1. Subgroups.

If G is a group, a subset $H \subseteq G$ is a subgroup if

$$\{e\} \in H$$
$$x \in H \Rightarrow x^{-1} \in H$$
$$x, y \in H \Rightarrow xy \in H.$$

Examples: $2\mathbb{Z} \subseteq \mathbb{Z}$. $\rightsquigarrow$ If R is a ring $I \subseteq R$ an ideal then I is a subgroup.

$\{e\} \subseteq G$ always a subgroup (trivial).

$G \subseteq G$ always a subgroup

$O(2) \leq GL_2(\mathbb{R})$

Any matrix group $\subseteq GL_n(K)$.

$SL_n(K) \subseteq GL_n(K)$

$\overset{\shortparallel}{\det = 1}$ matrices

or stabilizer of $e_1 \wedge \dots \wedge e_n$.

$\mathbb{R}_{>0} \leq \mathbb{R}^\times$

$D_{2n} \leq O(2) \leq GL_2(\mathbb{R})$.

$\llcorner$ linear transformations that preserve distance.

Example/Def'n:

If G is a group & $S \subseteq G$ is a subset, can consider

$\langle S \rangle$ the subgroup generated by S.

= the smallest subgroup of G containing S.

$$\langle S \rangle = \bigcap_{S \subseteq H \leq G} H$$

↑ Non-empty because $S \subseteq G$.

Intersection of subgroups is a subgroup.

<u>Example</u> If G is a group

$g \in G$ $\quad \langle g \rangle = \langle \{g\} \rangle.$

$$= \{ g^n : n \in \mathbb{Z} \}.$$

$\mathbb{Z}$ if $g^n \neq e$ for any n.

$\mathbb{Z}/n\mathbb{Z}$ for n the smallest pos integer s.t. $g^n = \{e\}$

<u>Definition</u> G is generated by $S \subseteq G$ if $\langle S \rangle = G$.

<u>Example</u>: $\langle G \rangle = G$. $\quad n\mathbb{Z} = \langle n \rangle$

$\mathbb{Z} = \langle 1 \rangle$

$\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$

$PSL_2(\mathbb{Z}) = {}_{\pm I}\backslash SL_2(\mathbb{Z}) = \langle \text{elt of order 3}, \text{elt of order 2} \rangle$

A group is cyclic if it is generated by a single element, in which case

$$G \cong \mathbb{Z} \text{ or } \mathbb{Z}/n\mathbb{Z}.$$

A group is <u>finitely generated</u> if it is generated by a finite subset.

<u>Example:</u> • Any finite group (e.g. S_n).

<u>Example of a group that is not finitely generated:</u>

• $(\mathbb{Q}, +)$

If $S = \{\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\}$,

$d = b_1 \cdots b_n$.

$$\langle S \rangle \subseteq \langle \tfrac{1}{d} \rangle \neq \mathbb{Q}$$

because $\frac{1}{d+1}$ is not in $\frac{1}{d}$.

Abelian groups.

<u>Defn:</u> A group is abelian $ab = ba \ \forall a, b \in G$.

$\mathbb{Z}$-module.

PID

Finitely generated as a group for an abelian group = fin. gen. as a $\mathbb{Z}$-module.

Classification: Any fin. generate abelian group G is isomorphic to

$$\mathbb{Z}^n \oplus \mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}$$

for $m_1 | m_2 | \dots | m_k$.
(for a unique choice of $n, m_1, \dots, m_k$)

Direct sums (coproducts) vs. products.

If G_i are abelian groups for $i \in I$.

$$\begin{array}{ccc} \bigoplus_{i \in I} G_i & & \prod_{i \in I} G_i \\ \cong & & \cong \\ (g_i)_{i \in I} & \subseteq & (g_i)_{i \in I}. \end{array}$$

s.t. $g_i = e$ for all but finitely many i

Has a map $G_i \to \bigoplus_{i \in I} G_i$ for each i.

$$\prod_{i \in I} G_i \to G_i$$

$$\mathrm{Hom}\left(\bigoplus_{i \in I} G_i, H\right) = \prod_{i \in I} (G_i, H).$$

$$\mathrm{Hom}\left(H, \prod_{i \in I} G_i\right) = \prod_{i \in I} \mathrm{Hom}(H, G_i)$$

H another abelian group.

Hom = Hom of abelian group

If G is an abelian group and $S \subseteq G$

for each s get a map $f_s : \mathbb{Z} \to G$
$1 \mapsto s$.

together define a homomorphism

$$\bigoplus_{s \in S} \mathbb{Z} \to G.$$

Image $(f) = \langle S \rangle$.

Product of abelian groups = Product of groups.
(= product of sets + extra structure).

Not true for direct sum.

In sets: disjoint union $\bigsqcup_{i \in I} X_i$

in abelian groups: direct sum $\bigoplus_{i \in I} A_i$

in groups: Free product $\ast_{i \in I} G_i$.

one construction via "all possible words"

another via algebraic topology.
(Van Kampen theorem)

If I is a set then $F(I) = \ast_{i \in I} \mathbb{Z}$.

$$\mathrm{Hom}_{\mathrm{group}}(F(I), G) \cong \mathrm{Hom}_{\mathrm{sets}}(I, G)$$

(F is a left-adjoint functor)

If G is a group. $S \subseteq G$ a set.

$F(S) \to G$.

Image $= \langle S \rangle$.

If G is a group and $S \subseteq G$ s.t. $\langle S \rangle = G$.

$\operatorname{Ker}\pi \hookrightarrow F(S) \xrightarrow{\pi} G$.

$$G \cong F(S)/\ker\pi$$

$\ker\pi$ is a group.

if Fix generators P

then

$$F(P) \twoheadrightarrow \ker\pi$$

$$G = F(S)/\operatorname{Im} F(P).$$

presenting a group.

$$D_8 = \langle r, s \mid r^4 = 1 \quad s^2 = 1 \quad rsrs = 1. \rangle.$$

$$\begin{tikzcd} \ker\pi \arrow[r, hook] & \mathbb{Z} * \mathbb{Z} \arrow[r, two heads, "\pi"] & D_8 \\ \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \arrow[u, bend left, "r^4"] \arrow[u, "s^2"] \arrow[u, bend right, "rsrs"'] & & \end{tikzcd}$$