Recap:

- Analogs of the integer. $\rightsquigarrow \mathbb{F}_p[t]$ $\rightsquigarrow$ Integral closure of $\mathbb{Z}$ in $K/\mathbb{Q}$ finite extension $\mathcal{O}_K$.
(Dedekind domains)
- Basic structure of these analogs.
- Factorization ~ failure to be a UFD / class group / unique factorization of ideals.
- Units

2 things we've noticed:

1) Working mod p is very useful! E.g. for factorization in a monogenic ring of integer

2) Embedding K into real / complex numbers was very powerful $\rightsquigarrow$ Minkowski's theorem
Unit theorem.

Where we're going $\rightsquigarrow$ refine working mod p so it looks like embedding K into $\mathbb{R}$ or $\mathbb{C}$.

Definition: If K is a field, an absolute value is a function

$$|\cdot| : K \to \mathbb{R} \quad \text{s.t.}$$

(1) $|x| \geq 0$ and $|x| = 0 \iff x = 0$.

(2) $|xy| = |x||y|$.

(3) $|x+y| \leq |x| + |y|$ (triangle inequality).

Note: $|\cdot|$ is a homomorphism of groups from $(K^\times, \times) \to (\mathbb{R}_{>0}, \times)$.
multiplication

Examples:

- $\mathbb{R}$ with the standard absolute value

$$|\ |_{\mathbb{R}} : x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

- $\mathbb{C}$ with $|\ |_{\mathbb{C}} : z \mapsto |z\bar{z}|_{\mathbb{R}}^{1/2}$

$$x + yi \mapsto \sqrt{x^2+y^2}$$

- $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ $\quad |\cdot|_{\mathbb{R}}$ or $|\cdot|_{\mathbb{C}}$.

- Any field K $\quad |\cdot|_{triv}: \begin{array}{l} x \mapsto 1 \text{ if } x \neq 0 \\ 0 \mapsto 0. \end{array}$ restrict it to $\mathbb{Q}$.

- For k any field $K = k(t)$.

$$|\cdot|_\infty : \frac{f}{g} \mapsto 2^{(\deg f - \deg g)}$$

$$|\cdot|_0 : t^n \frac{f}{g} \mapsto 2^{-n}$$ small if the function vanishes to high order at 0.

for $f(t)$ and $g(t)$ coprime to t in $k[t]$

- $|\cdot|_p$ p-adic absolute value on $\mathbb{Q}$

$$\left|p^n \frac{r}{s}\right|_p = p^{-n}$$

$r, s \in \mathbb{Z}$ coprime to p.

small if the rational number vanishes to high order at p.

- $K = \mathbb{Q}[x]/x^2 - 2$

$$\iota_+ : K \to \mathbb{R}, \quad x \mapsto \sqrt{2}$$
$$\iota_- : K \to \mathbb{R}, \quad x \mapsto -\sqrt{2}$$

$$|a|_+ = |\iota_+(a)|_{\mathbb{R}}$$
$$|a|_- = |\iota_-(a)|_{\mathbb{R}}$$

$$|1+x|_+ = |1+\sqrt{2}|_{\mathbb{R}} \sim 2.4$$
$$|1+x|_- = |1-\sqrt{2}|_{\mathbb{R}} \sim .4$$

- $K = \mathbb{Q}[x]/x^2 + 1$

$$\iota_1 : K \to \mathbb{C}, \quad x \mapsto i$$
$$\iota_2 : K \to \mathbb{C}, \quad x \mapsto -i$$

$$|a|_1 = |\iota_1(a)|_{\mathbb{C}}$$
$$|a|_2 = |\iota_2(a)|_{\mathbb{C}}.$$

$$|\cdot|_1 = |\cdot|_2$$ because $\iota_2(a) = \overline{\iota_1(a)}$

Def'n: An absolute value $|\cdot|$ on K is

archimedean if $\{|m| \quad m \in \mathbb{Z}\}$ is unbounded,

$m = \underbrace{1 + 1 + \cdots + 1}_{m \text{ times}}$ in K.

Otherwise **non-archimedean**.

Example: $|\ |_{\mathbb{R}}$, $|\ |_{\mathbb{C}}$ are archimedean

$|\ |_{triv}$ is non-archimedean.

$|\ |_0$, $|\ |_t$ on $\mathbb{C}(t)$ are non-archimedean.

$|\ |_p$ on $\mathbb{Q}$ is non-archimedean

Any $|\cdot|$ on a field of characteristic $\neq 0$ is non-archimedean.

Fact Every archimedean absolute value comes from $K \hookrightarrow \mathbb{C}$

Thm: $|\cdot|$ is non-archimedean $\iff$ it satisfies (Exercise on worksheet).

$$|x+y| \leq \max(|x|, |y|)$$

strong triangle inequality.

Prop: For K a field, $|\ |_1$, $|\ |_2$ absolute values on K. TFAE:

(a) The metrics $d_1(x,y) = |x-y|_1$, $d_2(x,y) = |x-y|_2$ define the same topology on K.

(b) $|\alpha|_1 < 1 \Rightarrow |\alpha|_2 < 1 \quad \forall\ \alpha \in K$.

(c) $|\ |_2 = |\ |_1^a$ for some $a > 0$.

Proof: Prop. 7.8 in Milne.

We say $|\cdot|_1$ and $|\cdot|_2$ are equivalent if they satisfy these conditions

Absolute values (multiplicative valuations)

$$|\cdot| \longleftrightarrow \log_c |\cdot| \quad c>1$$

$\leftarrow$ behaves like order of vanishing

$\uparrow$ additive valuation.

Main source of non-archimedean absolute values:

A is a DVR. $\mathfrak{m} \subseteq A$ $\quad \mathfrak{m} = (\pi)$.

$\uparrow$ irreducible

e.g. the localization of a DD at nonzero prime $\mathfrak{p}$.

$K = \operatorname{Frac} A = A[\frac{1}{\pi}]$. any element $b \in K$.

$b = \pi^n a \quad a \in A^\times$.

$|b| = c^{-n}$ for $c > 1$

<u>Example</u> If K is a number field $\quad \mathfrak{p} \subseteq \mathcal{O}_K$

and $A = \mathcal{O}_{K,\mathfrak{p}}$ $\quad (\operatorname{Frac} A = \operatorname{Frac} \mathcal{O}_K = K)$

$|\cdot|_{\mathfrak{p}}$ the resulting valuation with $c = N\mathfrak{p} = |\mathcal{O}_{K,\mathfrak{p}}/\mathfrak{p}|$ is called the normalized absolute value associated to $\mathfrak{p}$.

<u>Def'n</u>: A <u>place</u> of K is an equivalence class of absolute values on K.

<u>Theorem</u>: If K is a number field then the places of K are exactly.

Finite places: $|\cdot|_{\mathfrak{p}}$ for $\mathfrak{p}$ nonzero prime in $\mathcal{O}_K$

r real places: $|\cdot|_\iota$ for $\iota: K \hookrightarrow \mathbb{R}$ $\quad |x|_\iota := |\iota(x)|_{\mathbb{R}}$.

c complex places: $|\cdot|_\iota$ for $\iota: K \hookrightarrow \mathbb{C}$. $\quad |x|_\iota := |\iota(x)|_{\mathbb{C}}$

Not factoring through $\mathbb{R}$ up to conjugation $\iota \leftrightarrow \bar{\iota}$

Proof: For $\mathbb{Q}$, Ostrowski's theorem.
See later how to get for arbitrary K.

If K is a field and $|\cdot|$ is an absolute value define $\widehat{K}$ to be the completion of K

$= $ equivalence classes of Cauchy sequences in K.

Still a field.

Example $\mathbb{Q}$ $|\ |_{\mathbb{R}}$ $\rightsquigarrow$ $\widehat{\mathbb{Q}} = \mathbb{R}$.

$\mathbb{C}(t)$ $|\ |_0$ $\rightsquigarrow$ $\mathbb{C}((t))$
$\mathbb{C}(t)$ $|\ |_\infty$ $\rightsquigarrow$ $\mathbb{C}((\frac{1}{t}))$ } Laurent series.

(Inclusion is Taylor expansion at a point).

$\mathbb{Q}$ $|\ |_p$ $\rightsquigarrow$ $\mathbb{Q}_p$

$\sum_{k=-N}^{\infty} a_k p^k$ $\quad a_k \in \{0, \dots, p-1\}$.