6320 - Modern Alg II.

Groups, Galois theory, Representation theory.

Field extensions.

Rough schedule:

Week 1 - Motivation, First concepts

Weeks 2-5 - The structure of finite groups.

Week 6 - Character theory of abelian group

Hom$(G, \mathbb{C}^*)$.

Week 7 - Review, midterm.

Week 8 - Spring Pause.

Weeks 9-12 - Galois theory.

Weeks 13-15 - Rep'n theory of finite groups.

Digression:

D_{2n} dihedral group of order $2n$.

Good teacher

Bad teacher

Rigid symmetries of the regular n-gon in the plane.

$\langle r, f \mid r^4 = 1, \ f^2 = 1, \ frf = r^{-1} \rangle$

regular 4-gon

$0°, 90°, 180°, 270°$ rotation

Reflect about any axis or diagonal

Automorphisms = Set w/
preserving structure!
||
Symmetries $\longleftrightarrow$ Structure.

Fewer symmetries. = More structure

$GL_2(\mathbb{R}) \longleftrightarrow \mathbb{R}^2$ as a vector space

$\uparrow$
2×2 invertible matrices

All maps from $\mathbb{R}^2$ to $\mathbb{R}^2$. $\longleftrightarrow \mathbb{R}^2$ as a set

"**Defn**" A group is a set G equipped with

$$m : G \times G \to G$$
$$(a, b) \mapsto ab$$
$$i : G \to G$$
$$a \mapsto a^{-1}$$
$$e : \{*\} \to G$$
$$* \mapsto e.$$

satisfying some compatibilities:
associativity, existence of inverses

$\downarrow$ e.g.

$aa^{-1} = e$

$m(a, \iota(a)) = e.$

Map of groups = maps of sets respecting extra structure (homomorphism).

$$f(ab) = f(a) f(b)$$
$$f(a^{-1}) = f(a)^{-1}$$
$$f(e) = e.$$

Examples: $\mathbb{Z}/n\mathbb{Z}$ $\leftarrow$ cyclic group of order n

$GL_n(\mathbb{R})$ $\leftarrow$ General linear group over $\mathbb{R}$

S_n $\leftarrow$ Symmetric group on n-elts

$SL_n(\mathbb{R})$

$U(n)$

$\mathrm{Aut}_{\text{cover space}}(\underset{\text{universal covering space}}{\tilde{X}} \to X)$

$\mathbb{R}^\times = GL_1(\mathbb{R})$

$\pi_1(X, x)$ $\leftarrow$ Fundamental group.

D_{2n} $\leftarrow$ Dihedral group.

$\mathrm{Diff}/\mathrm{Diff}^0$ $\leftarrow$ Mapping class group.

$R^\times$ for R any ring.
= units in R.

$$S_n = \mathrm{Aut}_{\text{set}}(\{1,2,\ldots,n\})$$

$$GL_n(\mathbb{R}) = \mathrm{Aut}_{\mathbb{R}\text{-vector space.}}(\mathbb{R}^n)$$

$$GL_n(R) = \mathrm{Aut}_{R\text{-module}}(R^n).$$

(R commutative).

$$GL_1(R) = \mathrm{Aut}_{R\text{-module}}(R)$$
$$\| $$
$$R^\times$$

$$O(n) = \mathrm{Aut}_{\substack{\text{inner}\\\text{product}\\\text{spaces}}}(\mathbb{R}^n, \underbrace{\langle\,,\rangle}_{\text{standard inner product.}})$$

$$SL_n(\mathbb{R}) = \mathrm{Aut}_{\substack{\text{vectors}\\\text{spaces w/}\\\text{alt. multilinear form.}}}(\mathbb{R}^n, e_1^* \wedge e_2^* \wedge \cdots \wedge e_n^*)$$

basis for $\Lambda^n(\mathbb{R}^n)^*$

$$= \mathrm{Aut}_{\substack{\mathbb{R}\text{-vector spaces}\\\text{with volume +}\\\text{orientation}}}(\mathbb{R}^n)$$

$S^1 = |z| = 1$

$\mathbb{R} \xrightarrow{\pi} S^1$

$t \mapsto e^{2\pi i t}$

$\mathbb{Z} = \mathrm{Aut}(\mathbb{R} \to S^1)$

acting by translation

"σ" Aut of topological space $\mathbb{R}$ s.t. $\pi(\sigma(t)) = \pi(t)$.

$n = t \mapsto t+n$,

G a group. G is a right G-set.

$x \cdot g = xg$.

$\mathrm{Aut}_{\substack{\text{Right} \\ G\text{-set}}}(G) = G$

G acting by multiplication on the left.