Section 6.2: Compound Interest; Geometric Sequences

Definitions:

- If $P$ is invested at an interest rate of $r$ per year, **compounded annually**, the future value $S$ at the end of the $n$th year is
  \[ S = P(1 + r)^n \]
  Note: Instead of annually, we may have semianually, quarterly, monthly, daily, etc.

- If $P$ is invested for $t$ years at a **nominal interest rate** (rate per year) $r$, compound $m$ times per year, then the total number of compounding periods is
  \[ n = mt \]
  the interest rate per compounding period is
  \[ i = \frac{r}{m} \quad \text{(decimal)} \]
  and the future value is
  \[ S = P(1 + i)^n = P \left(1 + \frac{r}{m}\right)^{mt} \]

Comparison between the compound interest and the simple interest (P=$10,000, 10% annual interest)

<table>
<thead>
<tr>
<th>Year</th>
<th>Compound interest</th>
<th>Future value</th>
<th>Simple interest</th>
<th>Future value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1,000</td>
<td>$11,000</td>
<td>$1,000</td>
<td>$11,000</td>
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<tr>
<td>2</td>
<td>$1,100</td>
<td>$12,100</td>
<td>$1,000</td>
<td>$12,000</td>
</tr>
<tr>
<td>3</td>
<td>$1,210</td>
<td>$13,310</td>
<td>$1,000</td>
<td>$13,000</td>
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<td>...</td>
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<td>...</td>
<td>...</td>
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<tr>
<td>10</td>
<td>$2,358</td>
<td>$25,937</td>
<td>$1,000</td>
<td>$20,000</td>
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<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>30</td>
<td>$15,863</td>
<td>$174,494</td>
<td>$1,000</td>
<td>$40,000</td>
</tr>
</tbody>
</table>

Ex.1 (#6): What is the future value if $8600 is invested for 8 years at 10% compounded semiannually? What interest will be earned?
Ex.2 (#12) What lump sum should be deposited in an account that will earn 9%, compounded quarterly, to grow to $100,000 for retirement in 25 years?

Note: For the same $P$ and same compounding periods, the higher the interest rate is, the greater the future value is (see Fig.6.2 on p.418). You can see the dramatical difference after 15-20 years. The graphs of the future values versus time have exponential growth.

**Definition:** If $P$ is invested for $t$ years at a nominal rate $r$ **compounded continuously**, the future value is given by

$$S = Pe^{rt}$$

Note: For the same compound interest rate and the same $P$, the more frequent the compounding period is, the greater the interest is.

Ex.3 (#22) How much more interest will be earned if $5000$ is invested for 6 years at 7% compounded continuously, instead of at 7% compounded quarterly?
Definition: Let \( r \) be the annual (nominal) interest rate for an investment. Then the annual percentage yield (APY) (or effective annual rate) is given as follows:

- **Periodic compounding:** If \( m \) is the number of compounding periods per year, then \( i \) is the interest rate per period, and
  \[
  \text{APY} = \left(1 + \frac{r}{m}\right)^m - 1 = (1 + i)^m - 1 \quad \text{(decimal)}
  \]

- **Continuous compounding:**
  \[
  \text{APY} = e^r - 1 \quad \text{(decimal)}
  \]

Why is APY useful?

Comparing two nominal rates with different compounding periods does not give us any useful information. Instead we can compare their APYs to see which one have better interest.

**Ex.4** What is the annual percentage yield (or effective annual rate) for a nominal rate of (a) 10.5% compounded quarterly and (b) 10% compounded continuously?

**Ex.5** (#36) At what nominal rate, compounded annually would $10,000 have to be invested to amount to $14,071 in 7 years?
Definition: A geometric sequence, \( a_1, a_2, a_3, \ldots \) is given by
\[
a_n = r a_{n-1}, \quad (n > 1)
\]
where \( r \) is the common ratio. The \( n \)th term is given by
\[
a_n = a_1 r^{n-1}
\]

Ex. 6 (#48) (b) For the geometric sequence given, write the next three terms.

\[32, 40, 50, \ldots\]

Formula: The sum of the first \( n \) terms of a geometric sequence is given by
\[
s_n = \frac{a_1(1 - r^n)}{1 - r}.
\]

Why is this true?
Ex.7 (#58) Find the sum of the first 31 terms of the geometric sequence 9, −6, 4, . . .

Ex.8 (#74) If Sherri must repay a $9000 interest-free loan by making monthly payments of 15% of the unpaid balance, what is the unpaid balance after 1 year?