Derived equivalence, Albanese varieties, and the zeta functions of 3–dimensional varieties

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June 29, 2016

Abstract

We prove that derived equivalent smooth, projective varieties over any algebraically closed field have isogenous Albanese varieties, extending the proof of Popa and Schnell for varieties over \( \mathbb{C} \). We apply this result to show that for any derived equivalent smooth, projective varieties of dimension 3 over a finite field \( \mathbb{F}_q \), there is some \( n \in \mathbb{N} \) such that their base changes to \( \mathbb{F}_{q^n} \) have equal zeta functions.

The problem of characterizing the bounded derived category of coherent sheaves of a variety has connections to birational geometry, the minimal model program, mirror symmetry (in particular, the conjecture of Kontsevich [13]), and motivic questions.

Orlov has conjectured that derived equivalent smooth, projective varieties have isomorphic motives [18]. This conjecture predicts that smooth, projective varieties over a finite field that are derived equivalent have equal zeta functions. It has been verified that derived equivalent smooth, projective varieties over a finite field that are abelian or of dimension 2 or less have equal zeta functions [7]. The case of curves was already known since derived equivalent curves over a finite field are isomorphic: proof in the genus 1 case is given by Antieau, Krashen and Ward [2, Example 2.8], and proof in all other cases is a consequence of Bondal and Orlov [4, Theorem 2.5], which shows that derived equivalent varieties with ample or anti-ample canonical bundle must be isomorphic.

In this paper, we prove the following extension to these results:

**Theorem A.** Let \( X, Y/\mathbb{F}_q \) be derived equivalent smooth, projective varieties of dimension 3, where \( \mathbb{F}_q \) is a finite field with \( q \) elements. Then there is some \( n \in \mathbb{N} \) such that the base changes of \( X \) and \( Y \) to \( \mathbb{F}_{q^n} \) have equal zeta functions.

The equality of zeta functions over \( \mathbb{F}_{q^n} \) described in Theorem A is equivalent to \( X \) and \( Y \) having equal numbers of \( \mathbb{F}_{q^{mn}} \)-points for all \( m \in \mathbb{N} \), and is a weaker assertion than \( X \) and \( Y \) having equal zeta functions, which is equivalent to \( X \) and \( Y \) having equal numbers of \( \mathbb{F}_{q^m} \)-points for all \( m \in \mathbb{N} \).

However, the proof of Theorem A is similar to the argument in [7] proving that derived equivalent smooth, projective surfaces over any finite field have equal zeta functions: it is accomplished by comparing the eigenvalues of the
The geometric Frobenius morphism acting on the ℓ-adic étale cohomology groups of the varieties in question.

The crucial ingredient for making this comparison between the point-counts of three-dimensional varieties is the following theorem, which implies that if X and Y are derived equivalent smooth, projective varieties over a finite field, then there is an isomorphism $H^1_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell) \cong H^1_{\text{ét}}(\bar{Y}, \mathbb{Q}_\ell)$, which, for some $n \in \mathbb{N}$, is compatible with the action of the $q^n$-th power geometric Frobenius morphism.

**Theorem B.** Derived equivalent smooth, projective varieties X and Y over any algebraically closed field have isogenous Albanese varieties.

The proof of Theorem B replicates Popa and Schnell’s proof that if $X, Y/\mathbb{C}$ are derived equivalent varieties, then $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ must be isogenous [20]. In generalizing to positive characteristic, we only need to change the argument insofar as to replace $\text{Pic}^0(X)$ and $\text{Pic}^0(Y)$ by their associated reduced schemes: Given an integral, projective variety $V/k$, $\text{Pic}^0(V)$ is in general reduced if $\text{char} k = 0$, but it may be non-reduced if $\text{char} k > 0$.

An alternate proof of Theorem B over $\mathbb{C}$ has also been obtained by R. Abuaf in his Theorem 3.0.14 of [1]. It is conceivable that similar methods to those in loc. cit. can be used over algebraically closed fields of arbitrary characteristic.

In Section 1, we provide some background. In Section 2, we prove Theorem B. In Section 3, we prove Theorem A.

### Acknowledgements

Thanks to Aaron Bertram, Christopher Hacon, Luigi Lombardi, Mihnea Popa, Raphaël Rouquier, Christian Schnell and Sofia Tirabassi for helpful comments, conversations and correspondence. Thanks in particular to Charles Vial and Martin Olsson for alerting me to the presence of an error in a previous version of this paper, and Daniel Litt for conversations about the current corrected version.

### 1 Background

#### 1.1 Definitions

We take a variety to be a separated, integral scheme of finite type. A group scheme is a scheme that is a group object in the category of schemes, that is, equipped with morphisms giving composition, identity and inverse satisfying the appropriate diagram. We take an algebraic group to be a smooth group scheme of finite type.

Let X be a smooth, projective variety or an algebraic group. The Albanese variety of X, $\text{Alb}(X)$ and the accompanying Albanese morphism $\text{alb}_X : X \to \text{Alb}(X)$ are characterized uniquely up to isomorphism by the universal property that any morphism from X to an abelian variety factors uniquely through $\text{alb}_X$. The existence of the Albanese variety of an algebraic group is established by
Chevalley’s theorem (see for instance [5, Theorem 1.1] for a modern treatment). See Serre [22] for an algebraic construction of the Albanese variety of a smooth, projective variety.

Fix $X$ be a smooth, projective variety over an algebraically closed field $k$. The absolute Picard functor $F_X : \text{Scheme}_k \to \text{Set}$ sends any scheme $S/k$ to the Picard group of $X \times_k S$, by which we mean the group of line bundles on $X \times_k S$ under the operation of the tensor product. This functor is represented by a presheaf. Its sheafification is the Picard scheme of $X$, which we denote by $\text{Pic}_k(X)$.

The $k$-points of $\text{Pic}_k(X)$ are in bijection with the Picard group $\text{Pic}_k(X)$ of $X$ (the $k$ is suppressed when it is understood).

We define $\text{Pic}_k^0(X)$ to be the connected component of the Picard scheme containing the identity. Its $k$-rational points form the subgroup $\text{Pic}_k^0(k)$ of $\text{Pic}_k(X)$.

When $k$ is of characteristic 0, $\text{Pic}_k^0(X)$ is an abelian variety, but, over fields of positive characteristic it is an algebraic group and possibly non-reduced; see Igusa [10] for a first example and Liedtke [14] for further discussion of surfaces that may have non-reduced Picard schemes.

The associated reduced scheme $\text{Pic}_k^0(X)_{\text{red}}$ is in general an abelian variety and dual to the Albanese variety of $X$ (Badescu [3, Chapter 5]).

The discussion of the Picard scheme in this section suffices for the purposes of this paper, but is otherwise very minimal; see [12] for a history and further development.

Let $X$ be a smooth, projective variety over a field $k$. We may consider the functor

$$A_X : \text{Scheme}_k \to \text{Group}$$

that sends any scheme $S/k$ to the group of $S$-automorphisms of $S \times_k X \to S$. By [16] Theorem 3.7], $A_X$ is represented by a group scheme of finite type, which we denote by $\text{Aut}_k(X)$, and the reduced scheme $\text{Aut}_k(X)_{\text{red}}$ associated to $\text{Aut}_k(X)$ is thus an algebraic group. The $k$-points of $\text{Aut}_k(X)$ are in bijection with the group of automorphisms of $X$.

1.2 A theorem of Nishi and Matsumura

The following results on algebraic groups will be used in the proof of Theorem A

**Theorem 1.1** (Nishi, Matsumura [15, Theorem 2]). *Let $V$ be a variety with a connected algebraic group $G$ acting faithfully on it. By the universal property of the Albanese variety, this action induces a morphism $h : \text{Alb}(G) \to \text{Alb}(V)$ (unique after choice of Albanese morphisms $\text{alb}_G$, $\text{alb}_V$). This morphism has a finite kernel.*

**Remark 1.1.1** (cf. [15, p. 54], [20, Lemma 2.2]). *Let $G$ be a connected algebraic group acting on $V$ via automorphisms (this situation satisfies the hypothesis of Theorem 1.1). Given any $v_0 \in V$, we may choose $\text{alb}_V$ so that $h \circ \text{alb}_G(g) = \text{alb}_V(gv_0)$ for all $g \in G$.***
Proof of Remark. Fix Albanese morphisms alb$_G : G \to \text{Alb}(G)$ and alb$_V : V \to \text{Alb}(V)$. Let $\lambda : G \times V \to V$ be the morphism giving the action of $G$ on $V$. The universal property of the Albanese morphisms uniquely induces a map giving an action of $\text{Alb}(G)$ on $\text{Alb}(V)$. $\bar{\lambda} : \text{Alb}(G) \times \text{Alb}(V) \to \text{Alb}(V)$. The map $\bar{\lambda}$ acts identically on $\text{Alb}(V)$, and so $\bar{\lambda}$ is determined by its restriction to $\text{Alb}(G)$, which is $h : \text{Alb}(G) \to \text{Alb}(V)$.

Since $G$ acts on $V$ via automorphisms, $\text{Alb}(G)$ acts on $\text{Alb}(V)$ via automorphisms as well, which must be translations since $\text{Alb}(V)$ is abelian. In particular, any element of $\text{Alb}(G)$ acts on $\text{Alb}(V)$ by translation by its image under $h$. That is, for any $g \in G$ and $v \in V$, $\text{alb}_V(gv) = \text{alb}_V(v) + h \circ \text{alb}_G(g)$. Fix $g \in G$. Then $h(g) = \text{alb}_V(gv) - \text{alb}_V(v)$ for any $v \in V$.

Fix $v_0 \in V$. By altering our selection of $\text{alb}_V$ above by postcomposing with translation by $\text{alb}(v_0)$, we have $h(g) = \text{alb}_V(gv_0)$. □

1.3 Fourier–Mukai transforms

Definition 1.1.1. A functor $F$ between derived categories $D^b(X)$ and $D^b(Y)$ is a Fourier–Mukai transform if there exists an object $P \in D^b(X \times Y)$, called a Fourier–Mukai kernel, such that $F \cong p_Y^*(p_X^*(-) \otimes P) =: \Phi_P$, (1)

where $p_X$ and $p_Y$ are the projections $X \times Y \to X$ and $X \times Y \to Y$. A Fourier–Mukai transform that is an equivalence of categories is called a Fourier–Mukai equivalence. The pushforward, pullback, and tensor in (1) are all in their derived versions, but the notation is suppressed.

Theorem 1.2 (Orlov [19, Theorem 3.2.1]). Let $X$ and $Y$ be smooth projective varieties and $F : D^b(X) \to D^b(Y)$ an exact equivalence. Then there is an object $\mathcal{E} \in D^b(X \times Y)$ such that $F$ is isomorphic to the functor $\Phi_{\mathcal{E}}$, and the object $\mathcal{E}$ is determined uniquely up to isomorphism.

The full statement of [19, Theorem 3.2.1] is stronger than what is given here, but the statement in Theorem 1.2 is sufficient for the purposes of this paper.

1.4 Rouquier’s isomorphism

Theorem 1.3 (Rouquier’s Isomorphism [21 Théorème 4.18]). Let $X$ and $Y$ be derived equivalent smooth, projective varieties over an algebraically closed field $k$. Then there is an isomorphism of algebraic groups

$$F : \text{Aut}^0(X) \times \text{Pic}^0(X) \cong \text{Aut}^0(Y) \times \text{Pic}^0(Y).$$ (2)

Rouquier’s proof induces $F$ by demonstrating that under the hypotheses of the above theorem there is an isomorphism between the following functors $\mathcal{P}_X$...
and \( P_Y \), which are represented by \( \text{Aut}(X) \times \text{Pic}^0(X) \) and \( \text{Aut}(Y) \times \text{Pic}^0(Y) \):

\[
P_X : \text{Var}_k \to \text{Set}
\]

\[
S \mapsto \left\{ \text{Coherent } \mathcal{O}_{X \times X \times S} \text{ modules } M \text{ locally free over } S \middle| p_{1*}M(s) \text{ and } p_{2*}M(s) \text{ are numerically trivial line bundles on } X \text{ for all } s \in S \right\} / \text{Pic}(S)
\]

The functor \( P_Y \) is defined in the same way with \( Y \) in place of \( X \).

**Remark 1.3.1.** The isomorphism (2) may be characterized in another way that we will find useful in proving Theorem B.

By Theorem 1.2, the equivalence \( D^b(X) \cong D^b(Y) \) is isomorphic to the Fourier–Mukai functor \( \Phi_E := p_2^* (p_1^*(-) \otimes \mathcal{E}) \). In the course of the proof of Theorem 1.3, Rouquier shows that for any \((\varphi, L) \in \text{Aut}^0(X) \rtimes \text{Pic}^0(X)\), the equivalence \( \Phi_E \circ \Phi_\text{id,}\psi)_* L \circ \Phi^{-1}_E : D^b(Y) \to D^b(Y) \) has a kernel of the form \((\text{id,}\psi)_* M\) for a unique (up to isomorphism) pair \((\psi, M) \in \text{Aut}^0(Y) \rtimes \text{Pic}^0(Y)\), which is precisely the image of \((\varphi, L)\) under \( F \).

Furthermore, by Mukai’s formula for composing Fourier–Mukai transforms (see Huybrechts [9, Proposition 5.10]), we may relate \((\varphi, L)\) and \((\psi, M)\) as follows:

\[
p_1^* L \otimes (\varphi \times \text{id})^* \mathcal{E} \cong p_2^* M \otimes (\text{id} \times \psi)^* \mathcal{E}
\]

where \( p_1 \) and \( p_2 \) are the first and second projections from \( X \times Y \).

**Corollary 1.3.1.** Let \( X \) and \( Y \) be derived equivalent smooth, projective varieties over a field \( k \). Then there is an isomorphism:

\[
F : \text{Aut}^0(X)_{\text{red}} \times \text{Pic}^0_{\text{red}}(X) \cong \text{Aut}^0(Y)_{\text{red}} \times \text{Pic}^0_{\text{red}}(Y).
\]

**Proof.** Rouquier’s isomorphism \( F \) restricts to the isomorphism

\[
F : \text{Aut}^0(X)_{\text{red}} \times \text{Pic}^0_{\text{red}}(X) \cong \text{Aut}^0(Y)_{\text{red}} \times \text{Pic}^0_{\text{red}}(Y)
\]

when we take the reduced schemes associated to both sides of (2).

The action of \( \text{Aut}^0(X)_{\text{red}} \) on \( \text{Pic}^0(X)_{\text{red}} \) is trivial: For any \( \psi \in \text{Aut}^0(X)_{\text{red}} \), its action on \( \text{Pic}^0(X)_{\text{red}} \) is given by a translation since it is an automorphism of an abelian group. For any automorphism \( \psi \in \text{Aut}^0(X) \), \( \psi^* \mathcal{O}_X \cong \mathcal{O}_X \): the action of \( \psi \) is that of a translation fixing the origin, and so must be trivial. Hence, we may use the direct product rather than the semidirect product in (6). However, we must exchange \( \text{Pic}^0(X) \) and \( \text{Pic}^0(Y) \) for \( \text{Pic}^0_{\text{red}}(X) \) and \( \text{Pic}^0_{\text{red}}(Y) \) as this argument uses the fact that \( \text{Aut}^0(X) \) and \( \text{Aut}^0(Y) \) are acting on abelian varieties.

\[\square\]

**2 Proof of Theorem B**

**Theorem B.** Derived equivalent smooth, projective varieties \( X \) and \( Y \) over any algebraically closed field have isogenous Albanese varieties.
Proof. Let $X, Y$ be smooth, projective varieties defined over a field $k = \bar{k}$ such that $D^b(X) \cong D^b(Y)$. By Corollary 1.3.1 there is an isomorphism

$$F : \text{Aut}^0_{\text{red}}(X) \times \text{Pic}^0_{\text{red}}(X) \cong \text{Aut}^0_{\text{red}}(Y) \times \text{Pic}^0_{\text{red}}(Y). \quad (5)$$

Consider the following maps:

$$\beta : \text{Pic}^0_{\text{red}}(X) \to \text{Aut}^0(Y), \quad \beta(L) = p_1(F(\text{id}, L))$$
$$\alpha : \text{Pic}^0_{\text{red}}(Y) \to \text{Aut}^0(X), \quad \alpha(M) = p_1(F^{-1}(\text{id}, M))$$

where $p_1$ is the appropriate projection onto the first factor. Define $B := \text{im}(\beta)$ and $A := \text{im}(\alpha)$.

$A$ and $B$ are images of abelian varieties, and so are abelian. We see that (5) restricts to an isomorphism

$$F : A \times \text{Pic}^0_{\text{red}}(X) \cong B \times \text{Pic}^0_{\text{red}}(Y) \quad (6)$$

because there is a bijection of closed points: Fix arbitrary closed points $\alpha \in A$ and $L \in \text{Pic}^0(X)_{\text{red}}$. By the definition of $A$ there is some $N \in \text{Pic}^0(X)$ and $M \in \text{Pic}^0(Y)$ such that $(\alpha, N) = F^{-1}(\text{id}, M)$. Hence, denoting the group operations of $A \times \text{Pic}^0(X)_{\text{red}}$ and $B \times \text{Pic}^0(Y)_{\text{red}}$ by $\ast$, we have

$$F(\alpha, L) = F(\alpha, N) \ast F(id, L \otimes N^{-1}) = (id, M) \ast F(id, L \otimes N^{-1}).$$

Both $(id, M)$ and $F(id, L \otimes N^{-1})$ are members of $B \times \text{Pic}^0(Y)_{\text{red}}$. The proof that $F^{-1}$ maps closed points of $B \times \text{Pic}^0(Y)_{\text{red}}$ to $A \times \text{Pic}^0(X)_{\text{red}}$ is analogous.

The remainder of the proof of Theorem B is split into proving the following three facts:

(i) $A$ and $B$ have equal dimensions.

(ii) $A$ and $B$ are isogenous.

(iii) $\text{Pic}^0(X)_{\text{red}}$ and $\text{Pic}^0(Y)_{\text{red}}$ are isogenous.

(i) By Theorem 1.2, the derived equivalence $D^b(X) \cong D^b(Y)$ is isomorphic to a Fourier–Mukai functor $\Phi_{\mathcal{E}} := p_2_*(p_1^*(-) \otimes \mathcal{E})$.

Fix a point $(x, y)$ in $\text{Supp} \mathcal{E} \subseteq X \times Y$. We construct a morphism $A \times B \to X \times Y$ given by $(\phi, \psi) \mapsto (\phi(x), \psi(y))$. By the universal property of the Albanese variety, this map induces a morphism $f : A \times B \to \text{Alb}(X) \times \text{Alb}(Y)$.

By Remark 1.1.1 and Theorem 1.1. $f$ has a finite kernel, hence is an isogeny. Call the dual isogeny of $f$

$$f^* : \text{Pic}^0_{\text{red}}(X) \times \text{Pic}^0_{\text{red}}(Y) \to A \times B.$$

We construct the following morphism:

$$\pi : A \times \text{Pic}^0_{\text{red}}(X) \to A \times \text{Pic}^0_{\text{red}}(Y) \cong B \times \text{Pic}^0_{\text{red}}(Y) \to A \times B \times A \times B$$

$$(\varphi, L) \mapsto (\varphi, L^{-1}, F(\varphi, L)) := (\psi, M) \mapsto (\varphi, \psi, f^*(L^{-1}, M))$$
Additionally, define the maps \( \pi_1, \pi_2 \) to be \( \pi \) postcomposed with projection to the first two and last two factors, respectively. Since \( f^* \) is an isogeny, it surjects onto \( A \times B \), hence \( \dim(\text{im}(\pi)) \geq \dim(A) + \dim(B) \).

As a consequence of Rouquier’s isomorphism \([3]\), \((\varphi, \psi)^*\mathcal{E} \cong (L^{-1} \otimes M) \otimes \mathcal{E}\), hence \( t_{(\varphi, \psi)}^* (f^* \mathcal{E}) \cong f^* (L^{-1} \otimes M) \otimes f^* \mathcal{E} \), where \( t_{(\varphi, \psi)} \in \text{Aut}^0(X \times Y) \) denotes translation by \((\varphi, \psi)\). Since \( \pi_1 \) is surjective, each cohomology object \( \mathcal{H}^i(f^* \mathcal{E}) \) is semi-homogeneous. By Mukai \([17]\), the semi-homogeneity of \( \mathcal{H}^i(f^* \mathcal{E}) \) is equivalent to the fact that the following set has dimension exactly \( \dim(A) + \dim(B) \):

\[
\{(x, \alpha) \in (A \times B) \times (\hat{A} \times \hat{B}) \mid t_{(\varphi, \psi)}^* \mathcal{H}^i(f^* \mathcal{E}) \cong \mathcal{H}^i(f^* \mathcal{E}) \otimes \alpha \}.
\]

The image of \( \pi \) is contained in this set, hence \( \dim(\text{im}(\pi)) = \dim(A) + \dim(B) \).

By the above argument, \( \dim(\ker(\pi)) = \dim(A) + \dim(\text{Pic}^0(X)) - \dim(A) - \dim(B) = \dim(\text{Pic}^0(X)) - \dim(B) \). Furthermore, \( \ker \pi \) is contained in the kernel of the morphism \( \text{Pic}^0(X) \to A \) (a component of \( f^* \)). Since \( \text{Pic}^0(X) \to A \) is surjective, \( \dim(\ker(\text{Pic}^0(X) \to A)) = \dim(\text{Pic}^0(X)) - \dim(A) \). So \( \dim(\ker(\pi)) = \dim(\text{Pic}^0(X)) - \dim(B) \leq \dim(\text{Pic}^0(X)) - \dim(A) \), hence \( \dim(A) \leq \dim(B) \).

By symmetry, \( \dim(A) = \dim(B) \).

(ii) Consider the projection \( p : \text{im}(\pi) \to A \times \hat{A} \). By its construction, \( p \) is a surjection. The domain and codomain of \( p \) have equal dimensions, implying that \( p \) is an isogeny. By symmetry, \( \text{im}(\pi) \) and \( B \times \hat{B} \) are isogenous as well. Hence \( A \times \hat{A} \) and \( B \times \hat{B} \) are isogenous, which implies that \( A \) and \( B \) are isogenous.

(iii) In defining \( A \) and \( B \) we constructed the following short exact sequences:

\[
0 \to \ker(\beta) \to \text{Pic}^0_{\text{red}}(X) \xrightarrow{\beta} B \to 0, \quad 0 \to \ker(\alpha) \to \text{Pic}^0_{\text{red}}(Y) \xrightarrow{\alpha} A \to 0.
\]

Since \( A \) and \( B \) are isogenous and \( \ker(\alpha) \) and \( \ker(\beta) \) are isomorphic via \( F \), \( \text{Pic}^0(X) \) and \( \text{Pic}^0(Y) \) are isogenous as well. \( \square \)

\section{Zeta Functions}

\begin{claim}
Let \( X, Y / \overline{\mathbb{F}}_q \) be smooth, projective varieties, where \( \overline{\mathbb{F}}_q \) is a finite field with \( q \) elements. Let \( \ell \) be relatively prime to \( q \). If \( \text{Alb}(X) \) is isogenous to \( \text{Alb}(Y) \), then there is an isomorphism \( H^1(X, \mathbb{Q}_\ell) \cong H^1(Y, \mathbb{Q}_\ell) \). Furthermore, there is some \( n \in \mathbb{N} \) such that this isomorphism is compatible with the action of the geometric \( q^n \)-th power Frobenius morphism.
\end{claim}

\begin{proof}
Let \( n \in \mathbb{N} \). Set \( k = \overline{\mathbb{F}}_q \). The Kummer short exact sequence

\[
0 \to \mu_{\ell^n, X} \to \mathbb{G}_{m, X} \xrightarrow{(\ell^n)} \mathbb{G}_{m, X} \to 0
\]

yields the following long exact cohomology sequence:

\[
0 \to \mu_{\ell^n} \to k^* \xrightarrow{(\ell^n)} k^* \to H^1_{\text{et}}(X, \mu_{\ell^n}) \to \text{Pic}(X) \xrightarrow{(\ell^n)} \text{Pic}(X) \to H^2_{\text{et}}(X, \mu_{\ell^n}) \to \cdots
\]

(7)

\end{proof}
Proof. Let \( \Phi \) equal a Fourier–Mukai functor \( \Phi \) on \( H \) for some \( H \). Hence, (7) implies

\[
H^1_{\ell}(X, \mu_{\ell^n}) \cong \ker(\text{Pic}(X) \stackrel{(\cdot)^{\ell^n}}{\longrightarrow} \text{Pic}(X)) =: \text{Pic}(X)[\ell^n].
\]

By [12, Corollary 9.6.27], the quotient of the torsion of \( \text{Pic}(X) \) by \( \text{Pic}^0(X) \) is finite and its order is bounded, hence the inclusions \( \text{Pic}^0(X)[\ell^n] \subseteq \text{Pic}(X)[\ell^n] \) give rise to an isomorphism of vector spaces

\[
\lim_{\ell^n} \text{Pic}^0(X)[\ell^n] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \lim_{\ell^n} \text{Pic}(X)[\ell^n] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

As \( \lim_{\ell^n} \mu_{\ell^n} \) is identified canonically with \( \mathbb{Z}_\ell(1) \), we have

\[
H^1_{\ell}(X, \mathbb{Q}_\ell)(1) = \lim_{\ell^n} H^1_{\ell}(X, \mu_{\ell^n}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \lim_{\ell^n} \text{Pic}^0(X)[\ell^n] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
\]

By Theorem B, \( \text{Alb}(X) \) and \( \text{Alb}(Y) \) are isogenous, so their duals are isogenous as well: there exists a morphism \( \text{Pic}^0(X)_{\text{red}} \to \text{Pic}^0(Y)_{\text{red}} \) with finite kernel, and hence a map \( \text{Pic}^0(X)_{\text{red}} \to \text{Pic}^0(Y)_{\text{red}} \) with finite kernel \( K \). Since \( K \) is finite, \( \lim_{\ell^n} K[\ell^n] \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \) vanishes and so we have an inclusion \( H^1_{\ell}(X, \mathbb{Q}_\ell)(1) \subseteq H^1_{\ell}(Y, \mathbb{Q}_\ell)(1) \). Since isogeny is a symmetric relation, the inclusion is an isomorphism. Twisting by \( (-1) \) on both sides yields the desired isomorphism

\[
H^1(X, \mathbb{Q}_\ell) \cong H^1(Y, \mathbb{Q}_\ell). \tag{8}
\]

Any given isogeny between abelian varieties over \( \mathbb{F}_q \) descends to an isogeny over \( \mathbb{F}_{q^n} \) for some \( n \in \mathbb{N} \). Hence, there is some \( n \in \mathbb{N} \) such that the isomorphism (8) induced by the isogeny \( \text{Pic}^0(X)_{\text{red}} \to \text{Pic}^0(Y)_{\text{red}} \) is compatible with the action of the geometric \( q^n \)-th power Frobenius morphism. \( \square \)

**Theorem A.** Let \( X, Y/\mathbb{F}_q \) be derived equivalent smooth, projective varieties of dimension 3, where \( \mathbb{F}_q \) is a finite field with \( q \) elements. Then there is some \( n \in \mathbb{N} \) such that for any \( m \in \mathbb{N} \), the numbers of \( \mathbb{F}_{q^m} \)-points in \( X \) and \( Y \) are equal.

**Proof.** Let \( \bar{X}, \bar{Y} \) be the pullbacks of \( X \) and \( Y \) to the algebraic closure \( \bar{\mathbb{F}}_q \) of \( \mathbb{F}_q \). By the Lefschetz fixed-point formula for Weil cohomologies (see Proposition 1.3.6 and Section 4 of Kleiman [11]), to prove this theorem it is sufficient to show that, for some Weil cohomology theory \( H \) defined for \( \bar{X} \) and \( \bar{Y} \), and some \( n \in \mathbb{N} \), the traces of the geometric \( q^n \)-th power Frobenius map \( \varphi^n \) acting on \( H^i(\bar{X}) \) and \( H^i(\bar{Y}) \) are the same for each \( 0 \leq i \leq 6 \).

By Theorem 1.2, the derived equivalence \( D^b(X) \cong D^b(Y) \) is isomorphic to a Fourier–Mukai functor \( \Phi_F := p_{2*}(p_1^*(-) \otimes \mathcal{E}) \) for some \( \mathcal{E} \in D^b(X \times Y) \).

Any Fourier–Mukai transform gives a map on Chow groups: The functor \( \Phi_F \) induces a map

\[
\Psi_{\mathcal{E}}^{\text{CH}} = p_{2*}(v(\mathcal{E}) \cup p_1^*(-)) : \text{CH}(X) \to \text{CH}(Y)
\]
where \( v(\mathcal{E}) := \text{ch}(\mathcal{E}) \cdot \sqrt{\text{td}(X \times Y)} \) is the Mukai vector of \( \mathcal{E} \) (see for instance [9] Definition 5.28). Since \( \Phi \in \mathcal{E} \) where \( v \in \mathcal{E} \) 5.33].

Similarly, the cycle class of \( v(\mathcal{E}) \) inside any Weil cohomology theory \( H \) induces a map \( \Psi^H_{\mathcal{E}} = p_{2*}(\text{cl}(v(\mathcal{E}))) \cup p_{1*}(-) \). This map on cohomology does not necessarily preserve degree, and Tate twists must be accounted for if the map is to be compatible with the action of geometric Frobenius, so care must be taken with the domain and codomain of \( \Psi^H_{\mathcal{E}} \). The map \( \Psi^H_{\mathcal{E}} \) gives the following isomorphisms compatible with the action of geometric Frobenius \( \phi \) between the even and odd Mukai–Hodge structures [7,8], of \( X \) and \( Y \):

\[
\bigoplus_{i=0}^{3} H^{2i}(X)(i) \cong \bigoplus_{i=0}^{3} H^{2i}(Y)(i), \quad (9)
\]

\[
\bigoplus_{i=1}^{3} H^{2i-1}(X)(i) \cong \bigoplus_{i=1}^{3} H^{2i-1}(Y)(i). \quad (10)
\]

The presence of a Tate twist \( (l) \) has the effect of multiplying the eigenvalues of the action of \( \phi^* \) on cohomology by \( \frac{1}{q^l} \). Hence, from \([9,10] \), we have:

\[
\sum_{i=0}^{3} \frac{1}{q^l} \text{Tr}(\phi^*|H^{2i}(X)) = \sum_{i=0}^{3} \frac{1}{q^l} \text{Tr}(\phi^*|H^{2i}(Y)), \quad (11)
\]

\[
\sum_{i=1}^{3} \frac{1}{q^l} \text{Tr}(\phi^*|H^{2i-1}(X)) = \sum_{i=1}^{3} \frac{1}{q^l} \text{Tr}(\phi^*|H^{2i-1}(Y)). \quad (12)
\]

The values \( \text{Tr}(\phi^*|H^i(X)) \) and \( \text{Tr}(\phi^*|H^i(Y)) \) are trivially equal for \( i = 0, 6 \), so \((11)\) reduces to

\[
\frac{1}{q} \text{Tr}(\phi^*|H^2(X)) + \frac{1}{q^2} \text{Tr}(\phi^*|H^4(X)) = \frac{1}{q} \text{Tr}(\phi^*|H^2(Y)) + \frac{1}{q^2} \text{Tr}(\phi^*|H^4(Y)) \quad (13)
\]

Let \( \ell \in \mathbb{Z}^+ \) such that \((q, \ell) = 1 \) Fix \( H \) to be \( \ell \)-adic étale cohomology with coefficients in \( \mathbb{Q}_\ell \) (i.e., \( H(X) := H^\text{et}(\bar{X}, \mathbb{Q}_\ell) \)). Until now we have worked abstractly over any Weil cohomology theory, but now we need to use a result specific to \( \ell \)-adic étale cohomology: By Deligne’s Hard Lefschetz Theorem for \( \ell \)-adic étale cohomology [6 Théorème 4.1.1], we have the following lemma:

**Lemma 3.0.1** ([7 Lemma 4.2]). Let \( V/F_q \) be smooth, projective variety. If the set of eigenvalues (with multiplicity) of \( \phi^* \) acting on \( H^i_{\text{et}}(\bar{V}, \mathbb{Q}_\ell) \), \( 0 \leq i < 3 \), are \( \{\alpha_1, \ldots, \alpha_n\} \), then the set of eigenvalues of \( \phi^* \) acting on \( H^i_{\text{et}}(\bar{V}, \mathbb{Q}_\ell) \) are \( \{q^{d-i-1}\alpha_1, \ldots, q^{d-i-1}\alpha_n\} \).

By Lemma 3.0.1 ([13]) implies that \( \text{Tr}(\phi^*|H^2_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \text{Tr}(\phi^*|H^2_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)) \) and \( \text{Tr}(\phi^*|H^4_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \text{Tr}(\phi^*|H^4_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)). \)
By Lemma 3.0.1, (12) implies that
\[
\frac{2}{q} \text{Tr}(\phi^*|H^1_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) + \frac{1}{q} \text{Tr}(\phi^*|H^2_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \frac{2}{q} \text{Tr}(\phi^*|H^1_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)) + \frac{1}{q} \text{Tr}(\phi^*|H^3_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell))
\]
\[(14)\]

By Claim 3.0.1, there is some \( n \in \mathbb{N} \) such that
\[
\text{Tr}(\phi^*|H^1_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \text{Tr}(\phi^*|H^1_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)).
\]

So, by Lemma 3.0.1, we have \( \text{Tr}(\phi^*|H^0_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \text{Tr}(\phi^*|H^0_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)) \).

Since (9) and (10) are compatible with the action of \( \phi^* \), they are compatible with the action of \( \phi^{n*} \), and hence the above statements comparing the traces of the action of \( \phi^* \) also hold true if \( \phi^* \) is replaced by \( \phi^{n*} \). In particular, by (14), we have
\[
\text{Tr}(\phi^{n*}|H^1_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \text{Tr}(\phi^{n*}|H^1_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)),
\]
and now we have demonstrated that \( \text{Tr}(\phi^{n*}|H^i_{\text{et}}(\bar{X}, \mathbb{Q}_\ell)) = \text{Tr}(\phi^{n*}|H^i_{\text{et}}(\bar{Y}, \mathbb{Q}_\ell)) \) for all \( 0 \leq i \leq 6 \), as required.

References


