Graph Domination and the Lights Out Puzzle

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The puzzle game "Lights Out" is traditionally played on an \( n \times n \) grid where each square has a light which is also a button. The light in each square is either on or off. Pushing the button on a particular square switches the light in that square and any adjacent square. The goal of the lights out puzzle is to start with a grid where all of the lights are on and make successive button pushes until all of the lights are out. For example, the following figure demonstrates what happens when the indicated button is pressed.

Notice that pressing the same button twice effectively does nothing. We convince ourselves then, that a solution to the puzzle is a subset of the set of squares on the grid. Pressing the buttons associated with those squares in any order will turn out all of the lights.

This problem is interesting enough in itself, but we would like to extend it further. Notice that the game of lights out can be played on any connected graph with the same rules. Give each vertex a "light," and follow the same kind of rules. The game becomes much more complicated. Examine the following graph.

What is the simplest subset of vertices which, when "pressed," yields a lights out solution? Is a solution even guaranteed to exist?
To answer these questions, we begin by defining some terminology.

A vertex $v$ is said to **dominate** itself and every vertex in the neighborhood of $v$. Thus, the vertices which dominate $v$ are precisely the vertices in the set $N(v) \cup \{v\}$. For convenience, we will call this set the *closed neighborhood* $N[v]$ of $v$. $N[v] = N(v) \cup \{v\}$.

A **dominating set** of a graph $G$ is any subset $S$ of the vertices of $G$ such that every vertex $v \in V(G)$ is dominated by at least one vertex from $S$.

With this terminology in hand, we can rephrase the original lights our problem. There will be a solution to the lights out problem if and only if there exists a dominating set of $G$ such that every vertex in $G$ is dominated by an odd number of vertices. Such a dominating set is also known as an *odd parity cover*. To justify this is a valid restatement of the lights out problem, simply consider what happens when one presses the button on each of the vertices in the odd parity cover. Each time a vertex is pressed, every vertex dominated by that vertex switches from off to on or vise versa. Thus, if each vertex is dominated by an odd number of vertices, each of the vertices will switch an odd number of times, ending in the ‘off’ position.

It turns out that for any finite Graph $G$, the lights out game has a solution. The following demonstration is an elaboration on a paper by Klaus Sutner.

We define a **pattern** of a graph $G$ to be a function

$$X : V(G) \rightarrow \{0, 1\}$$

Essentially a pattern is just an association with each vertex either the number 1 or the number 0, which indicates that a light is on or off respectively.

If we denote the set of patterns on a graph $G$ by $C_G$, we find that $C_G$ is a vector space with addition defined for $X, Y \in C_G$ by

$$(X + Y)(v) = X(v) + Y(v) \mod 2$$

This will be called the **pattern space**.
Finally, we define a function
\[ \sigma : \mathbb{C}_G \rightarrow \mathbb{C}_G \]
by
\[ \sigma(X)(v) = 1 \iff \sum_{v \in N[v]} X(v) \text{ is odd} \]
The resultant pattern \( \sigma(X) = Y \) assigns a value of 1 to vertex \( v \) if and only if the pattern \( X \) assigns a one to an odd number of vertices in the closed neighborhood of \( v \). Thus, if we consider the set \( S \) defined by
\[ v \in S \iff X(v) = 1 \]
then we see that

\[ S \text{ is an odd parity cover of } G \iff \sigma(X)(v) = 1 \text{ for all } v \in V(G) \]

If we define \( \mathbf{1} \) to be the pattern such that
\[ \mathbf{1}(v) = 1 \text{ for all } v \in V(G) \]
then our problem reduces to proving there exists \( X \) such that \( \sigma(X) = \mathbf{1} \).

We should note that \( \sigma \) defined above is a linear transformation. In fact, it has a very nice form. To find this, we will formally associate each pattern \( X \) with a column vector in \( \mathbb{Z}_2^n \), where \( |V(G)| = n \), in the following fashion.

\[ X \sim \begin{bmatrix} X(v_1) \\ X(v_2) \\ \vdots \\ X(v_n) \end{bmatrix} \]

**Theorem:** If \( X \) is associated with a column vector as above, then
\[ \sigma(X) = (A + I)X \]
where \( A \) is the adjacency matrix for the graph \( G \). (entries are mod 2)

**Proof:** We must examine what happens to a generic entry of \( \sigma(X) \). The \( k^{th} \) entry of \( (A + I)X \) is the \( k^{th} \) row of \( A + I \) dotted with \( X \). Thus
\[ ((A + I)X)_k = \sum_{i=1}^{n} (A + I)_{k,i} \times X(v_i) \mod 2 \]
Now notice that \((A + I)_{k,i} = 1\) if and only if vertex \(v_i\) is in the closed neighborhood of the vertex \(v_k\). Otherwise it is 0. This is due to the definition of the adjacency matrix. Thus the above is equal to

\[
= \sum_{v \in N[v_k]} X(v) \quad \text{mod} \ 2
\]

This is clearly equal to 1 if and only if the number of vertices in the closed neighborhood of \(v_k\) for which \(X(v) = 1\) is odd. This is the exact definition of \(\sigma(X)(v_k)\). In symbols,

\[
(A + I)X = \begin{bmatrix}
\sigma(X)(v_1) \\
\sigma(X)(v_2) \\
\vdots \\
\sigma(X)(v_n)
\end{bmatrix} = \sigma(X)
\]

This concludes the proof.

As was mentioned before, the pattern space is a finite dimensional vector space. (with base field \(\{0, 1\}\)) We will define its standard basis to be

\[
\mathcal{B} = \{X_1, X_2, \ldots, X_n\}
\]

where \(X_i\) is defined by

\[
X_i(v_j) = \delta_{i,j} \quad \text{the Kronecker delta function.}
\]

For example, if the vertex set \(V(G) = \{v_1, v_2, v_3\}\), then the pattern \(X\) defined by \(X(v_1) = 1, X(v_2) = 0,\) and \(X(v_3) = 1\) is given by

\[
X(v_j) = \delta_{1,j} + \delta_{3,j}
\]

We now turn our attention to the dual space of linear functionals, \(C_G^*\), which has a dual basis

\[
\mathcal{B}^* = \{X_1^*, X_2^*, \ldots, X_n^*\}
\]

where \(X_i^*\) is defined by

\[
X_i^*(X_j) = \delta_{i,j}
\]

And now we define the associated bilinear form which comes naturally with the dual space:

\[
\langle \ , \ \rangle : C_G^* \times C_G \to \{0, 1\} \quad \text{by} \quad \langle X^*, Z \rangle = X^*(Z)
\]
From linear algebra, we know that any bilinear form defined in this way is non-degenerate.

We note at this point that $C_G$ is isomorphic to $C_G^*$ under the isomorphism

$$
\phi : C_G \to C_G^* \quad \text{by} \quad \phi(X_i) = X_i^*
$$

where $X_i \in \mathcal{B}$ and $X_i^* \in \mathcal{B}^*$ are basis elements for the respective spaces. Thus for notational convenience, we will actually consider the bilinear mapping to be acting on $C_G \times C_G$

$$
\langle \ , \ \rangle : C_G \times C_G \to \{0,1\} \quad \text{by} \quad \langle X,Z \rangle = X^*(Z)
$$

With a little more work, we can also show that if

$$
X = a_1X_1 + a_2X_2 + \ldots + a_nX_n
$$

and

$$
Z = b_1X_1 + b_2X_2 + \ldots + b_nX_n
$$

then

$$
\langle X,Z \rangle = \sum_{k=0}^{n} a_kb_k
$$

In other words, this just counts up the number of times where patterns $X$ and $Z$ are both evaluated to one at the same vertex. We now proceed to give some familiar definitions and prove the theorem.

$Z$ and $X$ are said to be perpendicular, written $Z \perp X$ if $\langle X,Z \rangle = 0$.

For any subspace $W$ of $C_G$, we define the orthogonal complement of $W$

$$
W^\perp = \{Z \in C_G \mid Z \perp X \text{ for all } X \in W\}
$$

Finally, we define the kernel of $\sigma$ by

$$
K_\sigma = \{X \in C_G \mid \sigma(X) = 0\}
$$

where $0$ is the pattern which sends every vertex to the number 0.
Theorem:

Let $G$ be an arbitrary finite graph. Then there exists a pattern $Y \in C_G$ such that $\sigma(Y) = X$ if and only if $X \in K_\sigma^\perp$.

Proof:

Since earlier we identified $\sigma$ with the matrix $A + I$, which is symmetric, we have that $\sigma$ is self-adjoint. Hence

$$\langle \sigma(X), Z \rangle = \langle X, \sigma(Z) \rangle$$

for all $X, Z \in C_G$. Recall that the bilinear form we are dealing with is non-degenerate:

$$X = 0 \iff \langle Y, X \rangle = 0 \text{ for all } Y \in C_G$$

If we let $W = \text{Img}(\sigma)$, then

$$Z \in W^\perp \iff \text{for all } X \in C_G \quad \langle \sigma(X), Z \rangle = 0$$

$$\iff \text{for all } X \in C_G \quad \langle X, \sigma(Z) \rangle = 0$$

$$\iff \sigma(Z) = 0$$

$$\iff Z \in K_\sigma$$

Thus $W^\perp = K_\sigma$. Recall that for any subspace $U$, $U^{\perp\perp} = U$, which yields that

$$W = K_\sigma^\perp$$

This completes the proof.

Theorem:

For any finite graph, there exists a pattern $X$ such that $\sigma(X) = 1$. Hence the lights out puzzle always has a solution.

Proof:

We would like to use the previous theorem. So instead of showing that $1$
is in the range of $\sigma$, we will show that it is in the orthogonal complement of the kernel of $\sigma$. So let $X \in K_\sigma$. Then we have

$$\sigma(X) = 0$$

We first claim that the set defined by

$$S = \{ v \in V(G) : X(v) = 1 \}$$

has even cardinality. Notice that every vertex in $S$ has odd degree. If a vertex $v$ in $S$ had even degree, then it would be adjacent to an even number of vertices $w$ with $X(w) = 1$. Now, since $v$ also satisfies $X(v) = 1$, the number of vertices in the closed neighborhood of $v$ satisfying this property would be odd, and hence we would have $\sigma(X)(v) = 1$. This is impossible. Now recall the handshaking lemma, which claims that the number of odd degree vertices must be even. This gives us that $S$ has even cardinality. Given our previous discussion on the bilinear form, it is easy to see that

$$\langle X, 1 \rangle = \text{cardinality of } S \text{ modulo } 2$$

which we just saw was zero. Thus $1 \in K_\sigma^\perp$. By the previous theorem, the result holds. Since the time of this proof, people have come up with purely graph theoretic proofs to this claim using directed graphs. Although the linear algebra approach is arguably less desirable, it is certainly easier. There is an easy proof that the game is playable on trees which is purely graph theoretic, which I will present in class.
References
