6320-001 - SPRING 2021 - WEEKS 13-15 (4/13, 4/15, 4/20, 4/22, 4/27)

1. The final weeks of 6320

4/13 – Gather.town test, discuss homeworks, Galois theory questions and examples.

4/15 (lecture to post 4/13 by 11:00pm) – Complex representations of finite groups: definitions, examples, and complete reducibility

4/20 (lecture to post 4/18 by 11:00pm) – Character theory I

4/22 (lecture to post 4/20 by 11:00pm) – Character theory II

4/27 (lecture to post 4/25 by 11:00pm) – Groups rings, representations over other fields

5/3 – Final exam (10:30am-12:30pm).

In Dummit and Foote the representation theory of finite groups is developed as a consequence of the general structure theory of finite dimensional semisimple algebras (Wedderburn's theorem). We will mostly eschew this perspective in favor of the more concrete approach where one starts withs actions of a group on a vector space that preserve the linear structure and maintains that language throughout. We state the main results and definitions here:

1.1. Group representations.

For G a group, K a field, and V a K-vector space, a *representation* of G on V is a map $\rho : G \to \mathrm{GL}(V)$ (here $\mathrm{GL}(V)$ denotes the group of vector space automorphisms of V). In what follows we will only consider finite dimensional representations. Sometimes we will refer to a representation using just the vector space V, sometimes just the map ρ, and sometimes the pair (ρ, V).

A homomorphism between representations (ρ_1, V_1) and (ρ_2, V_2) is a linear map $\varphi : V_1 \to V_2$ commuting with the action of G, i.e. such that

$$\varphi(\rho_1(g)v) = \rho_2(g)\varphi(v).$$

It is an isomorphism if it is bijective, or equivalently if it is an isomorphism of the underlying vector spaces. We write the space of homomorphisms between two representations (or G-homomorphisms) as $\mathrm{Hom}_G(V_1, V_2)$. It is a K-vector space, and can naturally be identified with the G-invariant subspace $\mathrm{Hom}(V_1, V_2)^G$ where $\mathrm{Hom}(V_1, V_2)$, the space of all K-linear maps, is equipped with the action $g \cdot \varphi = \rho_2(g) \circ \varphi \circ \rho_1(g^{-1})$. We write $\mathrm{End}_G(V)$ when $V = V_1 = V_2$ is the same representation.

Example. If G acts on a finite set X, write $K[X]$ for the free K-vector space on the basis X, equipped with the action

$$g \cdot \sum_{x \in X} a_x x = \sum_{x \in X} a_x (g \cdot x).$$

In particular, for $X = G$ with the left multiplication action, the representation $K[G]$ is called the *regular* representation of G. The assignment that sends a G-set X to the G-representation $K[X]$ is *functorial*, i.e. it sends maps of G-sets $X \to Y$ to maps of G-representations $K[X] \to K[Y]$ compatibily with composition.

Example. If (ρ, V) is a representation of G over a field of characteristic coprime to $|G|$, then

$$\frac{1}{|G|}\sum_{g\in G}\rho(g) \in \mathrm{End}(V)^G = \mathrm{End}_G(V)$$

is a G-invariant projection operator onto the subspace V^G of vectors invariant under G.

One can form duals, direct sums, tensor products, symmetric products, and exterior products of representations in the obvious way, and the usual identities among these are identities as representations. In particular, the canonical identification

$$\mathrm{Hom}(V_1, V_2) = V_1^* \otimes_K V_2$$

is an identification of representations. Subrepresentations are also defined in the obvious way. A representation (ρ, V) is *irreducible* if it has no subrepresentations except $\{0\}$ and V.

The fundamental result in the representation theory of finite groups is:

Theorem (Complete reducibility). *If K is of characteristic zero and G is a finite group then any representation is a direct sum of irreducible subrepresentations.*

Combined with existence of eigenvalues, we obtain

Corollary (Schur's lemma). *If K is algebraically closed and ρ is an irreducible representation of a finite group G on a K-vector space V then* $\mathrm{End}_G(V) = K$.

If K is not algebraically closed, one instead finds that $\mathrm{End}_G(V)$ is a division algebra.

1.2. **Characters.** The *character* of a representation ρ is the function

$$\chi_\rho : G \to K,\ g \mapsto \mathrm{Trace}(\rho(g)).$$

The following fact is amazing the first time you see it, but (at least when K is algebraically closed), is in fact a straightforward (but clever) consequence of complete reducibility and the formula for projection onto the G-invariant subspace of a representation.

Theorem (Characters determine representations uniquely). *If K is a field of characteristic zero and G is a finite group, then two representations ρ_1 and ρ_2 of G on K-vector spaces are isomorphic if and only if $\chi_{\rho_1} = \chi_{\rho_2}$.*

Characters are much more computable than representations themselves. The main results are encoded in the following theorem, which is most simply stated when $K = \mathbb{C}$:

Theorem (Characters of irreducible representations). *Consider the vector space $C(G)$ of complex valued* class *functions on G, that is functions which are constant on conjugacy classes, equipped with the Hermitian inner product*

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

The characters of the (isomorphism classes of) irreducible representations of G form an orthonormal basis for $C(G)$. In particular,

1. *if ρ is any complex representation of a finite group G and π is an irreducible complex representation of G, then the multiplicity of π in ρ is $\langle \chi_\rho, \chi_\pi\rangle$,*
2. *the number of isomorphism classes of complex irreducible representations of G is equal to the number of conjugacy classes in G, and*
3. *the sum of the squares of the dimensions of (representatives for each isomorphism class of) the irreducible complex irreducible representations of G is equal to $|G|$.*

2. Comments and suggested reading

Dummit and Foote §18.1-3 and §19.1-2 *and/or*
Serre - Linear representations of finite groups §1,2,3,5, and 6 *and/or*
Fulton and Harris - Representation theory 1-3.

Dummit and Foote put a lot of emphasis on the group ring from the start; the presentation we give will is much closer to that in Serre or Fulton and Harris.

3. Homework

Due Thursday, *April 29*, at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

3.1. Exercises.

Complete and turn in ALL exercises:
Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1. Suppose $H \leq G$ is a proper (i.e $H \neq G$) subgroup of a finite group G.

(1) Show that restriction induces a map from class functions on G to class functions on H.
(2) Show that this map is never injective.
(3) Give examples illustrating that it is sometimes surjective and sometimes not.

Exercise 2.

(1) Find all of the 1-dimensional complex irreducible representations of the quaternion group Q_8.
(2) Deduce from your computation in (1) that Q_8 has a unique 2-dimensional irreducible complex representation, and compute its character.
(3) Construct this representation explicitly; call it V.
(4) Can there be a two dimensional real representation W of Q_8 such that $V \cong W \otimes_{\mathbb{R}} \mathbb{C}$?

Exercise 3. The following is another important result in Galois theory that we admit here:

Theorem (Normal basis theorem)**.** *If L/K is a finite Galois extension of fields of characteristic zero with Galois group $G = \mathrm{Gal}(L/K)$, then there is an element $\alpha \in L$ such that*

$$\{\sigma(\alpha) \,|\, \sigma \in G\}$$

form a basis for L as a K-vector space.

(1) Explain how L can be thought of as a representation of G on a K-vector space, then show that the normal basis theorem is equivalent to the statement that L is isomorphic to the regular representation of G.

Exercise 4. In this exercise, we prove the following converse to Week 11-12 Exercise 3, which was used in the proof that a polynomial is solvable in radicals if and only if its Galois group is solvable.

Theorem (Kummer theory)**.** *Suppose K is of characteristic zero and contains all roots of $x^n - 1$. If L/K is Galois with $\mathrm{Gal}(L/K)$ cyclic of order n, then there exists $\alpha \in L$ and $a \in K$ such that $L = K(\alpha)$ and $\alpha^n = a$. In other words,*

$$L = K(\alpha) \cong K[x]/(x^n - a).$$

(1) We fix a primitive nth root of unity $\zeta \in K$ and a generator σ for $\mathrm{Gal}(L/K)$. Let $\beta \in L$ be a normal generator for L/K (i.e. β as in the statement of the normal basis theorem – see the previous exercise), and let

$$\alpha = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \zeta^{-k} \sigma^k(\beta).$$

In other words, up to a scalar, α is obtained from β via the standard projection formula onto the character space for the character $\sigma \mapsto \zeta$ of $\mathbb{Z}/k\mathbb{Z}$.
  (a) Show $\sigma(\alpha) = \zeta\alpha$.
  (b) Show $\alpha \neq 0$.

(2) Show that $a := \alpha^n \in K$, that the minimal polynomial of α over K is $x^n - a$, and that $L = K(\alpha)$.

Exercise 5. In this exercise, we prove

Theorem (Fourier inversion)**.** *Write $\hat{G}$ for the set of isomorphism classes of complex irreducible representations of a finite group G. Let f be a complex valued function on G, and write $\hat{f}$ for the "function" on $\hat{G}$ defined by*

$$\hat{f}(\rho) = \sum_{g \in G} f(g)\rho(g).$$

Then,

$$f(h) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} (\dim \rho) \mathrm{Trace}(\rho(h^{-1})\hat{f}(\rho)).$$

(1) Reduce to showing the inversion formula holds for the identity element (i.e. $h = e$).
(2) Conclude by computing the trace of left multiplication by the element

$$\sum_{g \in G} f(g) g$$

on $\mathbb{C}[G]$ in two different ways.

Exercise 6 – BONUS
Compute the character table of $\mathrm{GL}_3(\mathbb{F}_2)$ (the unique simple group of order 168).