Continue group action / G-sets.

**Last time:** Basic idea. Cosets, orbits, Lagrange's theorem ($\text{If } H \leq G \Rightarrow |H| \mid |G|$).

Classification of G-sets

Examples

If $G \circlearrowright X$ ($X$ is a G-set).

Orbits = equivalence classes for $x \sim y \iff \exists g \; gx = y$.

$G \circlearrowright X$ is transitive if there is a single orbit.

If $x \in X$ $\underbrace{\mathrm{stab}(x)}_{\text{stabilizer}} = \{g \in G \mid gx = x\} \leq G$

$G \circlearrowright X$ is free if $\mathrm{stab}(x) = \{e\} \leftarrow \forall x \in X$.

**Classification:** If $X$ is a G-set.

(1) $X = \bigsqcup_{\substack{\text{orbits} \\ \Theta}} \Theta$ and each orbit $\Theta$ is a transitive G-set.

(2) For any orbit $\Theta \subseteq X$ and $x \in \Theta$ there is a canonical isomorphism of G-sets

$$G/\mathrm{stab}(x) \xrightarrow{\sim} \Theta$$

sending $e \to x$.

**Q: What are the free transitive G-sets?**

Only one: $G$ itself (by left multiplication).
(up to isomorphism) $\quad = G/\{e\}$.

**Example:** $SL_2(\mathbb{R}) \circlearrowright \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

$SL_2(\mathbb{R}) \leq SL_2(\mathbb{C}) \circlearrowright \mathbb{P}^1(\mathbb{C})$

Transitive. $\mathrm{stab}(i) = SO(2)$. (= rotation matrices)

$\cup\!|$ $\qquad SL_2(\mathbb{R})/SO(2) = \mathbb{H}$

$\begin{pmatrix} z \\ i \end{pmatrix} \quad z \in \mathbb{H}$

This how we understand curved spaces

**Theorem (orbit-stabilizer)**

If $G \curvearrowright X$ and $x \in X$

$$|G| = |\mathcal{O}(x)|\,|\mathrm{Stab}(x)|.$$

$\mathcal{O}(x)$: orbit of $X$

← Most useful packaging of basic counting argument for group actions

**Proof:** $G/_{\mathrm{Stab}(x)} \cong \mathcal{O}(x)$.

$$|G/\mathrm{stab}(x)| = |\mathcal{O}(x)|$$

$$|G/\mathrm{stab}(x)| = |G|/|\mathrm{Stab}(x)|$$

$$\Longrightarrow |G| = |\mathrm{Stab}(x)|\,|\mathcal{O}(x)|.$$

↳ same argument as Lagrange's Theorem

↳ $|H|\,(\#\text{ of cosets}) = |G|$

Conjugation action:

If $G$ is a group then $G \curvearrowright G$

$$g \cdot x = g x g^{-1}.$$

If $x \in G$ $\mathrm{stab}(x)$ for this action is called the centralizer of $x$.

$$\mathrm{stab}(x) = \{g \in G \text{ s.t. } g x g^{-1} = x\}$$

$$g x = x g.$$

$C_G(x) = \mathrm{Cent}(x) =$ all of the elements of $G$ s.t. multiplication with $x$ commutes.

$$Z(G) = \text{Center of } G = \bigcap_{x \in G} \mathrm{cent}(x)$$

i.e. $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$.

= Kernel of the conjugation action.

More notation! $G \circlearrowright X \leftrightarrow G \to \mathrm{Aut}_{set}(X)$

Kernel of action = kernel of homomorphism.

$= \bigcap_{x \in X} \mathrm{stab}(x)$.

Orbits of the conjugation action are called conjugacy classes.

(2 elements in the same orbit are conjugate).

$x$ conjugate to $y \iff \exists g \in G$ s.t. $gxg^{-1} = y$.

Example If $G = GL_n(\mathbb{R})$

then conjugacy classes

" matrices that give the same linear transformation in different bases.

Example If $G = S_n = \mathrm{Aut}_{set}(\{1, 2, \ldots, n\})$.

conjugacy classes = elements with same cycle type

e.g. $(12) \sim (23)$

transpositions form a conjugacy class.

$\tau\Big((a_1 a_2 \ldots a_m)(b_1 \ldots b_n) \ldots\Big)\tau^{-1}$

$= (\tau(a_1)\ \tau(a_2)\ \tau(a_3) \ldots)(\tau(b_1) \ldots) \ldots$

(good exercise).

This is both conceptually & literally the same as matrix conjugation

$S_n \hookrightarrow GL_n(\mathbb{R})$

$S_n \to GL_n(\mathbb{R})$.
$\sigma \mapsto (e_i \mapsto e_{\sigma(i)})$.

!! Special:
If $H \leq G$
then there can be
$x, y$ not conjugate
in $H$ but conjugate
in $G$.

**Exercise:** What are the conjugacy classes in $Q_8$?

quaternion group
$\{\pm 1, \pm i, \pm j, \pm K\}$.
$ij = K \quad ji = -K \ldots$

**Answer:** $\{1\} \quad \{-1\} \quad \{i, -i\} \quad \{j, -j\} \quad \{K, -K\}$.

Example computation:
$\langle g \rangle \leq C_G(g)$.

Conjugacy class of $i$: can't be $\{i\}$ itself
because conjugacy class
1 elt. $\leftrightarrow$ $i$ is in $Z(G)$.

$\{i, -1, -i, 1\} \subseteq C_{Q_8}(i) \neq Q_8$
$|\text{orbit}| = 8 / |C_{Q_8}(i)| = 2$.

$jij^{-1} = -i$.

$g \in Z(G) \leftrightarrow$ Conjugacy class of $g = \{g\} \leftrightarrow C_G(g) = G$.
($g$ central)

**Class equation:** $G = \bigsqcup$ conjugacy classes.
$|G| = \sum$ size of conjugacy classes.

$$|G| = |Z(G)| + \Sigma \text{ size of nontrivial conjugacy class}$$

$$|G| = |Z(G)| + \underset{C_i \text{ non-trivial conj classes}}{\Sigma |C_i|}$$

Fix $x_i$ in each $C_i$

$$|G| = |Z(G)| + \Sigma |G|/|C_G(x_i)|$$

Each term divides $|G|$, & only the first term can be $=1$.

**Example** For $Q_8$ $\quad 8 = \underbrace{2}_{\substack{Z(Q_8) \\ \{1\}\ \{-1\}}} + \underbrace{2}_{\{\pm i\}} + \underbrace{2}_{\{\pm j\}} + \underbrace{2}_{\{\pm k\}}$

**Application:** If $p$ is prime any group of order $p^2$ is abelian (so $\cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$).

Class equation

$$p^2 = |Z(G)| + \Sigma |C_i|$$

$\neq 1$ divides $p^2$ so divisible by $p$.

mod $p$

$$0 = |Z(G)| \text{ mod } p.$$

i.e. $p \mid |Z(G)|$.

Case 1 $\quad |Z(G)| = p^2$ then $Z(G) = G$ so abelian.

Case 2 $\quad |Z(G)| = p$.

$Z(G)$ always normal!

$$\left|G/Z(G)\right| = p^2/p = p.$$

$\Rightarrow$ $G/Z(G)$ is abelian. ($\cong \mathbb{Z}/p\mathbb{Z}$).

**Exercise** For any $G$: if $G/Z(G)$ is cyclic ~~is abelian~~ then $G$ is abelian

(i.e. if $G$ is not abelian then $G/Z(G)$ not abelian).

corrected 2020-02-07.
Sorry! Counterexample to original statement: $Q_8$.

After class discussion sorting out $\sigma$ vs. $\sigma^{-1}$!

$$S_n \to GL_n(\mathbb{R})$$
$$\sigma \mapsto \underbrace{e_i \mapsto e_{\sigma(i)}}_{T_\sigma}.$$

$$\begin{aligned} T_\sigma(T_\tau(e_i)) &= T_\sigma(e_{\tau(i)}) \\ &= e_{\sigma(\tau(i))} \\ &= e_{\sigma\tau(i)} \\ &= T_{\sigma\tau}(e_i). \end{aligned}$$

Functor from sets to $\mathbb{R}$-vector spaces

$S \mapsto F(S)$
$\quad = \mathbb{R}$-vector space with basis $e_s$ $s \in S$.

covariant.

$$S_n \to \mathrm{Aut}_{\{set\}}(\{1,\dots,n\}) \xrightarrow{F} \mathrm{Aut}_{\mathbb{R}\text{-v.s.}}(F\{1,\dots,n\})$$

$$= \mathrm{Aut}(\mathbb{R}^n).$$

Functor from sets to $\mathbb{R}$-vector spaces

$$S \to \mathrm{Maps}(S, \mathbb{R}).$$

contravariant.

basis $\delta_1, \dots, \delta_n$.

$$\delta_i(j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd} S_n^{op} \arrow[r] \arrow[rr, bend right] & \mathrm{Aut}_{(set)}(\{1,\dots,n\})^{op} \arrow[r] & \mathrm{Aut}_{vect}(\mathrm{Maps}(S,\mathbb{R})) \end{tikzcd}$$

$$(\delta_i \cdot \sigma)(j) = \delta_i(\sigma(j)).$$

$$= \delta_{\sigma^{-1}(i)}.$$

Turn right into left by precomposing with inverse

$$\sigma \cdot \delta_i = \delta_i \cdot \sigma^{-1} = \delta_{(\sigma^{-1})^{-1}(i)} = \delta_{\sigma(i)}.$$

$$A \to (A^T)^{-1}$$

expresses pull back by $A^{-1}$ on the dual space. in the dual basis.