§1. Introduction

A simple continued fraction in the most basic sense is any fraction of the form

\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} \]

where \( a_0 \in \mathbb{Z} \) and for \( i > 0 \), \( a_i \in \mathbb{Z}^+ \). The fraction need not end at any prescribed point. In fact, what should seem obvious, a continued fraction will terminate at some point if and only if the continued fraction represents a rational number. We proceed with an example. The number 67/25 will have a representation as a continued fraction, which we will find below.

Example 1.1:

First we take the integer part of 67/25, which is 2. Then we write 67/25 as follows:

\[ 67/25 = 2 + 17/25 = 2 + 1 \frac{25/17}{17} \]

Now we repeat the same process of writing a number smaller than one as a reciprocal of a number larger than one to obtain:

\[ = 2 + \frac{1}{1 + \frac{17/8}{17/8}} \]

...and again with 17/8 we get our final result, which is

\[ \frac{67}{25} = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{8}}} = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{7 + \frac{1}{1}}} \}

Note that we have two equivalent representations of the number 67/25. One ends with an 8, and the other splits the 8 into \( 7 + 1/1 \). These are both acceptable continued fractions. In general, a rational number has exactly 2 continued fraction representations. One of those representations always ends in a 1. ”Ending in a 1“ means that the representation has the final \( a_i = 1 \).
List Notation:

For convenience throughout, it will be useful to have a shorthand notation for continued fractions. So for the rest of this paper, we will commonly denote the continued fraction

\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \]

by the list \([a_0; a_1, a_2, \ldots]\). The semicolon is used only after \(a_0\) and signifies that \(a_0\) is special since it is the only one of the numbers that can be zero. Using this notation, we can see that in the example above, \(67/25 = [2; 1, 2, 8] = [2; 1, 2, 7, 1]\).

Writing square roots as continued fractions:

Square roots of most integers, being irrational, do not have finite representations as continued fractions. However, we will see that they often exhibit pleasing patterns. Let us prove an amusing fact.

**Proposition 1.2:** \([6]\)

If \(q \in S = \{2, 5, 10, 17, 26, 37, \ldots\}\), then the continued fraction representation for \(\sqrt{q}\) is \(\sqrt{q} = [r, 2r, 2r, \ldots]\), where \(r\) is the greatest integer of \(\sqrt{q}\). ( \(r = \lfloor \sqrt{q} \rfloor\) )

**Proof:**

Note that the set we are talking about starts at 2, then adds 3 to get to the next element. It then adds 5 to get to the next element, then 7, and so on, using consecutive odd numbers to get to the next element in the set. More importantly, we can rewrite this familiar set in the following way.

\[ S = \{n^2 + 1 : n \in \mathbb{Z}^+\} \]

Now, let us find the continued fraction expansion for \(\sqrt{q}\). By our assumption, \(q = n^2 + 1\) for some \(n \in \mathbb{Z}^+\). Thus, we can write the following equation.

\[ \sqrt{q} = \sqrt{n^2 + 1} = r + 1/x \]

where \(x > 1\). Now, we solve this equation for \(x\), utilizing elementary algebra techniques, to obtain

\[ x = \frac{\sqrt{n^2 + 1} + r}{n^2 - r^2 + 1} \]

The reason why the pattern for \(\sqrt{q}\) is so "nice" is that for our particular case, \(n^2 = r^2\), which I will now prove. The proof relies on the fact that \(n < \sqrt{n^2 + 1} < n + 1\), which should be clear. Therefore, by the definition of the greatest integer function, we then have that

\[ \lfloor \sqrt{n^2 + 1} \rfloor = n \]

Thus in our case \(r = n\), or more importantly, \(r^2 = n^2\). So our equation for \(x\) simplifies to the following:

\[ x = \sqrt{n^2 + 1} + r = \sqrt{q} + r \]
Now notice the following
\[ x = \sqrt{q} + r = r + \frac{1}{x} + r = 2r + \frac{1}{x} \]

Now we can return to an earlier equation and substitute this value in for x to obtain
\[ \sqrt{q} = r + \frac{1}{2r + \frac{1}{x}} \]

One can see how to repeat this process using the equation above over and over again to obtain (recall \( r = \lfloor \sqrt{q} \rfloor \))
\[ \sqrt{q} = r + \frac{1}{2r + \frac{1}{2r + \frac{1}{\ldots}}} \]

Note that more generally it can be proven that any quadratic irrational number \( \alpha \) can be written as an infinite continued fraction which is eventually periodic. (i.e. the sequence of numbers in the list which represents its simple continued fraction will eventually repeat)

No continued fraction exposition would be complete without a reference to the golden ratio. Consider the infinite continued fraction
\[ \phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ldots}}} \]

Now notice that
\[ \phi - 1 = \frac{1}{1 + \frac{1}{1 + \frac{1}{\ldots}}}, \text{ so that } \frac{1}{\phi - 1} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ldots}}} = \phi \]

Now multiply both sides of the equation by \( \phi - 1 \) and rearrange to obtain \( \phi^2 - \phi - 1 = 0 \). Using the quadratic formula to solve for \( \phi \), we see that \( \phi = \frac{1 + \sqrt{5}}{2} \), the golden ratio. For now, we have no reason to believe that this will actually converge. Later, we will see a proof that it does.

§2. Euler’s continued fraction formula [5]

While browsing the literature, one who is interested in continued fractions will almost certainly encounter Euler’s formula for sums of products represented as continued fractions. I encountered this during my research without a proof, and have devised a proof. I in no way, shape, or form, claim originality in this proof. Before we can go about proving this formula, it is important to give a definition.
Definition 2.1: A generalized continued fraction is any number in the form

\[
a_0 + \frac{b_0}{a_1 + \frac{b_1}{a_2 + \frac{b_2}{\ldots}}}
\]

where \(a_i, b_i \in \mathbb{R}\). We shall be primarily concerned in the case where none of \(a_i\) and \(b_i\) are zero, except for possibly \(a_0\). The main difference here is that we no longer need to restrict ourselves to positive integers, and the \(b_i\) are not all 1’s. Note that often times the continued fraction is generalized even further so that its elements may be complex numbers or even functions. We will examine those possibilities later.

Theorem 2.1: Euler’s continued fraction formula

\[
a_0 + a_0 a_1 + \ldots + a_0 a_1 \cdots a_n = \frac{a_0}{1 - \frac{a_1}{1 + a_1 - \frac{a_2}{1 + a_2 - \frac{\ddots}{1 + a_{n-1} - \frac{a_n}{1 + a_n}}}}}
\]

for any \(n \in \mathbb{N}\).

Proof: The proof follows by induction.

\(n = 1\) case.

For \(n = 1\), the formula claims \(a_0 + a_0 a_1 = \frac{a_0}{1 - a_1}\). Work from the right hand side, getting a common denominator, we see that the right hand side is \(\frac{a_0}{1/(1+a_1)}\). Now flip the bottom fraction over and distribute to see that this indeed equals the left hand side. Now we assume that the formula holds for sums of products in the appropriate form for up to \(k\) terms. Now, it is sufficient to prove that the formula holds for \(k + 1\) terms. To this end, we examine the sum

\[
a_0 + a_0 a_1 + \ldots + a_0 a_1 \cdots a_k + a_0 a_1 \cdots a_k a_{k+1}
\]

The trick to the proof is to put \(a_0\) aside and apply the induction hypothesis to the remaining \(k\) terms. To do this, we consider \(a_0 a_1\) to be the "first" term in the series. In other words, we group the expression in the following way.

\[
a_0 + [(a_0 a_1) + (a_0 a_1) a_2 + (a_0 a_1) a_2 a_3 + \ldots (a_0 a_1) a_2 a_3 \cdots a_{k+1}]
\]

Now in the square brackets, we have \(k\) terms in the same form as in Euler’s formula, so we
can apply the induction hypothesis to these k terms to equate the above expression with:

\[
\begin{align*}
&\frac{a_0 + \frac{a_0a_1}{1 - \frac{a_2}{1 + a_2 - \frac{a_3}{1 + a_3 - \frac{\ddots}{1 + a_k - \frac{a_{k+1}}{1 + a_{k+1}}}}}} \end{align*}
\]

Now, for simplicity of computation, let

\[
f = 1 + a_2 - \frac{a_3}{1 + a_3 - \frac{\ddots}{1 + a_k - \frac{a_{k+1}}{1 + a_{k+1}}}}
\]

If we now return to our induction proof, we see that what we have so far is

\[
a_0 + a_0a_1 + \ldots + a_0a_1 \cdots a_k + a_0a_1 \cdots a_ka_{k+1} = a_0 + \frac{a_0a_1}{1 - \frac{a_2}{f}}
\]

and we can see that in order to complete the induction, our goal is to show that this is

\[
\frac{a_0}{1 - \frac{a_1}{1 + a_1 - \frac{a_2}{f}}}
\]

Starting by factoring out $a_0$ out of the first expression using $f$ above, we can see that this is true by simply doing the computation. By the principle of mathematical induction, Euler’s formula holds for all $n \in \mathbb{N}$.

**Example 2.2:**

Recall from Calculus Taylor’s expansion, which says that any function which is analytic at zero can be written as an infinite polynomial in the form:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]

This will allow us to come up with interesting continued fractions for a special subset of these functions in the following way. Assume that $f^{(n)}(0) \neq 0$ for all $n \in \mathbb{N}$. Then we have the following.

\[
f(x) = f(0) + f(0) \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{f^n(0)x}{n!f^{n-1}(0)}
\]
Essentially, this is putting the Taylor expansion into the form required by Euler’s continued fraction formula. Doing this, we see that

\[ a_0 = f(0) \quad a_1 = \frac{f'(0)x}{f(0)} \quad a_2 = \frac{f''(0)x}{2f'(0)} \quad \ldots \quad a_n = \frac{f^{(n)}(0)x}{nf^{n-1}(0)} \text{ if } n > 1 \]

Thus, due to Euler’s formula, any function \( f(x) \) which is analytic at zero with \( f^{(n)}(0) \neq 0 \) for all \( n \in \mathbb{N} \) can be written in the following way.

\[
f(x) = \frac{f(0)x}{1 - \frac{f'(0)x}{f(0)}} \frac{f(0)x}{1 + \frac{f''(0)x}{2f'(0)}} \frac{f(0)x}{1 + \frac{f''(0)x}{2f'(0)}} \ldots
\]

Of course, this is quite an unwieldy and strange formula. However, it can take on a nicer form by doing some transformations which amount to multiplying by certain numbers in the numerator and denominator of the fraction above. Multiply the top numerator and denominator by \( f(0) \), the next one by \( 2f'(0) \), and so on to obtain

\[
f(x) = \frac{f(0)x}{f(0) + f'(0)x - \frac{f(0)f''(0)x}{2f'(0) + f''(0)x - \frac{2f(0)f'''(0)x}{3f''(0) + f'''(0)x - \ldots}}}
\]

Example 2.3

For the function \( f(x) = e^x \), the above formula simplifies quite a bit since \( f^{(n)}(0) = 1 \) for all \( n \). If also, we let \( x = 1 \), we get a continued fraction for the number \( e \).

\[
e = \frac{1}{1 - \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}}
\]

§3. Convergents [1]

Definition 3.1 (Convergent)
Start with a continued fraction \( \alpha = [a_0; a_1, a_2, \ldots, a_n] \) with \( a_i > 0 \) for \( i \neq 0 \). If \( 0 \leq k \leq n \) with \( k \in \mathbb{N} \), then the \( k^{th} \) convergent \( c_k \) is defined to be \( c_k = [a_0; a_1, a_2, \ldots, a_k] \). Convergents for infinite continued fractions are defined analogously.
In the following, we will establish what are called the fundamental recurrence relations. To do this in the easiest way possible, we introduce 2x2 matrices and examine a certain sequence of entries. Credit for this section of the work is given to Eric Skouson and his thesis on Continued Fractions.

Let \( A_i = \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} \), and for \( i \geq 0 \), let \( B_i = A_iA_{i-1} \ldots A_1A_0 = \begin{bmatrix} h_i & k_i \\ r_i & s_i \end{bmatrix} \).

If \( B_{i-1} = \begin{bmatrix} h_{i-1} & k_{i-1} \\ r_{i-1} & s_{i-1} \end{bmatrix} \), then \( B_i = \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_{i-1} & k_{i-1} \\ r_{i-1} & s_{i-1} \end{bmatrix} = \begin{bmatrix} a_i h_{i-1} + r_{i-1} & a_i k_{i-1} + s_{i-1} \\ h_{i-1} & k_{i-1} \end{bmatrix} \).

Notice that from this we get two relations, \( r_i = h_{i-1} \) and \( s_i = k_{i-1} \). Thus we get that

\[
\begin{bmatrix} a_i h_{i-1} + r_{i-1} & a_i k_{i-1} + s_{i-1} \\ h_{i-1} & k_{i-1} \end{bmatrix} = \begin{bmatrix} a_i h_{i-1} + h_{i-2} & a_i k_{i-1} + k_{i-2} \\ h_{i-1} & k_{i-1} \end{bmatrix} = \begin{bmatrix} h_i & k_i \\ h_{i-1} & k_{i-1} \end{bmatrix}
\]

(1)

Notice that if we consider the natural sequence \( \{a_i\} \) of the continued fraction \( \alpha \), we get two derived sequences \( \{h_i\} \) and \( \{k_i\} \).

**Theorem 3.1 : Fundamental recurrence relation**

Let \( [a_0; a_1, a_2, \ldots, a_n] \) be a continued fraction with positive \( a_i \), with the sequences \( \{h_i\} \) and \( \{k_i\} \) as above. Then for \( m \leq n \), the \( m^{th} \) convergent \( c_m = \frac{h_m}{k_m} \).

**Proof:**

We will prove the result by induction, first for the cases where \( m = 0 \) and \( m = 1 \), and then for every other case. For \( m = 0 \), notice \( B_0 = A_0 = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \). Thus the theorem asserts that the 0\(^{th} \) convergent \( c_0 = \frac{a_0}{1} = a_0 \), which is obviously true. For the \( m = 1 \) case, notice that

\[
B_1 = A_1A_0 = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 a_0 + 1 & a_1 \\ a_0 & 1 \end{bmatrix}.
\]

We see \( h_1 = a_1 a_0 + 1 \), and \( k_1 = a_1 \), thus according to the theorem, we should have that \( c_1 = \frac{h_1}{k_1} = \frac{a_1 a_0 + 1}{a_1} \). Notice that

\[
c_1 = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1} = \frac{h_1}{k_1}
\]

as desired. Thus the result holds for \( m = 1, 2 \). Now assume that the result holds for \( m \leq m - 1 \), then we get that

\[
c_{m-1} = \frac{h_{m-1}}{k_{m-1}} \Rightarrow c_{m-1} = \frac{a_{m-1} h_{m-2} + h_{m-3}}{a_{m-1} k_{m-2} + k_{m-3}} \quad \text{by equation (1) above.}
\]
The key trick in this proof is to notice that if we have the convergent $c_{m-1}$, we can form the $m^{th}$ convergent by replacing the last entry in $c_{m-1}$. Recall that $c_{m-1} = [a_0; a_1, a_2, \ldots a_{m-1}]$. If we replace $a_{m-1}$ with $\tilde{a}_{m-1} = a_{m-1} + \frac{1}{a_m}$, we get a new convergent, which we will call $\tilde{c}_{m-1}$. We can easily see that $\tilde{c}_{m-1} = c_m$, for

$$\tilde{c}_{m-1} = [a_0; a_1, \ldots, \tilde{a}_{m-1}] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\vdots + \frac{1}{\tilde{a}_{m-1}}}}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\vdots + \frac{1}{a_{m-1} + \frac{1}{a_m}}}}} = c_m$$

It is also important to realize that

$$\tilde{c}_{m-1} = \frac{\tilde{a}_{m-1} h_{m-2} + h_{m-3}}{\tilde{a}_{m-1} k_{m-2} + k_{m-3}}$$

holds due to the induction assumption (which doesn’t require that $a_{m-1}$ be an integer.) Now observe that...

$$\tilde{c}_{m-1} = \frac{\tilde{a}_{m-1} h_{m-2} + h_{m-3}}{\tilde{a}_{m-1} k_{m-2} + k_{m-3}} = \frac{(a_{m-1} + \frac{1}{a_m}) h_{m-2} + h_{m-3}}{(a_{m-1} + \frac{1}{a_m}) k_{m-2} + k_{m-3}}$$

and now multiply the top and the bottom by $a_m$ to obtain

$$= \frac{(a_{m}a_{m-1} + 1) h_{m-2} + a_m h_{m-3}}{(a_{m}a_{m-1} + 1) k_{m-2} + a_m k_{m-3}} = c_m \quad (2)$$

Now we return to the matrices to notice that

$$B_m = A_m B_{m-1} = A_m A_{m-1} B_{m-2} = \begin{bmatrix} a_m & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{m-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_{m-2} & k_{m-2} \\ h_{k-3} & k_{m-3} \end{bmatrix} =$$

$$\begin{bmatrix} (a_{m}a_{m-1} + 1) h_{m-2} + a_m h_{m-3} & (a_{m}a_{m-1} + 1) k_{m-2} + a_m k_{m-3} \\ ** & ** \end{bmatrix} = \begin{bmatrix} h_m & k_m \\ ** & ** \end{bmatrix}$$

We notice that the ratio of $h_m$ to $k_m$ in the matrix above is exactly the $c_m$ in equation (2). Thus the $(m-1)^{st}$ case implies the $m^{th}$ case. By the principle of mathematical induction, the result holds for all $m \in \mathbb{N}$. It is important to realize that the fundamental recurrence relations exist independent of the matrices. We desire formulas which, in essence, capture the structure of the continued fraction in such a way that to make it unnecessary to refer back to what a continued fraction actually is. In other words, this theorem is so important that almost all of the theory developed about continued fractions derives from it. The fundamental recurrence relations relate $c_{m-1}$ and $c_m$. Using the theorem just developed, we see that $B_m = A_m B_{m-1}$, or

$$\begin{bmatrix} h_m & k_m \\ h_{m-1} & k_{m-1} \end{bmatrix} = \begin{bmatrix} a_m & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_{m-1} & k_{m-1} \\ h_{m-2} & k_{m-2} \end{bmatrix} = \begin{bmatrix} a_m h_{m-1} + h_{m-2} & a_m k_{m-1} + k_{m-2} \\ ** & ** \end{bmatrix}$$
Equating the top left and top right entries of the matrices above, we get the standard recurrence relations for the numerators and denominators of the convergent \( c_m = \frac{h_m}{k_m} \).

\[
h_m = a_m h_{m-1} + h_{m-2} \\
k_m = a_m k_{m-1} + k_{m-2}
\]

**Example 3.2**

Find the 4th convergent of \( \pi \).

Using the method of finding the greatest integer of a real number and isolating the remaining part, taking the reciprocal of that, and continuing as outlined in section 1, we can get a continued fraction for \( \pi \). In fact, we can establish that \( \pi = [3; 7, 15, 1, 292, ...] \). Using the method outlined in the previous theorem, we can compute the 4th convergent of \( \pi \). We will need to compute \( B_4 = A_4 A_3 A_2 A_1 A_0 \).

\[
B_4 = \begin{bmatrix}
292 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
15 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
7 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
3 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
103993 & 33102 \\
355 & 113
\end{bmatrix}
\]

Thus the 4th convergent for \( \pi \) is

\[
c_4 = \frac{103993}{33102} = 3.1415926530119 \text{ whereas } \pi = 3.1415926535898...
\]

Notice that this approximation is accurate to 9 digits, and to store the information needed to compute this convergent, we only needed 7 digits (add up the number of digits from 3, 7, 15, 1 292. We see that continued fractions thus represent \( \pi \) better than a decimal representation in some sense. Continued fractions converge to their limits quickly, which is why they could be used as tools for approximation of irrational numbers. In fact, as we shall see, they are the best approximation to irrational numbers in the sense that any better rational approximation will have a larger denominator. First, we will show that any infinite simple continued fraction (as in section 1) has a limit. Recall that

\[
B_n = \begin{bmatrix}
h_n & k_n \\
h_{n-1} & k_{n-1}
\end{bmatrix} = A_n A_{n-1} A_{n-2} \cdots A_0 \text{ therefore,}
\]

\[
\det (B_n) = \det (A_n A_{n-1} \cdots A_0) = \det (A_n) \det (A_{n-1}) \cdots \det (A_0) = (-1)^{n+1}
\]

since \( \det \begin{bmatrix}
a_i & 1 \\
1 & 0
\end{bmatrix} = -1 \) for all \( i \). Computing the determinant of \( B_n \) yields the following.

\[
h_n k_{n-1} - k_n h_{n-1} = (-1)^{n+1}
\]

(3)

Now divide both sides of equation (3) above by \( k_n k_{n-1} \) to obtain \( c_n - c_{n-1} = \frac{(-1)^{n+1}}{k_n k_{n-1}} \). We’ll take this one step further and examine

\[
c_n - c_{n-2} = \frac{h_n}{k_n} - \frac{h_{n-2}}{k_{n-2}} = \frac{h_n k_{n-2} - k_n h_{n-2}}{k_n k_{n-2}}
\]
Now apply the fundamental recurrence relations and simplify to get:
\[
\frac{(a_nh_{n-1} + h_{n-2})k_{n-2} - (a_nk_{n-1} + k_{n-2})h_{n-2}}{k_nk_{n-2}} = \frac{a_n(h_{n-1}k_{n-2} - k_{n-1}h_{n-2})}{k_nk_{n-2}}
\]
Finally, apply the result we found using determinants (equation (3)) to obtain
\[
c_n - c_{n-2} = \frac{a_n(-1)^{n+1}}{k_nk_{n-2}}
\]
We know that \(a_i > 0\) and \(k_i > 0\) for all \(i > 0\), thus the above implies that if \(n\) is even, \(c_n - c_{n-2} > 0\), and thus \(c_n > c_{n-2}\). Similarly, if \(n\) is odd, \(c_n < c_{n-2}\). Also, using an above equation,
\[
c_n - c_{n-1} = \frac{(-1)^{n+1}}{k_nk_{n-1}} \Rightarrow c_n < c_{n-1} \text{ for } n \text{ even, and } c_n > c_{n-1} \text{ for } n \text{ odd. (★★)}
\]
In other words, the even convergents are monotonically increasing and the odd convergents are monotonically decreasing. We also see that any of the odd convergents is greater than every even convergent. Using these facts, we can establish the following inequality chain.
\[
c_0 < c_2 < c_4 < \cdots < c_{2n} < c_{2n+2} < c_{2n+1} < \cdots < c_5 < c_3 < c_1 \quad (4)
\]
Using this chain in conjunction with the following result, we will prove that every simple infinite continued fraction has a limit.

**Lemma 3.3:** For all \(n > 1\), \(|c_n - c_{n-1}| \leq \frac{1}{n}\).

Proof (by induction):

First, we prove that \(k_n \geq n\) for all \(n \in \mathbb{N}\)
\[
c_1 = a_0 + \frac{1}{a_1} = \frac{a_0a_1 + 1}{a_1} = \frac{h_1}{k_1} \text{ thus } k_1 = a_1 \geq 1
\]
Now assume that \(m > 1\) and \(k_q \geq q\) for all natural numbers \(q \leq m \in \mathbb{N}\). (induction assumption) We use the recurrence relations to try to show that it will also work for the \((m + 1)^{st}\) case.
\[
k_{m+1} = a_mk_m + k_{m-1} \geq k_m + k_{m-1} \geq m + (m - 1) \geq m + 1
\]
where the first inequality relies on \(a_i \geq 1\) and the second inequality relies on the induction assumption. The third inequality above is due to the fact that we are assuming \(m > 1\). Thus the \(m^{th}\) case implies the \((m + 1)^{st}\) case. By the principle of mathematical induction, \(k_n \geq n\) for all \(n \in \mathbb{N}\). Now examine
\[
c_n - c_{n-1} = \frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} = \frac{h_nk_{n-1} - k_nh_{n-1}}{k_nk_{n-1}}
\]
Now apply the formula derived from the determinant of \(B_n\), followed by what we just proved.
\[
\frac{(-1)^{n+1}}{k_nk_{n-1}} \leq \frac{1}{k_nk_{n-1}} \leq \frac{1}{n(n - 1)} \leq \frac{1}{n}
\]
The first inequality above is true because $k_i > 0$, the second one is due to the inductive proof above, and the third inequality is true because $n > 1$. This concludes our lemma.

**Theorem 3.4:** Every simple infinite continued fraction has a finite limit.

**Proof:**

Examine the sequence $\{c_{2n}\}_{n \in \mathbb{N}}$ of even convergents and the sequence $\{c_{2n+1}\}_{n \in \mathbb{N}}$ of odd convergents. The inequality chain exhibited in (4) above shows that the even sequence is monotonically increasing and bounded by $c_1$. We know that any bounded monotonic sequence converges to a finite limit. Say that this limit is $l_1$. Similarly, the odd convergents are monotonically decreasing and bounded below by $c_0$. So the sequence of odd convergents also converges to a finite limit, say $l_2$. By the lemma above, we have

$$|c_{2n+1} - c_{2n}| \leq \frac{1}{2n + 1} \implies \lim_{n \to \infty} |c_{2n+1} - c_{2n}| \leq \lim_{n \to \infty} \frac{1}{2n + 1} = 0$$

Thus the even convergents and odd convergents have the same limit. ($l_1 = l_2$) Due to a theorem from calculus, we know that the limit of the sequence $\{c_n\}_{n \in \mathbb{N}}$ has that same limit.

§4. Approximation of irrational numbers [1]

Earlier we mentioned that continued fractions are the best tool for approximating irrational numbers in the sense that any better approximation has a bigger denominator. Here we will prove this. (this section also from Eric Skouson)

**Theorem 4.1:**

Let $\alpha = [a_0; a_1, a_2, \ldots]$ be an irrational number, and let $c_n = \frac{h_n}{k_n}$ be its $n^{th}$ convergent. Also let $\frac{r}{s} \in \mathbb{Q}$ be a reduced fraction with $s > 0$.

If $\left| \frac{\alpha - r}{s} \right| < \left| \alpha - \frac{h_n}{k_n} \right|$, then $s > k_n$

**Proof:**

From the last theorem, we know that the sequence of convergents $\{c_n\}_{n \in \mathbb{N}}$ converges to $\alpha$. Each convergent gets closer and closer to $\alpha$, thus

$$\left| \alpha - \frac{h_n}{k_n} \right| < \left| \alpha - \frac{h_{n-1}}{k_{n-1}} \right|$$

Also, $\alpha$ is between the odd convergents and the even convergents, so we know that

$$\left| \frac{h_n - r}{k_n - s} \right| < \left| \frac{h_n}{k_n} - \frac{h_{n-1}}{k_{n-1}} \right| = \left| \frac{h_n k_{n-1} - h_{n-1} k_n}{k_n k_{n-1}} \right| = \frac{1}{k_n k_{n-1}}$$
Now get a common denominator on the left hand side of this inequality to get that
\[
\left| \frac{h_{n-1}s - k_{n-1}r}{k_{n-1}s} \right| < \frac{1}{k_n k_{n-1}}
\]

Now multiply both sides by \(k_{n-1}\) to obtain
\[
\frac{|k_{n-1}s - k_{n-1}r|}{s} < \frac{1}{k_n}
\]

The numerator of this fraction is greater than 0 because otherwise we would have \(\frac{h_{n-1}}{k_{n-1}} = \frac{r}{s}\), which would mean that \(\frac{r}{s}\) is the \((n-1)\)st convergent. Since successive convergents always get closer to the limit value of the continued fraction \(\alpha\), this would mean that the \(n\)th convergent is closer to \(\alpha\) than \(\frac{r}{s}\). This contradicts our assumption. Since we know that \(|k_{n-1}s - k_{n-1}r|\) is an integer greater than 0, we can say that
\[
\frac{1}{s} \leq \frac{|k_{n-1}s - k_{n-1}r|}{s} < \frac{1}{k_n} \quad \text{or} \quad \frac{1}{s} < \frac{1}{k_n}
\]

which implies \(s > k_n\), as desired.

The next theorem tells us that if we can get a "good" rational approximation to the irrational continued fraction, then it must be a convergent in the continued fraction expansion for \(\alpha\).

**Theorem 4.2:**

Let \(\frac{h}{k} \in \mathbb{Q}\), with \(k > 0\). If
\[
\left| \frac{h}{k} - \alpha \right| < \frac{1}{2k^2}
\]
then \(\frac{h}{k}\) is a convergent in the simple continued fraction list for \(\alpha\).

**Proof:**

If we let \(\frac{h}{k}\) satisfy our assumptions, then there exists \(\delta\) such that \(|\delta| < 1/2\) and
\[
\frac{h}{k} - \alpha = \frac{\delta}{k^2}.
\]

Now let \([a_0; a_1, a_2, \ldots, a_m]\)
be the continued fraction for \(\frac{h}{k} = \frac{h_m}{k_m}\). \(m\) can be made even or odd as we wish by letting the final number in the continued fraction be a 1 or not, as in section 1). We want to match the sign of \(\delta\) to \(m\) such that
\[
\frac{h}{k} - \alpha = \frac{(-1)^{m+1}|\delta|}{k^2}
\]

Let \(\frac{h_n}{k_n}\) be the \(n\)th convergent of \(\frac{h}{k}\), and let \(\beta = \frac{h_{m-1} - \alpha k_{m-1}}{\alpha k_m - h_m}\)
Solving for \( \alpha \) yields \( \alpha = \frac{\beta h_m + h_{m-1}}{\beta k_m + k_{m-1}} \)

which is in the form of the fundamental recurrence relations. Hence we can consider \( \alpha \) as the \((m + 1)^{st}\) convergent of the continued fraction \([a_0; a_1, a_2, \ldots, a_m, \beta]\). Now if we set the continued fraction for \( \beta \) as \( \beta = [a_{m+1}; a_{m+2}, \ldots] \), then we can say that

\[
\alpha = [a_0, a_1, \ldots, a_m, a_{m+1}, \ldots]
\]

The above equation for \( \alpha \) contains the string of \( a_0, a_1, \ldots, a_m \), which would yield the desired result, that \( \frac{h}{k} \) is a convergent in the continued fraction expansion for \( \alpha \). All that remains to be proved is that the above equation for \( \alpha \) is a simple continued fraction expansion. (ie. we need to prove that \( \beta > 1 \) to allow us to only use natural elements for \( a_{m+1}, a_{m+2}, \ldots \)).

So why is \( \beta > 1 \)? We can see that from the beginning of the proof,

\[
\frac{(-1)^{m+1}|\delta|}{k^2} = \frac{h}{k} - \alpha = \frac{h}{k} - \frac{\beta h_m + h_{m-1}}{\beta k_m + k_{m-1}}
\]

Now we can simplify the above, getting a common denominator and using equation (3) from section 3 to get

\[
= \frac{h_{km-1} - kh_m}{k (\beta k + k_{m-1})} = \frac{(-1)^{m+1}}{k (\beta k + k_{m-1})}
\]

From this long string of equations, cancel the \((-1)^{m+1}\) and \( k \). Finally "invert" both sides to obtain

\[
\frac{1}{|\delta|} = \frac{\beta k + k_{m-1}}{k} \Rightarrow \frac{1}{|\delta|} - \frac{k_{m-1}}{k} = \beta > 1
\]

where the last inequality is because \( \frac{1}{|\delta|} > 2 \) and the sequence \( \{k_n\} \) is increasing as we saw earlier. (giving \( \frac{k_{m-1}}{k} < 1 \)) Thus \( \beta > 1 \), completing the proof.

Now we will prove another result which relates the closeness of the convergent \( c_n \) and the value of the continued fraction \( \alpha \). This lemma will be useful in the next section.

**Lemma 4.3**

The value \( \alpha \) of a convergent infinite continued fraction satisfies the inequality

\[
\left| \alpha - \frac{h_n}{k_n} \right| < \frac{1}{k_n k_{n+1}}
\]

where \( \frac{h_n}{k_n} \) is the \( n^{th} \) convergent of the continued fraction \( \alpha \).
Proof:

Recall equation (★) in section 3, which states

$$c_n - c_{n-1} = \frac{(-1)^{n+1}}{k_n k_{n-1}}$$

We also know that all of the even convergents are greater than all of the odd convergents, so that the difference between successive convergents is more than the difference between one of those convergents and $\alpha$. Thus

$$|\alpha - c_n| < |c_n - c_{n-1}| = \frac{1}{k_n k_{n-1}}$$

This seemingly innocent result will be key to proving the existence of transcendental numbers in the next section.

§5: Algebraic irrational numbers and Liouville’s transcendental numbers [2]

The goal of this section is to provide an application of continued fractions to algebra and number theory. Credit for this section is due mainly to Continued Fractions by A. Ya. KHINCHIN.

Definition 5.1: A number $\alpha$ is said to be algebraic over $\mathbb{Z}$, or more commonly, algebraic, if and only if $\alpha$ is a root of a polynomial in the polynomial ring $\mathbb{Z}[x]$.

For example, all rational numbers are algebraic because any rational number $\frac{p}{q}$ is a root of the polynomial $f(x) = qx - p \in \mathbb{Z}[x]$.

For another example, $\sqrt{3}$ is algebraic because it is a root the polynomial $g(x) = x^2 - 3$.

In more specific terms, we say that $\alpha$ is algebraic of degree $n$ if $\alpha$ is a root of an irreducible polynomial of degree $n$ with integral coefficients. In the example above, we see that $\sqrt{3}$ is algebraic of degree 2. Any number which is not algebraic is said to be transcendental.

For example, $\pi$ is a transcendental number (which historically was difficult to prove). The following theorem, dubbed Liouville’s theorem, was the first theorem to demonstrate the existence of such numbers.

Theorem 5.1: Liouville’s theorem

For each irrational algebraic number $\alpha$ of degree $n$, there exists a positive number $C$ such that for arbitrary integers $p$ and $q$, ($q > 0$),

$$\left| \alpha - \frac{p}{q} \right| > \frac{C}{q^n}$$

Proof:

Suppose $\alpha$ is an irrational algebraic number of degree $n$, and that $f(\alpha) = 0$, where

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$
Then due to the factor theorem, we can write \( f(x) = (x - \alpha) q(x) \), where \( g(x) \) is a polynomial of degree \( n - 1 \). Since \( \alpha \) is algebraic of degree \( n \), we cannot have \( q(\alpha) = 0 \), because then we would have that \((x - \alpha)^2\) is a factor of \( f(x) \), yielding that \( f'(x) \) also has a factor of \((x - \alpha)\). (apply product rule). Hence \( f'(x) \) would be a polynomial of degree \( n - 1 \) with integer coefficients and \( f'(\alpha) = 0 \). But this is impossible since the lowest degree polynomial \( \alpha \) can be a zero for is \( n \).

Due to the fact that all polynomials are continuous over the entire real line, and the fact that \( g(\alpha) \neq 0 \), we get an entire interval of size \( \delta \) around \( \alpha \) for which

\[
g(x) \neq 0 \quad \text{whenever} \quad |x - \alpha| < \delta
\]

Now take two random integers \( p \) and \( q \) such that \( q > 0 \), and let them ”reside” in this interval. In other words, let \( |\alpha - \frac{p}{q}| < \delta \), which will give us that \( g\left(\frac{p}{q}\right) \neq 0 \).

Now rewriting the result of the factor theorem as \((x - \alpha) = \frac{f(x)}{g(x)}\), we get that

\[
\frac{p}{q} - \alpha = \frac{f\left(\frac{p}{q}\right)}{g\left(\frac{p}{q}\right)} = \frac{a_0 + a_1 \left(\frac{p}{q}\right) + \cdots + a_n \left(\frac{p}{q}\right)^n}{g\left(\frac{p}{q}\right)}
\]

Now multiply the numerator and denominator by \( q^n \) to obtain

\[
\frac{p}{q} - \alpha = \frac{a_0 q^n + a_1 pq^{n-1} + \cdots + a_n p^n}{q^n g\left(\frac{p}{q}\right)}
\]

The numerator of this fraction is an integer and is also nonzero (because otherwise \( \alpha = \frac{p}{q} \), which is impossible since \( \alpha \) is irrational). Thus the absolute value of the numerator is at least one. Since \( g(x) \) is a polynomial, it is bounded in the interval \((\alpha - \delta, \alpha + \delta)\), and we let \( M \) be the least upper bound for \( g(x) \) in this interval. Using the above equation along with the fact that the numerator is at least one, we get

\[
\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^n g\left(\frac{p}{q}\right)} \geq \frac{1}{M q^n} \quad \text{whenever} \quad \left|\alpha - \frac{p}{q}\right| < \delta
\]

If, on the other hand, we have \( \left|\alpha - \frac{p}{q}\right| > \delta \), then we easily see that \( \left|\alpha - \frac{p}{q}\right| > \frac{\delta}{q^n} \).

To pick a \( C \) which satisfies the conclusion of the theorem in both cases, we let

\[
C = \frac{1}{2} \min\left(\delta, \frac{1}{M}\right) \implies \left|\alpha - \frac{p}{q}\right| > \frac{C}{q^n}
\]

\( C \) is positive, as claimed, which completes the proof.

Often times this theorem is useful in the light of its contrapositive, which I will state below.

If, for any \( C > 0 \) and any \( n \in \mathbb{N} \), there exist integers \( p \) and \( q \) \((q > 0)\), such that

\[
\left|\alpha - \frac{p}{q}\right| \leq \frac{C}{q^n} \quad \text{then} \quad \alpha \text{ is transcendental.}
\]
Essentially this is saying that if a number can be approximated "well" using rational numbers, then that number is transcendental. Using the continued fractions, it will actually be quite easy to show the existence of transcendental numbers in this way.

To find a transcendental number using this theorem, we create a continued fraction "inductively." If we have already chosen the elements \( a_0, a_1, \ldots, a_n \), we find the convergent \( c_n = \frac{h_n}{k_n} \). Next, we choose \( a_{n+1} \) so that it satisfies

\[ a_{n+1} > (k_n)^{n-1} \]

This gives us that

\[ \left| \alpha - \frac{h_n}{k_n} \right| < \frac{1}{k_n k_{n+1}} < \frac{1}{(k_n)^2 a_{n+1}} < \frac{1}{(k_n)^{n+1}} \quad (\dagger) \]

where the first inequality is due to lemma 4.3. The last inequality is easily derived from how we have chosen \( a_{n+1} \). The second inequality is due to the fundamental recurrence relations, which give that

\[ k_{n+1} = a_{n+1} k_n + k_{n-1} < a_{n+1} k_n \Rightarrow k_n k_{n+1} > (k_n)^2 a_{n+1} \Rightarrow \frac{1}{k_n k_{n+1}} < \frac{1}{(k_n)^2 a_{n+1}} \]

The derived inequality (\dagger) above satisfies the condition for \( \alpha \) to be transcendental because by making \( n \) large enough, we can make the right hand side as small as we want (in other words, the inequality will hold regardless of the value of \( C \) or \( n \)).

§6: Hypergeometric functions and Gauss’s continued fraction [3]

We shall now head in a different direction, to see how continued fractions can be used as a tool in analysis. It is assumed that the reader is unfamiliar with the notion of hypergeometric functions, and as such we begin with a definition.

Definition 6.1

A hypergeometric function is defined on the complex plane for \( |z| < 1 \) by

\[ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \]

where \((x)_n\) is the so-called Pochhammer symbol or rising factorial defined for \( n > 0 \) by

\[ (x)_n = x(x+1)(x+2)\cdots(x+n-1) \quad (x)_0 = 1 \]

The notation for the hypergeometric function is quite bulky but is easy to understand. The 2 to the left and 1 to the right of the F represent how many of the Pochhammer symbols there are in the numerator and denominator, respectively. When Gauss first studied these functions, there were 2 of these in the numerator and 1 in the denominator. Therefore, we will
occasionally write \( F = _2F_1 \). In the notation \( _2F_1 (a, b; c; z) \), the \( a \) and \( b \) represent Pochhammer symbols in the numerator, which are separated by a semicolon from the denominator terms. The \( z \) is the only input to the function. We can see how this function is quite naturally generalized to include as many "terms" in the numerator and denominator as one could want.

**Example 6.2:**

Examine the hypergeometric function

\[
_2F_1 (1, b; b; z) = \sum_{n=0}^{\infty} \frac{(1)_n (b)_n}{(b)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(1)_n}{n!} z^n = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}
\]

which simplifies to a geometric series. (We recall that \(|z| < 1\) means that the sum converges to \(\frac{1}{1-z} \).) This example shows us that hypergeometric functions are defined by hypergeometric series which are natural generalizations of geometric series.

**Example 6.3:**

We will again use the geometric series to make a simplification (for now we are pretending that all of the derivatives and integrals make sense in complex numbers. The notation is formal here)

\[
\ln(1-z) = \int_0^z \frac{-1}{1-t} \, dt = -z \left( 1 + \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \cdots \right)
\]

The goal is to express the infinite sum in the parentheses in terms of a hypergeometric function. By simplifying, we see that

\[
_2F_1 (1, 1; 2; z) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(1)_n}{(2)_n} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n+1}
\]

where the sum on the right hand side is equal to the sum in the parentheses we were talking about earlier. If we now put all of this together, we obtain that

\[
_2F_1 (1, 1; 2; z) = \frac{1}{z} \ln \left( \frac{1}{1-z} \right) \quad \text{valid for } |z| < 1
\]

Now, we wish to be able to express ratios of hypergeometric functions as continued fractions. Gauss came up with several formulas accomplishing this. The most commonly seen are his continued fraction formulas involving ratios of \(_0F_1, _1F_1\), and \(_2F_1\).

Since the function \(_0F_1\) best illustrates how the continued fractions are developed, we will concentrate on these. It should be noted that all of the continued fractions developed depend on the following lemma.
Lemma 6.4

Let \( f_0, f_1, f_2, \ldots \) be any sequence of analytic functions satisfying 
\[
  f_i - f_{i-1} = k_i z f_{i+1}
\]
for all \( i > 0 \)

where each \( k_i \) is constant. Then we have
\[
  \frac{f_i}{f_0} = \frac{1}{1 + \frac{k_1 z}{1 + \frac{k_2 z}{1 + \frac{k_3 z}{1 + \cdots}}}}.
\]

Proof:

Let the functions be related as in the hypothesis. Then
\[
  \frac{f_{i-1}}{f_i} = 1 + k_i z \frac{f_{i+1}}{f_i} \quad \Rightarrow \quad \frac{f_i}{f_{i-1}} = \frac{1}{1 + k_i z \frac{f_{i+1}}{f_i}}
\]

Now plug in \( i = 1 \) and keep re-using the above relation to get
\[
  \frac{f_1}{f_0} = \frac{1}{1 + k_1 z \frac{f_2}{f_1}} = \frac{1}{1 + \frac{k_1 z}{1 + \frac{k_2 z}{1 + \frac{k_3 z}{1 + \cdots}}}} = \ldots = \frac{1}{1 + \frac{k_1 z}{1 + \frac{k_2 z}{1 + \frac{k_3 z}{1 + \cdots}}}}
\]

The fact that ratios of hypergeometric functions can be represented as continued fractions relies on proving that the hypergeometric functions satisfy the assumptions in the previous lemma and finding the \( k_1, k_2, \ldots \)

A continued fraction involving \( {}_0 F_1 \)

First, we need to prove an important identity. Examine
\[
  {}_0 F_1 \left( ; a - 1; z \right) - {}_0 F_1 \left( ; a; z \right) = \sum_{n=0}^{\infty} \frac{1}{(a - 1)_n} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!}
\]

Now combine the sums, and do some algebraic manipulations to obtain
\[
  = \sum_{n=0}^{\infty} \frac{(a)_n - (a - 1)_n}{(a)_n(a - 1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{n(a)_n}{(a)_n(a - 1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{a(a - 1)(a + 1)_{n-1}} \frac{z^n}{(n - 1)!}
\]
Now factor out some constants and re-index to see that
\[
\frac{z}{a(a-1)} \sum_{n=0}^{\infty} \frac{1}{(a+1)_n} n! n^z = \frac{z}{a(a-1)} {}_0F_1 (; a + 1; z)
\]

To summarize,
\[
{}_0F_1 (; a - 1; z) - {}_0F_1 (; a; z) = \frac{z}{a(a-1)} {}_0F_1 (; a + 1; z)
\]

Now, in terms of our lemma, if we take
\[
f_i = {}_0F_1 (; a + i; z) \quad k_i = \frac{1}{(a+i)(a+i+1)}
\]
we get that \( f_{i-1} - f_i = k_i z f_{i+1} \), exactly as required by the lemma. Hence we can use the \( k_i \) to write a continued fraction and get
\[
\frac{f_i}{f_0} = \frac{{}_0F_1 (; a + 1; z)}{{}_0F_1 (; a; z)} = \frac{1}{1 + \frac{1}{a(a+2)}z + \frac{1}{(a+2)(a+3)}z + \frac{1}{1 + \ddots}}.
\]

We can clean this up a bit by multiplying by \( a \) in the top numerator and denominator, then multiply by \( (a+1) \) in the next numerator/denominator, then \( (a+2) \) in the next, and so on.. to get
\[
\frac{{}_0F_1 (; a + 1; z)}{{}_0F_1 (; a; z)} = \frac{a}{a+1} \frac{z}{a+2} \frac{z}{(a+2)+} \frac{z}{(a+3)+} \ddots.
\]

**Conclusion:**

Continued fractions have many more applications beyond what is seen here. One thing not mentioned here is the solution to Pell’s equation. One of the more interesting things that I have come across during my research is that there is no known pattern for continued fractions for irrational numbers which are algebraic of degree greater than 2. For degree 2, the pattern is always periodic, but for higher degrees, the pattern eludes us. I regret not spending more time trying to figure out this difficult pattern. Perhaps this will be a topic of future research for me.

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