6370-001 - FALL 2021 - WEEK 3 (9/7, 9/9)

We missed a day last week due to the football hoopla, so we will also continue discussing problems from Week 2.

Exercise 1.

(1) Show that, for any finite extension L/K, $N_{L/K}(ab) = N_{L/K}(a)N_{L/K}(b)$.
(2) Suppose K is a finite extension of $\mathbb{Q}$. Show that $a \in \mathcal{O}_K$ is a unit (i.e. $1/a \in \mathcal{O}_K$) if and only if $N_{K/\mathbb{Q}}(a) = \pm 1$.
(3) Show $1 \pm \sqrt{-5}, 2,$ and 3 are all irreducible in $\mathbb{Z}[\sqrt{-5}]$.
(4) Find the factorizations of the *ideals* (2) and (3) in $\mathbb{Z}[\sqrt{-5}]$.

Exercise 2 (Milne 2-1).
Recall that the ring of integers in $\mathbb{Q}(\sqrt{5})$ is not $\mathbb{Z}[\sqrt{5}]$. Thus, $\mathbb{Z}[\sqrt{5}]$ is not integrally closed, and by an exercise from last week it therefore cannot be a UFD. Find an element with two distinct irreducible factorizations to illustrate this.

Exercise 3 (Milne 3-2).
Recall that $\mathbb{Z}[\sqrt{3}]$ is the ring of integers in $\mathbb{Q}[\sqrt{3}]$ and $\mathbb{Z}[\sqrt{7}]$ is the ring of integers in $\mathbb{Q}(\sqrt{7})$. Show, however, that $\mathbb{Z}[\sqrt{3}, \sqrt{7}]$ is *not* the ring of integers in $\mathbb{Q}(\sqrt{3}, \sqrt{7})$.

Exercise 4 (Milne 3-1).
Let k be a field. Is $k[X, Y]$ a Dedekind domain?

Exercise 5.
Let $A = \mathbb{R}[x, y]/(y^2 - x)$.

(1) Show that A is a Dedekind domain. Is A a PID?
(2) How do the prime ideals of $\mathbb{R}[x]$ factor in A?

Exercise 6.
Let A be a Dedekind domain and let $I \subset A$ be an ideal. Show that I can be generated by two elements. (Hint: take $a \in I$, then apply unique factorization of ideals to (a) and I and use the Chinese Remainder Theorem).

Exercise 7.

(1) Show that for a domain A, $A = \cap A_{\mathfrak{m}}$ where the intersection is over all maximal ideals $\mathfrak{m}$ of A. (Hint: consider the annihilator of $a \in \bigcap A_{\mathfrak{m}}/A$.)
(2) Show that a domain A is integrally closed if and only if $A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p}$ in A.

Exercise 8.
Read the proof of Proposition 3.2 in Milne, which shows that a Dedekind domain with exactly one nonzero prime ideal is a PID (a PID with exactly one non-zero prime ideal is called a discrete valuation ring). The key point is to show the prime ideal is principal, and at a certain point in the argument, a variant on the integrality lemma we used last week is invoked – the proof of that lemma

is essentially the same as Exercise 4 from week 2.

Exercise 9.
Let $f(x,y) \in \mathbb{C}[x,y]$ be such that, for every $(z_1,z_2) \in V(f) \subset \mathbb{C}^2$, $\nabla(f)(z_1,z_2) \neq 0$. Let $A = \mathbb{C}[x,y]/(f)$, which you should interpret as the "ring of polynomial functions on the plane curve $V(f)$". Here $V(f)$ is the vanishing locus of f, i.e. the set of $(z_1,z_2) \in \mathbb{C}^2$ such that $f(z_1,z_2) = 0$.

(1) Recall that A is Noetherian (why?).
(2) What are the maximal ideals of A?
(3) Show that for any maximal ideal of A, $\mathfrak{m}/\mathfrak{m}^2$ is a one-dimensional $\mathbb{C}$-vector space.
(4) Show that for any maximal ideal $\mathfrak{m}$ of A, $\mathfrak{m}A_{\mathfrak{m}}$ is principal in $A_{\mathfrak{m}}$. (Hint: Nakayama's lemma)
(5) Show that for any maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is a PID.
(6) Show that every non-zero prime ideal is maximal.
(7) Show that A is integrally closed.
(8) So A is a Dedekind domain!