Computation of Invariants for Harish-Chandra Modules of $SU(p, q)$ by Combining Algebraic and Geometric Methods.

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Slides and notes are available at

\[\text{www.math.utah.edu/\textasciitilde housley \rightarrow talks.}\]
Definition: Lie Group

- $G$ is a differentiable manifold with a group operation.
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- Smooth inverse.
- Complex and real flavors.
Symmetry

Lie groups: smooth symmetries.
Symmetry

Examples:

- $\text{GL}(n, \mathbb{C})$ is the set of all invertible complex linear transformations on $\mathbb{C}^n$. (Complex)
- $\text{GL}(n, \mathbb{R})$ is the set of all invertible real linear transformations on $\mathbb{R}^n$. (Real)
- $\text{SO}(n)$ is the set of orientation preserving isometries of the $(n-1)$-sphere. (Real)
- E.g., $\text{SO}(3)$ is the set of rotations of the 2-sphere.
- Has applications to quantum mechanics.
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  - Has applications to quantum mechanics.
Symmetry

Examples:

- $SU(p, q)$ is the set of invertible complex linear transformations of $\mathbb{C}^{p+q}$ that preserve the hermitian form given by

$$\langle v, w \rangle = v^* \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} w.$$  

(real)
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We’ll assume this from now on.
Start with a Lie group $G$. Define $\mathfrak{g}$ to be the tangent space to $G$ at $id$. 

![Diagram showing a sphere and a tangent plane]
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The structures of $G$ and $g$ are closely related.
Let $V$ be a (complex) vector space.
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  - Reductive: more or less the whole group acts on $V$. 

Finite dimensional representations arise naturally.
Standard representation for matrix groups.
Adjoint representation: $G$ acts on $g$.
Subs: restriction to an invariant subspace.
Quotients.
Generating new representations. (Tensor products, exterior powers, etc.)
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  - Generating new representations. (Tensor products, exterior powers, etc.)
- Finite dimensional representations of real and complex Lie groups are well understood.
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- Roughly: irreducible representations are simplest actions of the symmetries in $G$. 
If $V$ is a $G$ representation, it becomes a $g$ representation via differentiation.
Lie Algebra Representations

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- We complexify $\mathfrak{g}$ to $\mathfrak{g}_\mathbb{C}$. 
Lie Algebra Representations

- If $V$ is a $G$ representation, it becomes a $g$ representation via differentiation.
- We complexify $g$ to $g_C$.
- Irreducible representations of $g_C$ are important in classifying irreducible representations of $G$. 
Infinite Dimensional Motivations

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- Find a real Lie group \( G \) that acts on \( M \) and preserves the measure.
  - Irreducible representations of complex Lie groups are finite dimensional.
- Consider the action of \( G \) on \( L^2(M) \).
Harish-Chandra Modules

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- We’ll ignore this distinction.
- We can study infinite dimensional representations using algebraic and algebro-geometric techniques.
Geometric Invariants

Two geometric invariants to consider:

- Associated variety $AV(X)$: a variety contained in $g_C$.
- Associated cycle $AC(X)$: finer invariant that attaches an integer (multiplicity) to each component of $AV(X)$.

We'd like to calculate the multiplicities.
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$SU(p, q)$

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We only need to compute one multiplicity to get $AC(X)$. 

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\[ W = N_G(T)/Z_G(T). \]

The Weyl group for \( SU(p, q) \) is the symmetric group \( S_{p+q} \).
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Cells

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- Take the formal $\mathbb{Z}$-span of the elements of $\mathcal{C}$.

- $\text{span}_{\mathbb{Z}} \mathcal{C}$ becomes an irreducible representation of the Weyl group $S_{p+q}$. 
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Take the formal $\mathbb{Z}$-span of the elements of $C$.

$\text{span}_\mathbb{Z} C$ becomes an irreducible representation of the Weyl group $S_{p+q}$.

Let $m_{X_i}$ denote the multiplicity in the associated variety of $X_i$. 
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- The representation relates the multiplicities $m_{X_i}$ for the various $X_i$ in $C$.
- If we know $m_{X_j}$ for some $X_j$ we can calculate $m_{X_i}$ for the other $X_i$ in the cell $C$. 
Strategy

- For $SU(p, q)$ and any cell $C$ of representations, we can always find an $X_i \in C$ so that $m_{X_i}$ can be computed by geometric means.
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- Compute the $S_{p+q}$ representation on $\text{span}_\mathbb{Z} C$.
- Compute $m_{X_i}$ for other $X_i$ in $C$. 
Problem

It can be difficult to compute the $S_{p+q}$ representation on $\text{span}_\mathbb{Z} C$. 
Symmetric Group Representations

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For example, acting by (12):

$e_1 - e_2 \rightarrow e_2 - e_1$ and $e_3 - e_1 \rightarrow e_3 - e_2$. 
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This is the standard representation $V$ of $S_n$. 
More Representations of $S_n$

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Construction: take subspaces of tensor powers of the standard representation.
Hook Type Representations

Hook type diagram: upside down $L$. 
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Two rows with one box on the bottom row: standard representation.
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General hook type with $m + 1$ rows: $\wedge^m V$, where $V$ is the standard representations.
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Example: (12) action on $(e_1 - e_2) \wedge (e_2 - e_3)$ for $\wedge^2 V$: 

\[
\begin{align*}
(e_2 - e_1) \wedge (e_1 - e_3) &= (e_2 - e_1) \wedge (e_2 - e_3) \\
&= - (e_1 - e_2) \wedge (e_1 - e_3) \wedge (e_2 - e_3) \\
&= - (e_1 - e_2) \wedge (e_1 - e_3) - (e_2 - e_3) \wedge (e_1 - e_3) \\
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(e_2 - e_1) \wedge (e_1 - e_3) = -(e_1 - e_2) \wedge (e_1 - e_2 + e_2 - e_3)
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- Find a hook type cell $C$.
- Find one $X_j$ in the cell such that computation of $m_{X_j}$ is easy.
- Find $m_{X_i}$ for other $X_i$ in $C$ by using the $S_n$ representation on $\text{span}_\mathbb{Z} C$. 


Result

We get a formula for $m_{X_i}$ where $X_i$ is any infinite dimensional representation in a hook type cell:
We get a formula for $mX_i$ where $X_i$ is any infinite dimensional representation in a hook type cell:

$$mX_i = A_m \frac{1}{\prod |\tau_k|!} \sum_{\sigma \in S_{\tau}} \text{sgn}(\sigma)\sigma \cdot \left( \sum_{\sigma' \in S_m} \text{sgn}(\sigma')\sigma' \cdot \left( \prod_{i=1\ldots m} \chi_{\tau(i)}^{m-i+1} \right) \right)$$

where

$$A_m = \frac{1}{m! \cdot (m-1)! \cdots 1}.$$
Details are available at www.math.utah.edu/~housley → research.

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