Announcements:

① At some point I said

"$G/Z(G)$ ~~abelian~~ cyclic $\Rightarrow$ G is abelian" (counterexample for abelian is Q_8).

② Problem 2: $n \neq 6$.

③ Cauchy's theorem on HW: Here's an easier/better way to deduce Cauchy from Sylow.

Warmup exercise: Show the only group of order 15 is $\mathbb{Z}/15\mathbb{Z}$.

$n_5 \equiv 1 \bmod 5$, $n_5 \mid 3$ $\Rightarrow n_5 = 1$

$n_3 \equiv 1 \bmod 3$, $n_3 \mid 5$ $\Rightarrow n_3 = 1$

$\boxed{H, K \text{ normal!}}$

Sol. 1.

$$G/_{H\cap K} \cong \underbrace{G/H}_{\text{cyclic}} \times \underbrace{G/K}_{\text{cyclic}}.$$

$H \cap K = \{e\}$ all by order.

Fact: If G is a finite group w/ a unique subgroup of order d for $d \mid |G|$, then G is cyclic. (Fun exercise: proof by induction)

Sol. 2 Invoke this fact.

Sol. 3. $H \trianglelefteq G$ of order 5.

$M < G$ of order 3.

$MH = G$. need to show elements of M commute with elements of G.

$M \circlearrowright H$ by conjugation.

$$M \xrightarrow{\text{group hom}} \mathrm{Aut}_{\text{Groups}}(H)$$

$$\mathbb{Z}/3\mathbb{Z} \cong M \to \mathrm{Aut}_{\text{Grp}}(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$$

$$\mathrm{Aut}_{\text{Grp}}(\mathbb{Z}/5\mathbb{Z}) = (\mathbb{Z}/5\mathbb{Z})^\times$$

the image is trivial.
(kernel is $\mathbb{Z}/3\mathbb{Z}$).
So elements of M commute with elements of H.
(See more of this Thursday for semidirect products).

Finish proof of Sylow's theorem.

Recall: Last time we showed existence of Sylow p-subgroup.

(G finite group $|G| = p^a K$, $p \nmid K$,
then $\exists\, H \leq G$ $|H| = p^a$).

Left to show:

- (# of Sylow p-subgroups) $\mid |G|$ and $\equiv 1 \mod p$.
- Any p-subgroup is contained in a Sylow p-subgroup.
- All Sylow p-subgroups are conjugate.

Proof: G is a finite group $|G| = p^a K$. $p \nmid K$.

Fix a Sylow p-subgroup $P_0 \leq G$.
Let X be the orbit of P_0 under the conjugation action of G.
(Proof is by studying the action of p-subgroups on X).

Lemma: If $P \leq G$ is a Sylow p-subgroup of G & $Q \leq G$ is a p-subgroup of G then $N_G(P) \cap Q = P \cap Q$

$\shortparallel$

Stabilizer of P for the conjugating action of Q on Sylow p-subgroups of G.

<u>**Proof**</u> $H := N_G(P) \cap Q$

$H \geq P \cap Q$.

Have HP is a subgroup of G.

$|HP| = [H : H \cap P]\,|P|$

$\uparrow$ is a power of p because $H \leq Q$.

so $[H : H \cap P] = 1$ because P is a Sylow p-group

So $H \cap P = H$.

$H \subseteq P \qquad H \subseteq Q$.

$H \subseteq P \cap Q$ $\boxtimes$

Step 1: Show $|X| \equiv 1 \bmod p$.

$X = \{P_0, P_1, \ldots, P_r\}$.

distinct conjugates of P_0.

$P_0 \circlearrowright X$ by conjugation.

<u>Orbits</u>: orbit of $P_0 = \{P_0\}$.

orbit of P_i $i \neq 0$.

size of the orbit is

$[P_0 : \mathrm{stab}(P_i)]$

$$\begin{aligned} \mathrm{Stab}(P_i) &= P_0 \cap N_G(P_i). \\ &= P_0 \cap P_i \quad \text{by Lemma.} \\ &\neq P_0 \quad \text{because } P_i \neq P_0. \end{aligned}$$

So $[P_0 : \mathrm{stab}(P_i)] \mid p^k$ and $\neq 1$, thus is divisible by p.

Conclude step 1.

Step 2: Show any p-subgroup Q is contained in a conjugate of P_0. (i.e. an element of X)

$Q \circlearrowright X$.

orbit of P_i has size.

$[Q : \mathrm{Stab}\, P_i]$.

$$\begin{aligned} \mathrm{Stab}\, P_i &= Q \cap N_G(P_i). \\ &= Q \cap P_i \quad \text{(by lemma),} \\ &= Q \text{ if and only if } Q \subseteq P_i. \end{aligned}$$

either size divisible by p or $Q \subseteq P_i$

$$|X| = \sum_{\substack{\mathcal{O} \text{ orbit} \\ \text{of } Q}} |\mathcal{O}| \equiv 1 \bmod p.$$

If stab of each P_i is a proper subgroup of Q then this would be $\equiv 0 \bmod p$.

Theorem A_n is simple for $n \geq 5$.

Proof By induction on n.

Base case: $n=5$ ✓ (by counting conjugacy classes).

Inductive step: $n \geq 6$ and A_{n-1} is simple, $\leq S_n$.

$G_i = \mathrm{Stab}_{A_n}(i) \leq A_n \quad (\text{for } A_n \circlearrowright \{1,\dots,n\})$

$G_i \leq \mathrm{Stab}_{S_n}(i) \cong S_{n-1}$

$G_i \xrightarrow{\sim} A_{n-1} \leq S_{n-1}$

So G_i is simple.

Observation: $A_n = \langle G_1, G_2, \dots, G_n \rangle$

Why: If $\sigma \in A_n$. $\sigma = \tau_1 \cdots \tau_{2m}$

τ_i: transposition

$\sigma = (\tau_1\tau_2)(\tau_3\tau_4)\cdots(\tau_{2m-1}\tau_{2m})$.

each $\tau_i\tau_{i+1}$ is in G_j for some j.

Main tool: If $H \trianglelefteq A_n$ is normal and it contains a nontrivial σ s.t. $\sigma(i) = i$ for some i

then $H = A_n$.

Proof: $\sigma \in G_i$ so $H \cap G_i$ is a normal subgroup of G_i that is not trivial thus it is equal to G_i (because G_i is simple).

so $H \supseteq G_i$.

the G_j are all conjugate so $H \supseteq G_j$ for each j.

thus $H = A_n$.

Rest of argument: give yourself a non-trivial element, play around with conjugation to get one s.t. $\tau(i) = i$.