Goal this week: Sylow theorem(s).

$\uparrow$ structure of subgroups of prime power order in a finite group.

A few more adjectives for group actions:

$G \circlearrowright X$ faithfully if $G \to \mathrm{Aut}_{set}(X)$ is injective,

i.e. if kernel $= \{e\}$.

Q: If $H \leq G$, $G \circlearrowright G/H$

when is this faithful?

A: If and only if the largest normal subgroup of G contained in H $\{e\}$.

$$\mathrm{Ker} = \bigcap_{gH} \mathrm{stab}(gH) = \bigcap_{g\in G} \mathrm{stab}(gH) = \underline{\bigcap_{g \in G} gHg^{-1}}$$

(always have $\mathrm{stab}(gx) = g\,\mathrm{stab}(x)\,g^{-1}$).

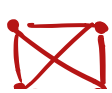
the largest normal subgroup of G contained in H.

Example $G \circlearrowright G$ by left multiplication

$S_n \circlearrowright \{1, \ldots, n\}$ is faithful.

$\mathrm{stab}(k) \cong S_{n-1}$ but intersection of all is trivial.

$G \circlearrowright G$ by conjugation is faithful $\Leftrightarrow Z(G) = \{e\}$.

$D_8 \circlearrowright$  action corners is faithful.

r_1 & r_2 diagonal reflections

12:50 pm.

stabilizers are $\langle r_1 \rangle$ and $\langle r_2 \rangle$.

$\langle r_1 \rangle \cap \langle r_2 \rangle = \{e\}$

Cayley's Theorem/Observation: Any finite group can be realized as a subgroup of S_n for n sufficiently large.

Proof: $G \hookrightarrow Aut_{set}(G) \cong S_n$ for $n = |G|$.

left multiplication action

(sometimes optimal sometimes not)

$\updownarrow$

Q_8

Last week we talked a lot about $G \circlearrowright G$ by conjugation.

$G \circlearrowright \{\text{Subsets of } G\}$ by conjugation.

$A \subseteq G$

$g \cdot A = gAg^{-1} = \{gag^{-1} \mid a \in A\}$.

(Note: If $G \circlearrowright X$ then $G \circlearrowright$ subsets of X.)

If $A \subseteq G$ is a subset

$N_G(A) =$ stabilizer of A under conjugation

$N_G(A) \circlearrowright A$ by conjugation.

$C_G(A) =$ Kernel of this action.

= all elements of G that commute with everything in A.

$\{\text{subgroups of } G\} \subseteq \{\text{subsets of } G\}$.

↑ Sub G-set for the conjugation action.

(Why?: $gabg^{-1} = ga\underbrace{g^{-1}g}_{e}bg^{-1} = (gag^{-1})(gbg^{-1})$.

If $H \leq G$ then for any $g \in G$

$H \to gHg^{-1}$

$h \mapsto ghg^{-1}$ is an isomorphism of groups.

(conjugate subgroups of G are isomorphic as groups).

If $H \leq G$ then $N_G(H) \circlearrowright H$

$N_G(H) \to \mathrm{Aut}_{\text{group}}(H)$.

1:05pm.

$N_G(H)/C_G(H) \hookrightarrow \mathrm{Aut}_{\text{group}}(H)$.

Normal subgroups.

Definition/theorem: A subgroup $H \leq G$ is <u>normal</u> if any of the following equivalent conditions hold:

(1) H is the kernel of a homomorphism $G \to G'$.

(2) $N_G(H) = G$ (i.e. $gHg^{-1} = H \;\; \forall g \in G$).

(3) ...

(3) $G/H = H\backslash G$ $\left(\Leftrightarrow gH = Hg \text{ for any } g \in G \right)$.

$\subseteq$

$\{\text{Subsets of } G\}$. ($\Leftrightarrow$ any left coset is a right coset
$\Leftrightarrow$ any right coset is a left coset).

(4) H is a union of conjugacy classes in G.

To show (2)-(4) $\Rightarrow$ (1)

Show G/H $(= H\backslash G)$ has group structure

$(g_1H)(g_2H) = g_1g_2H$.

works for a normal subgroup.

Just multiplication of subsets

$AB = \{ab \mid a \in A,\ b \in B\}$

$$g_1H\,g_2H = g_1g_2g_2^{-1}Hg_2H = g_1g_2HH = g_1g_2H.$$

For H normal:

$H = \ker G \to G/H$.

Exercise: If H_1, H_2 are subgroups of G then H_1H_2 is a subgroup of G
$\Leftrightarrow H_1H_2 = H_2H_1$.

Example: If $H_1 \leq G$ and $H_2 \leq N_G(H_1)$

then $H_1H_2 = H_2H_1$

(if $a \in H_1$, $b \in H_2$, $ab = bb^{-1}ab$

$$= \underbrace{b}_{\in H_2}\underbrace{(b^{-1}ab)}_{\in H_1}$$

diamond.

Second isomorphism theorem: If $H \le G$, $M \le N_G(H)$

then:

1. $HM \le G$
2. $H \trianglelefteq HM$.
3. $M/_{M \cap H} \xrightarrow{\sim} HM/_{H}$

$$\begin{tikzcd}[column sep=small]
& HM \arrow[dl, dash] \arrow[dr, dash] & \\
M \arrow[dr, dash] & & H \arrow[dl, dash] \\
& M \cap H &
\end{tikzcd}$$

(Numeric version of this is true $\forall$ M, H).

Jordan-Holder program: Classify finite groups.

Find building blocks

$\boxed{\text{finite Simple groups}}$

How to put them together.

$\boxed{\text{Extensions}}$

G is simple if the only normal subgroups are $\{e\}$ and G.

If G is not simple. Take N to be a normal subgroup $\neq \{e\}$ or G.

$$1 \to N \to G \to G/N \to 1$$

<u>Jordan Holder theorem</u>: G has a filtration by subgroups

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \cdots \trianglelefteq G_N = G.$$

$$\underbrace{G_i/G_{i+1} \text{ simple.}}_{\text{the isomorphism classes of these don't depend on the choice.}}$$