MATH 1050 SECTION 2
FALL 2009
PRACTICE EXAM II

1. General Guidelines

1.1. **Calculators.** You will not be permited to use calculators. As such, the problems will involve relatively simple arithmetic that you should be able to do by hand.

1.2. **Material Covered.** The exam will cover material through section 3.4 in the book. You should know how to do all questions from the homework.

1.3. **Showing Work.** You should show enough work so that the graders can follow your thought process. When in doubt, show your work. I also highly recommend that you check your answer for consistency where possible. For example, when you solve a logarithmic equation, plug your answer back in to verify it when you finish.

1.4. **Studying.** Mathematics is more a set of skills than a collection of facts. Practice is the key to mastering these skills. As such, I recommend that you work as many problems as possible rather than simply memorizing formulas from the book. Work the problems below and then rework as many homework problems as you can. If you have time, work extra problems from the book.

2. Practice Problems

(1) Compute the inverse of the function

$$f(x) = \frac{3x+2}{1-2x}.$$

Solution.

$$y = \frac{3x+2}{1-2x}.$$

Exchange x and y:

$$x = \frac{3y+2}{1-2y}.$$

Cancel the denominator:

$$x(1-2y) = 3y+2.$$

$$x - 2xy = 3y + 2.$$

Group all y-terms on one side. Put everything else on the other side:

$$-2xy - 3y = 2 - x.$$

Pull out y.

$$y(-2x-3) = 2 - x.$$

Solve for y.

$$y = \frac{2-x}{-2x-3}.$$

Simplify by multiplying top and bottom by -1:

$$y = \frac{x-2}{2x+3}.$$

□

(2) Find the vertex of the parabola given by

$$f(x) = 3x^2 + 2x - 8.$$

Rewrite $f(x)$ in standard form. That is, rewrite $f(x)$ in the form

$$a(x - h)^2 + k.$$

Does this parabola open upward or downward?

solution. We can most easily find the vertex of the parabola by applying the following formula: the x-coordinate of the vertex is given by $h = -\frac{b}{2a} = -\frac{1}{3}$. The y-coordinate of the vertex is given by $k = f(-1/3) = 1/3 - 2/3 - 8 = -25/3$. We get $a = 3$ directly from the definition of f. Putting f in standard form, we get

$$f(x) = 3(x + 1/3)^2 - 25/3.$$

Since a is positive, the parabola opens upward. If you prefer, you could also work through this problem by completing the square. □

(3) Describe the end behavior of the following polynomials: what happens as x goes to $+\infty$ and $-\infty$?

(a) $x^6 + 3x^2 - 2x + 4$
This polynomial goes to positive infinity in both directions since the degree is even and the leading coefficient is positive.

(b) $-3x^5 + 3x^4 - 12x^3 + x - 18$
Since the degree of this polynomial is odd and the leading coefficient is negative, this polynomial goes to negative infinity as we go to the right and positive infinity as we go to the left.

At most, how many turning points can each of these polynomials have?
In each case, the maximum number of turning points is $n - 1$, where n is the degree of the polynomial.

(4)

$$f(x) = (x + 2)^3(7x - 2)(x + 3)^2$$

What are the roots of f? At each root, determine whether or not the graph of f crosses the x-axis. Put the zeros of f on a number line and show the regions where f is positive and negative.

solution. To find the roots, we find the root of each linear factor. These are -2, $2/7$ and -3. To determine whether or not the graph crosses the x-axis at each root, we look at the root multiplicities. The roots -2 and $2/7$ have odd multiplicity, so the graph crosses at these points. Since -3 has even multiplicity, the graph does not cross here. The roots divide the real line into the intervals $(-\infty, -3)$,$(-3, -2)$,$(-2, 2/7)$ and $(2/7, \infty)$. By evaluating, we find that $f(0)$ is negative. We can find the sign of f on each interval by realizing that the sign of f changes only where it crosses the x-axis. Thus, f is positive on $(\infty, -3)$, positive on $(-3, -2)$, negative on $(-2, 2/7)$ and positive on $(2/7, \infty)$. □

(5) Evaluate $3x^6 + 7x^5 - 3x^4 - 12x^3 + 3x^2 + 15x - 5$ at $x = -2$ by using synthetic division.

solution. This is an application of the remainder theorem: $f(-2)$ is the remainder when we divide by $x - (-2)$, which is the same as applying synthetic division with -2. Doing this, we get the remainder -7. □

(6) Find all the roots of $f(x) = x^4 + x^3 - 8x^2 - 2x + 12$ given that 2 and -3 are roots.

solution. We perform successive synthetic division with the roots 2 and -3 and obtain the polynomial $x^2 - 2 = 0$. Solving this quadratic gives the roots $\pm\sqrt{2}$. □

(7) Compute $x^4 + 11x^3 + 3x^2 - 2x - 5 \div x^2 + 2$ by long division.

solution. The long division yields $x^2 + 11x + 1$ with remainder $-24x - 7$. □

(8) Rewrite $\dfrac{8-2i}{2+3i}$ in standard form i.e., in the form $a + bi$. You will get fractions for a and b.

solution. We multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{(8-2i)}{(2+3i)}\frac{(2-3i)}{(2-3i)} = \frac{16-24i-4i-6}{4+9} = \frac{10-28i}{13} = \frac{10}{13} - \frac{28}{13}i.$$

□

(9) Find the roots of $x^2 - 2x + 8$ by using the quadratic formula. Simplify the roots by pulling squares out of square roots and replacing $\sqrt{-1}$ with i.

solution.

$$\frac{2 \pm \sqrt{4-32}}{2} = \frac{2 \pm \sqrt{-28}}{2} = \frac{2 \pm 2\sqrt{7}i}{2} = 1 \pm \sqrt{7}i.$$

□

(10) Let $f(x) = 6x^5 - x^4 + 3x^3 - 11x^2 + 8x - 35$. List all the possible rational roots of f according to the rational root theorem. *Do not attempt to factor f.*

solution. Possible values for the numerator: $p = 1, 5, 7, 35$. Possible values for the denominator: $q = 1, 2, 3, 6$. Possible rational roots:

$$\pm1, \pm5, \pm7, \pm35, \pm1/2, \pm5/2, \pm7/2, \pm35/2, \pm1/3, \pm5/3, \pm7/3, \pm35/3, \pm1/6, \pm5/6, \pm7/6, \pm35/6$$

□

(11) Use the rational root theorem to find all the roots of $g(x) = 3x^3 + 19x^2 + 16x - 20$. Write g as a product of linear factors.

solution. Using the rational root theorem, we get the following possible rational roots:

$$\pm1, \pm2, \pm4, \pm5, \pm10, \pm20, \pm1/3, \pm2/3, \pm4/3, \pm5/3, \pm10/3, \pm20/3.$$

We test rational roots by synthetic division until we find a root. I found the root $x = -2$ first and got the quadratic polynomial $3x^2 + 13x - 10$. This factors as $(3x - 2)(x + 5)$. Thus, the list of roots is -2, $2/3$ and -5, all with multiplicity 1. Writing g as a product of linear factors, we have:

$$g(x) = (x + 2)(3x - 2)(x + 5).$$

□

(12) Let $f(x) = x^4 - 3x^3 - 5x^2 + 29x - 30$. Let $a = 2 - i$. In this problem, we will factor f completely by combining several skills developed in the homework. If a problem like this appears on the test, I will walk you through each step just as I do here.

(a) What other complex number must be a root of f? Call this root b.
Since f has real coefficients, $b = 2 + i$, the complex conjugate of a, must also be a root.

(b) Compute $(x-a)(x-b)$. You should put parentheses around a and b to do this computation. Once you've simplified, you should have a quadratic polynomial with real integer coefficients. Call this polynomial $g(x)$.

$$g(x) = (x - (2 - i))(x - (2 + i)) = x^2 - (2 - i)x - (2 + i)x + 4 + 1 = x^2 - 4x + 5.$$

(c) Compute $f \div g$ by long division. You should get no remainder and end up with a quadratic polynomial with integer coefficients. Call this result $h(x)$.

$$h(x) = x^2 + x - 6.$$

(d) In the previous step, we basically divided f by $(x-a)$ and $(x-b)$ simultaneously. Find the remaining roots of f by finding the roots of h.
Factoring, we get $h(x) = (x+3)(x-2)$. The remaining roots are -3 and 2.

(13) Use Descarte's rule of signs to determine how many positive and negative real zeros $f(x) = x^4 - 3x^2 + x - 3$ can have. *Hint: remember to count sign changes and count by 2. For negative roots, look at $f(-x)$.*

solution. To compute the possible number of positive roots, we count the sign changes and get 3. We count down from here by 2 and get that f has 3 or 1 positive real roots.

To determine how many negative real roots f could have, we compute $f(-x) = x^4 - 3x^2 - x - 3$. There is only one sign change, so f must have exactly one negative real root. □

(14) Find the horizontal and vertical asymptotes of

$$h(x) = \frac{7x^2 - 5x + 4}{x^2 + x - 12}.$$

Graph $h(x)$.

solution. The numerator and denominator of f have the same degree. The horizontal asymptote is given by the quotient of the leading coefficients, which is 7. To find the vertical asymptotes, we find the roots of the denominator. It factors as $(x+4)(x-3)$, so we have vertical asymptotes at -4 and 3.

To graph h, we need to find the roots of the numerator. This is most easily done by using the quadratic formula.

$$\frac{5 \pm \sqrt{25 - 112}}{14} = \frac{5 \pm \sqrt{-87}}{14}.$$

We see that the numerator has no real roots. The vertical asymptotes divide the real line into three intervals: $(-\infty, -4), (-4, 3)$ and $(3, \infty)$. To graph h, evaluate it at one point on each interval. These values will determine the sign of h on each interval and give you few points to put on your graph. □

(15) Simplify: $\log_5 25$.
Answer: 2.

(16) Rewrite in terms of the natural logarithm: $\log_2 35$.
$\dfrac{\ln 35}{\ln 2}$

(17) Use the properties of the logarithm to expand the following expression as a sum, difference, and/or constant multiple of logarithms:

$$\log_5 \left(\frac{\sqrt{x-4}}{5x} \right), \ x > 4.$$

solution.

$$\log_5 \left(\frac{\sqrt{x-4}}{5x} \right) = \log_5 \sqrt{x-4} - \log_5(5x) = \frac{1}{2}\log_5(x-4) - \log_5 5 - \log_5 x = \frac{1}{2}\log_5(x-4) - \log_5 x - 1.$$

□

(18) Condense the expression to the logarithm of a single quantity:

$$\ln 5 - \frac{1}{2}\ln(x-5) + \ln(x).$$

solution.

$$\ln 5 - \frac{1}{2}\ln(x-5) + \ln(x) = \ln 5 - \ln\sqrt{x-5} + \ln(x) = \ln\frac{5}{\sqrt{x-5}} + \ln(x) = \ln\frac{5x}{\sqrt{x-5}}.$$

□

(19) Solve the given equations. Where needed, express your answers in terms of logarithms.

(a) $e^{x^2-x} = e^{x+15}$

solution.

$$x^2 - x = x + 15$$
$$x^2 - 2x - 15 = 0.$$
$$(x-5)(x+3) = 0.$$
$$x = 5, -3.$$

□

(b) $e^{2x} + e^x = 2$

solution. This exponential equation can be converted to a quadratic equation. Let $y = e^x$. Then

$$y^2 + y = 2.$$
$$y^2 + y - 2 = 0.$$
$$(y+2)(y-1) = 0.$$
$$y = -2, 1.$$
$$e^x = -2.$$

We see that this solution doesn't make sense because e^x must be positive.

$$e^x = 1.$$
$$x = 0.$$

□

(c) $7 \cdot 5^{2-x} = 13$

solution.

$$5^{2-x} = \frac{13}{7}$$

Apply the logarithm with base 5:

$$2 - x = \log_5 \frac{13}{7}.$$
$$x = 2 - \log_5 \frac{13}{7}.$$

□

(d) $\ln\sqrt{x-8} = 5$

solution. Exponentiate with base e:

$$\sqrt{x-8} = e^5.$$

Square each side:

$$x - 8 = e^{10}.$$
$$x = e^{10} + 8.$$

□

(e) $\log x + \log(x+2) = \log(x+6)$

solution. First, we combine the logarithms on the left side:

$$\log(x(x+2)) = \log(x+6).$$
$$x^2 + 2x = x + 6.$$
$$x^2 + x - 6 = 0.$$
$$(x+3)(x-2) = 0.$$
$$x = -3, 2.$$

Plugging the possible solutions into the original equation, we see that -3 is extraneous and 2 is the only solution. □