,G]$) is the subgroup generated by $(gh)(hg)^{-1}$ $\forall\, g,h \in G$.

$$[H_1, H_2] = \langle (h_1h_2)(h_2h_1)^{-1},\ h_1 \in H_1,\ h_2 \in H_2 \rangle$$

Simple lemma: $[G,G]$ is normal.

$G^{ab} := G/[G,G]$ ⟵ maximal abelian quotient.

$ab(ba)^{-1} = e$. (G^{ab})

so $gh = hg$. (in G),

Easy to check: any map

$$\begin{tikzcd} G \arrow[r] \arrow[d] & H \\ G^{ab} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where H is an abelian group factors uniquely through G^{ab}

$$1 \to [G,G] \to G \to G^{ab} \to 1$$

normal subgroup

abelian

$G^{(0)} = G$

$G^{(1)} = [G,G]$

$G^{(2)} = [G^{(1)}, G^{(1)}]$

$G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

$$\cdots \trianglelefteq G^{(2)} \trianglelefteq G^{(1)} \trianglelefteq G$$

$$G^{(i)}/G^{(i+1)} = G^{(i)}/[G^{(i)}, G^{(i)}] = (G^{(i)})^{ab}$$ is an abelian group.

Def'n 0. This is called the derived or commutator series of G.

1. G is solvable if $\exists$ $i \geq 0$ s.t. $G^{(i)} = \{e\}$

$\Leftarrow$

$\exists$ any chain

$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_m = G.$

s.t. H_i / H_{i-1} is abelian.

(The chain $G^{(i)}$ is the shortest possible way to do this).

<u>Exercise</u>: Compute the derived series for S_3.

$$\begin{array}{ccccccc}
\cdots \trianglelefteq & S_3^{(2)} & \trianglelefteq & S_3^{(1)} & \trianglelefteq & S_3^{(0)} \\
 & \| & & \| & & \| \\
\cdots & \{e\} & \trianglelefteq & A_3 & \trianglelefteq & S_3 \\
 & & & \| & & \\
 & & & \mathbb{Z}/3\mathbb{Z} & &
\end{array}$$

$S_3 \xrightarrow{\text{sgn}} \{\pm 1\}$. $[g,h] = ghg^{-1}h^{-1}$ even. so $[S_3, S_3] \trianglelefteq A_3$

is a map to an abelian group.

can't be $\{e\}$ because S_3 is abelian

So its Kernel contains $[S_3, S_3]$.

A_3 //

<u>Example</u> Derived series of S_5:

$$\begin{array}{ccccccc}
\cdots \trianglelefteq & S_5^{(2)} & \trianglelefteq & S_5^{(1)} & \trianglelefteq & S_5^{(0)} \\
 & & & & & \| \\
\cdots & A_5 \trianglelefteq & A_5 & \trianglelefteq & A_5 & \trianglelefteq & S_5
\end{array}$$

So S_5 is <u>not</u> solvable.

(Note: If G is simple and not abelian then $\underset{G^{(1)}}{[G, G]} = G$.).

A nice property: If G is a group $H \trianglelefteq G$, then

G is solvable $\Longleftrightarrow$ H and G/H are solveable.

2 nice results.

Theorem (Burnside): If $|G| = p^a q^b$ for p, q prime then G is solveable.

(Come back to this in rep theory section).

Theorem (Feit-Thompson): If $|G|$ is odd then G is solveable. (!).

Nilpotent groups.

$G^0 = G \quad G^1 = [G, G] \quad G^2 = [G, G^1]$

$\ldots \; G^i = [G, G^{i-1}]$

"Lower central series"

$G = G^0 = G^{(0)} \quad G^1 = G^{(1)} = [G, G]$ for $i \geq 2$,

$G^i \geq G^{(i)}$.

A group is nilpotent if $G^i = \{e\}$ for i sufficiently large.

nilpotent $\Rightarrow$ solveable.

Example S_3 is solveable but not nilpotent.

$G^{(2)} = [A_3, A_3] = \{e\}.$

$G^2 = [S_3, A_3] = A_3.$

Upper central series:

$Z_0(G) = \{e\}$

$Z_1(G) = Z(G) \leftarrow$ center.

$Z_2(G) = \pi^{-1}(Z(G/Z(G)))$ $\quad G \xrightarrow{\pi} G/Z(G)$

$Z(G/Z(G))$

$\vdots$

$Z_i(G) = \pi^{-1}(Z(G/Z_{i-1}(G))).$

$Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \dots$

G is nilpotent $\iff Z_i(G) = G$ for i sufficiently large.

Example: S_3:

$$\begin{array}{ccccc} Z_0(S_3) & \trianglelefteq & Z_1(S_3) & \trianglelefteq & \dots \\ \| & & \| & & \| \\ \{e\} & & Z(S_3) & & \\ & & \| & & \end{array}$$

$\{e\}$. $\{e\}$.

Theorem: A finite group G is nilpotent
$\iff$
$$G \cong P_1 \times P_2 \times \dots \times P_n.$$
where $|G| = \prod_{i=1}^{n} p_i^{a_i}$ $p_i \neq p_j$ prime.

P_i = Sylow p_i-subgroup (unique).

$G = A \rtimes \mathbb{Z}/3\mathbb{Z}$.

$\hookrightarrow$ Sylow 5 group

$|A| = 25$

$A = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

or $\mathbb{Z}/25\mathbb{Z}$.

For each map
$$\mathbb{Z}/3\mathbb{Z} \xrightarrow{\varphi} GL_2(\mathbb{F}_5)$$
you get
$$(\mathbb{Z}/5)^2 \rtimes_{\varphi} \mathbb{Z}/3$$