G is simple if it has no normal subgroups except $\{e\}$ and G.

Example: $\{e\}$ $\quad \mathbb{Z}/p\mathbb{Z}$ p prime $\quad A_n$ for $n \geq 5$.

Only simple abelian groups (every subgroup of an abelian group is normal).

We'll talk about a lot!

Recall:

$$\begin{tikzcd} S_n \arrow[r, hook] \arrow[dr, "sgn"'] & GL_n(\mathbb{C}) \arrow[d, "\det"] \\ & \mathbb{C}^* \end{tikzcd}$$

(permuting the coordinates).

$\mathrm{Im}(sgn) = \pm 1$

$A_n = \mathrm{Ker}\ sgn.$

Note: Depending on how you define det this may be cyclic reasoning!

Def'n of det: $GL_n(\mathbb{C}) \circlearrowright \mathbb{C}^n$ linear

$GL_n(\mathbb{C}) \circlearrowright \Lambda^n \mathbb{C}^n$ linear

1-dim'l

Fixing a basis for $\Lambda^n \mathbb{C}^n$ $\longleftrightarrow$ Fixing a nonzero alternating n-linear form $\langle , \ldots , \rangle$ on $\mathbb{C}^n$

so $GL_n(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}}(\Lambda^n \mathbb{C}^n) = \mathbb{C}^*$

$\det$.

$g^* \langle , \ldots , \rangle = \det g \langle , \ldots , \rangle.$

$e_1 \wedge \ldots \wedge e_n$ is a basis for $\Lambda^n \mathbb{C}^n$ express in the basis then get the expression in terms of columns / sgn

Corollary of the def'n:

$$\sigma = \tau_1 \tau_2 \cdots \tau_m$$

τ_i are transposition (2-cycle). e.g. $\tau_i = (12)$

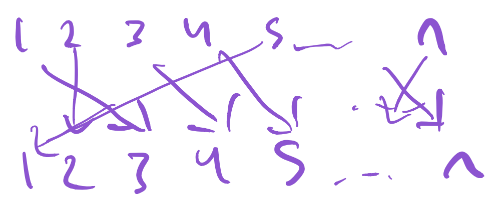
$$\mathrm{sgn}(\sigma) = \prod_{i=1}^{m} \mathrm{sgn}(\tau_i) = (-1)^m$$

τ_i swaps 2 coordinates in $\mathbb{C}^n$ = reflection. so det -1.

Transpositions generate S^n — so every permutation is a product of transpositions

$$\mathrm{sgn}(\sigma) = (-1)^{\#\text{ inversions}}$$

$\leftarrow$ crossing of 2 arrows



<u>Def'n</u> $A_n = \mathrm{Ker}\ \mathrm{sgn} \subseteq S_n$

for $n \geq 2$ $\quad S_n / A_n = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$

$[S_n : A_n] = 2.$

= "even permutations" $\leftarrow$ If written as product of transposition there are an even #.

<u>Example</u> A_5 is simple by a counting argument.

<u>2 Facts</u> (1) Any normal subgroup is a union of conjugacy classes.

(2) Lagrange: order divides $|A_5| = \frac{|S_5|}{2}$

$$= \frac{5!}{2} = 60.$$

Conjugacy classes have sizes:

1,	15,	20,	12,	12
$\{e\}$	$(ab)(cd)$	(abc)	conj. of 5-cycles	

Labelled wrong in the lecture at first! p.127 DF.

Try to add these up and get a non-trivial divisor of 60

.... Can't so group is simple.

Sylow's Theorem(s):

A p-group is a group of order a power of p
↑ a prime number.

$H \leq G$ is a p-subgroup if it's a p-group

it's a <u>sylow p-subgroup</u> if it it's a p-subgroup.

$p \nmid [G:H]$

i.e. if $|G| = p^a K$, $p \nmid K$

then H is a sylow p-subgroup if $|H| = p^a$.

<u>Theorem:</u> If G is a finite group.

① G has a Sylow p-subgroup and they are all conjugate.

② If $H \leq G$ is a p-subgroup then it is contained in a Sylow p-subgroup

(3) If n_p denotes the # of Sylow p-subgroups, then

$$n_p \mid K \qquad (|G| = p^a K \quad p \nmid K),$$

$$n_p \equiv 1 \bmod p.$$

Observation: $\Rightarrow$ a Sylow p-subgroup is normal $\Leftrightarrow \underline{\underline{n_p}} = 1.$

some control over this # by 3.

Example There are no simple groups of order $448 = 2^6 7$

Suppose $|G| = 448.$

$$n_7 \mid 2^6 \qquad \equiv 1 \bmod 7$$

$$n_7 = 1 \qquad n_7 = 8 \qquad n_7 = 64$$

$$n_2 \mid 7 \qquad n_2 \equiv 1 \bmod 2 \leftarrow \text{no information}$$

$$n_2 = 1 \qquad n_2 = 7.$$

If $n_2 = 1$ then Sylow 2-subgroup is normal

If $n_2 = 7$ then

$$G \circlearrowright \{\text{sylow 2-subgroups}\} \leftarrow 7 \text{ elements}$$

$$G \xrightarrow{\phi} S_7$$

action non-trivial because transitive

$$\operatorname{Ker}\phi \trianglelefteq G, \quad \operatorname{Ker}\phi \neq G$$

Want to show $\operatorname{Ker}\phi \neq \{e\}.$

(then it's a non-trivial normal subgroup).

If $\operatorname{Ker}\phi = \{e\}$,

$\phi: G \hookrightarrow S_7$.

so $|G| \mid |S_7| = 7!$

$|G| = 2^6 \cdot 7$

$= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 2^4 \cdot 7 \cdot 3 \cdot 5 \cdot 3 \cdot 1$

$2^6 \nmid 2^4$.

So $\operatorname{Ker}\phi \neq \{e\}$, G is a nontrivial normal subgroup.

Proof

part of ①

Existence: Induction on the order of the group, $|G|$.

Base case: $|G| = p$ prime.

$G = \mathbb{Z}/p\mathbb{Z}$. G is itself a Sylow p-subgroup.

Inductive step: $|G| = p^a K$. $a \geq 1$.

Use class equation.

Fix representatives $g_1, \dots, g_m$ for the non-central conjugacy classes.

$$|G| = |Z(G)| + \sum_{i=1}^{m} [G : C_G(g_i)].$$

Suppose $p \nmid [G : C_G(g_i)]$.

then $|C_G(g_i)| = \frac{|G|}{[G : C_G(g_i)]}$

$[G : C_G(g_i)]$

$= p^a \ell < |G|.$

Ind hypothesis $\Rightarrow$

$C_G(g_i)$ has

a Sylow p-subgroup.

thus so does G.

If $p \mid [G : C_G(g_i)]$ $\forall$ i

since $p \mid |G|$.

so $p \mid |Z(G)|$.

$Z(G)$ is a finite abelian group.

So it contains a subgroup of order p.

$H \leq Z(G)$.

H is a normal subgroup of G.

of order p

$H \to G \xrightarrow{\pi} G/H$

$|G/H| = p^{a-1} k < |G|.$

so it has a Sylow p-subgroup.

M

$$|\pi^{-1}(M)| = |H||M|$$

$$= p \, p^{a-1}$$

$$= p^a. \quad \boxtimes$$

Discussion after class:

Average # of k-cycles in a permutation in S_n.

$$= \frac{\sum_{\sigma \in S_n} \#\ k\text{-cycles in } \sigma}{n!}$$

$$\sum_{\sigma \in S_n} \#\ k\text{-cycles in } \sigma$$

$$= \sum_{k\text{-cycles } \tau} \#\text{ of } \sigma \text{ in } S_n \text{ containing } \tau.$$

$$= \sum_{k\text{-cycles } \tau} (n-k)!$$

(123) $\underline{}\ \underline{}$ 4 5 6 7

$$= \left(\#\text{ of } k\text{-cycles}\right)(n-k)!$$

$$= \left(\binom{n}{k}/k\right)(n-k!)$$

$$= \frac{n!}{k}.$$