Coarse Geometry of Pure Mapping Class Groups of Infinite Graphs

George Domat, Hannah Hoganson, Sanghoon Kwak

Abstract

We discuss the large-scale geometry of pure mapping class groups of locally finite, infinite graphs, motivated from recent work by Algom-Kfir–Bestvina [1] and the work of Mann–Rafi [12] on the large-scale geometry of mapping class groups of infinite-type surfaces. Using the framework of Rosendal for coarse geometry of non-locally compact groups, we classify when the pure mapping class group of a locally finite, infinite graph is globally coarsely bounded (an analog of compact) and when it is locally coarsely bounded (an analog of locally compact).

Our techniques also give lower bounds on the first integral cohomology of the pure mapping class group for some graphs and show that some of these groups have continuous actions on simplicial trees.

1 Introduction

The mapping class group of a surface, Map(S), is the group of orientation-preserving homeomorphisms of the surface up to isotopy. Mapping class groups of finite-type surfaces have been a classical field of study for several decades, and within the past decade there has been newfound interest in the study of mapping class groups of infinite-type, also known as big, surfaces. See [3] for a recent survey on the topic of mapping class groups of infinite-type surfaces.

The study of mapping class groups of finite-type surfaces has also been intimately connected with the study of outer automorphism groups of free groups, Out(F_n). There is a rich dictionary between Map(S) and Out(F_n) when S is of finite-type and this has led to numerous results in both fields. See [6] and [19] for an in-depth survey of Out(F_n) and its connections to mapping class groups.

This connection and the recent interest in mapping class groups of infinite-type surfaces begs the question: What is an appropriate “big” or “infinite-type” analogue of Out(F_n)? Recent work of Algom-Kfir–Bestvina proposes such a definition.

Definition 1.1 ([1, Definition 1.1]). Let Γ be a locally finite, connected graph. The mapping class group of Γ, denoted Map(Γ), is the group of proper homotopy equivalences of Γ up to proper homotopy.

When Γ is finite this definition exactly recovers the group Out(F_n), where n is the rank of Γ. Thus, when Γ is infinite we obtain a version of a “big Out(F_n).” In [1] the authors prove a version of Nielsen realization for these groups.
Mapping class groups are topological groups with a topology coming from the compact-open topology on \( \text{Homeo}(S) \). In the finite-type setting these groups are finitely generated, countable, and discrete. In the infinite-type setting these groups are not compactly generated, but they are Polish and homeomorphic to the irrationals. The same holds for \( \text{Map}(\Gamma) \) when \( \Gamma \) is an infinite graph.

This presents a challenge in the application of the tools of geometric group theory and coarse geometry to these groups. However, recent work of Rosendal [17] has established a framework for studying the coarse geometry of non-locally compact Polish groups. Rosendal introduces a notion of “coarsely bounded” sets which serves as a replacement for compact subsets. Rosendal proves that if a group is generated by a coarsely bounded generating set, then the group admits a well-defined quasi-isometry type with respect to the word metric induced by that generating set, i.e., the identity map with a different coarsely bounded generating set is a quasi-isometry. Within this framework, Mann-Rafi [12] have begun the study of the coarse geometry of mapping class groups of infinite-type surfaces. Under a mild condition on infinite-type surfaces, they give a complete classification of surfaces which have mapping class groups that are coarsely bounded (have trivial geometry), have a coarsely bounded neighborhood about the identity, or have a coarsely bounded generating set (hence have a well-defined quasi-isometry type).

Our goal is to establish similar results as Mann-Rafi in the setting of infinite-type graphs. The pure mapping class group, \( \text{PMap}(\Gamma) \), of a graph \( \Gamma \) is the subgroup of \( \text{Map}(\Gamma) \) which fixes the set of ends of \( \Gamma \) pointwise. We give a complete classification of when these subgroups are coarsely bounded, i.e., have trivial geometry, and when these groups are locally coarsely bounded. We use \( E(\Gamma) \) to represent the end space and \( E_l(\Gamma) \) for the subspace of ends accumulated by loops, each of which are defined in Section 2. See Figure 1 for a summary of our classification of coarse boundedness of \( \text{PMap}(\Gamma) \).

**Theorem A.** Let \( \Gamma \) be a locally finite, infinite graph. Then \( \text{PMap}(\Gamma) \) is coarsely bounded if and only if one of the following holds.

1. \( \Gamma \) has rank zero.
2. \( \Gamma \) has rank one and has one end, i.e., \( \Gamma = \circlearrowright \).
3. \( \Gamma \) has exactly one end accumulated by loops, and \( E(\Gamma) \setminus E_l(\Gamma) \) is discrete.

We note that this classification is not the same classification as the one for surfaces. That is, there are surfaces with non-CB pure mapping class groups for which the “corresponding” graph has a CB pure mapping class group. The simplest example of this is the punctured Loch Ness Monster surface (one end accumulated by genus and one isolated puncture) versus the Hungry Loch Ness Monster graph (one end accumulated by loops and one other end).

We can also extend these results to the full mapping class group in some cases.

**Corollary B.** If \( \Gamma \) has at least two ends accumulated by loops, and has finite end space, then the full group \( \text{Map}(\Gamma) \) is not coarsely bounded. If \( \Gamma \) has a single end accumulated by loops and \( E(\Gamma) \setminus E_l(\Gamma) \) is discrete, then \( \text{Map}(\Gamma) \) is coarsely bounded.

The techniques that we use to prove Theorem A yield some results that are interesting on their own.
Theorem C. Let $\Gamma$ be a locally finite graph with $|E_\ell(\Gamma)| = 1$ such that $E(\Gamma) \setminus E_\ell(\Gamma)$ contains at least one accumulation point. Then $\text{PMap}(\Gamma)$ acts transitively and continuously on a simplicial tree.

The following result is analogous to a result of Aramayona-Patel-Vlamis ([2]) in the surface setting.

Theorem D. Let $\Gamma$ be a locally finite graph with $|E_\ell(\Gamma)| \geq 2$, then $\text{PMap}(\Gamma)$ has a nontrivial homomorphism to $\mathbb{Z}$. Furthermore, if $|E_\ell(\Gamma)| = n$ with $2 \leq n < \infty$, then $\text{rk}(H^1(\text{PMap}(\Gamma);\mathbb{Z})) \geq n - 1$. If $|E_\ell(\Gamma)| = \infty$, then $H^1(\text{PMap}(\Gamma);\mathbb{Z}) = \bigoplus_{i=1}^{\infty} \mathbb{Z}$.

In particular, Theorem D tells us that $\text{PMap}(\Gamma)$ is indicable. The techniques used to prove Theorem A also allow us to give a complete classification of graphs for which $\text{PMap}(\Gamma)$ is locally CB, that is, $\text{PMap}(\Gamma)$ has a CB neighborhood of the identity.

Theorem E. Let $\Gamma$ be a locally finite, infinite graph, then $\text{PMap}(\Gamma)$ is locally coarsely bounded if and only if one of the following holds.

1. The rank of $\Gamma$ is finite, or
2. $\Gamma$ has finitely many ends accumulated by loops and the complement of the core graph of $\Gamma$ has only finitely many components with infinite end space.
In order to begin to get a feel for how these groups function we highlight two key differences between mapping class groups of infinite graphs and surfaces:

Map(Γ) always displaces finite subgraphs of Γ when Γ has infinite rank. To briefly see why, let Γ be a graph with infinite rank, and Δ be any finite subgraph. The complement of Δ in Γ is still infinite rank, so we can find a subgraph Δ′ of the same rank as Δ, but disjoint from Δ. Both Δ and Δ′ deformation retract onto a rose of equal rank. Thus, we can interchange Δ and Δ′ by a proper homotopy equivalence to displace Δ. When Γ has nonzero finite rank r, this is not the case. For example, we can choose a subgraph of rank \( r/2 \), which must be nondisplaceable. Note that the use of nondisplaceable subsurfaces was integral to the arguments of Mann and Rafi.

The restriction of a (proper) homotopy equivalence may not be a homotopy equivalence. For example, consider a rose with two loops labeled by \( a \) and \( b \). The map defined by \( a \mapsto a \) and \( b \mapsto ab \) is a homotopy equivalence, but it maps a loop labeled by \( b \) of rank 1 to a subgraph of rank 2, while homotopy equivalences induce isomorphisms on fundamental groups. This contrasts with the fact that the restriction of a surface homeomorphism to a subsurface is still a homeomorphism. This observation demonstrates that there is no analogous change of coordinates principle for graphs, which is a commonly used technique in the field of mapping class groups of surfaces.

Outline

In Section 2, we give the necessary background. In particular, we review some of the known facts about locally finite graphs of infinite type and their end spaces, as well as some basic facts on coarse structures on groups. Section 3 gives the reader a hands-on introduction to different types of elements that exist in PMap(Γ). Section 4 concerns graphs with one end accumulated by loops. In Section 5 we use the structure of PMap(Γ) for graphs with finite, positive rank, established in [1], to show that almost all of these groups are not CB. In Section 6 we define length functions on a large class of graphs, showing their pure mapping class groups are not CB, and prove Theorem C. We define flux maps in Section 7 and use them to prove that if Γ has at least two ends accumulated by loops then PMap(Γ) is not CB, as well as Theorem D. The results of Sections 4 through 7 together prove Theorem A. Finally, in Section 8 we use the previously established techniques to classify graphs for which PMap(Γ) is locally CB, proving Theorem E. We include an appendix in which we give an explicit proof that ultrametric spaces are 0-hyperbolic.

Acknowledgement

The authors are grateful to Mladen Bestvina for suggesting this project and for reading the first draft and providing thoughtful comments. The authors would also like to thank Ryan Dickmann, Priyam Patel, and Brian Udall, for helpful conversations. In addition, the authors acknowledge support from NSF grants DMS–1906095 (Hoganson), DMS–1905720 (Domat, Kwak), and RTG DMS–1840190 (Domat, Hoganson).
2 Preliminaries

2.1 Infinite Locally Finite Graphs

Let Γ be a locally finite, infinite, connected graph. We often will forget about the actual graph structure of Γ and regard it simply as its underlying topological space. Since Γ can be realized as the direct limit of nested finite graphs, the fundamental group of Γ is free. We define the rank of Γ, denoted by \( \text{rk}(\Gamma) \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \), to be the rank of its
fundamental group. The **space of ends** $E(\Gamma)$ of $\Gamma$ is the inverse limit:

$$E(\Gamma) := \lim_{\leftarrow K \subset \Gamma} \pi_0(\Gamma \setminus K),$$

where the limit runs over the compact subsets $K \subset \Gamma$. Equipped with the usual inverse limit topology, $E(\Gamma)$ becomes a totally disconnected compact metrizable space, so is homeomorphic to a closed subset of the Cantor set. Also, $E(\Gamma)$ compactifies $\Gamma$ in the union $\Gamma \cup E(\Gamma)$ (sometimes referred to as the “end compactification” or Freudenthal compactification). This allows us to define the neighborhood of an end in the ambient graph $\Gamma$ by picking a neighborhood in the union first then taking the intersection with $\Gamma$. Note this is different from the neighborhood of an end in the end space $E(\Gamma)$.

We define the **space of ends accumulated by loops**, denoted by $E_\ell(\Gamma)$, to be the subspace of $E(\Gamma)$ consisting of ends for which every neighborhood in $\Gamma$ is infinite rank. When no confusion will arise we sometimes refer to $(E(\Gamma), E_\ell(\Gamma))$ as simply $(E, E_\ell)$. Any end which is accumulated by ends accumulated by loops is itself accumulated by loops, and thus $E_\ell$ is a closed subspace of $E$. Observe that the rank of $\Gamma$ is infinite if and only if $E_\ell$ is nonempty. We remark that that $E_\ell$ is called **non-$\infty$-stable ends** in [4], and **ends accumulated by genus** in [1].

We will consider the homeomorphism type of the pair $(E, E_\ell)$, where we say $(E, E_\ell) \cong (E', E_\ell')$ when there is a homeomorphism $h : E \to E'$ which restricts to a homeomorphism $h|_{E_\ell} : E_\ell \to E'_\ell$. Now we define the **characteristic triple** of $\Gamma$ as the triple $(\text{rk}(\Gamma), E(\Gamma), E_\ell(\Gamma))$. We say two characteristic triples $(r, E, E_\ell)$ and $(r', E', E'_\ell)$ are **isomorphic** if $r = r'$ and $(E, E_\ell)$ is homeomorphic to $(E', E'_\ell)$.

As Kerékjártó [11] and Richards’s [14] classification theorem for surfaces is the foundation for the study of infinite-type surfaces, we use the following classification for locally finite graphs established by Ayala–Dominguez–Márquez–Quintero:

**Theorem 2.1** ([4, Theorem 2.7], Proper Homotopy Classification of Locally Finite Graphs). An isomorphism $(r, E, E_\ell) \to (r', E', E'_\ell)$ of characteristic pairs of two locally finite, connected graphs $\Gamma$ and $\Gamma'$ extends to a proper homotopy equivalence $\Gamma \to \Gamma'$.

Conversely, by definition of ends, a proper homotopy equivalence yields an isomorphism between characteristic pairs. Therefore, it follows that two graphs $\Gamma$ and $\Gamma'$ have the same proper homotopy type if and only if they have isomorphic characteristic triples of end spaces.

For the remainder of the paper, we assume all of our graphs to be infinite and locally finite. For a graph $\Gamma$ we let $\Gamma_c \subset \Gamma$, the **core graph** of $\Gamma$, be the smallest subgraph that contains all immersed loops.

We will make use of the fact that connected graphs are $K(G, 1)$ spaces for free groups. In particular we will use the following proposition.

**Proposition 2.2** ([10, Proposition 1B.9]). Let $X$ be a connected CW complex and let $Y$ be a $K(G, 1)$. Then every homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y_0)$ is induced by a map $(X, x_0) \to (Y, y_0)$ that is unique up to homotopy fixing $x_0$.

A consequence of this proposition is the following lemma.
Lemma 2.3. Let $\Gamma$ be a connected, finite graph of rank at least 2 and let $x_1, \ldots, x_n$ be distinct points in $\Gamma$. Let $f, g : \Gamma \to \Gamma$ be two homotopy equivalences that both fix each of the $x_i$. If $f_{*i}, g_{*i} : \pi_1(\Gamma, x_i) \to \pi_1(\Gamma, x_i)$ are the same induced maps for all $i$ then $f$ is homotopic to $g$ via a homotopy fixing $x_1, \ldots, x_n$.

Proof. Let $\hat{\Gamma}$ be the graph obtained from $\Gamma$ by identifying all of the $x_i$ to a single point $z \in \hat{\Gamma}$. Note that since $f$ and $g$ fix all of the $x_i$ they induce maps $\hat{f}$ and $\hat{g}$ on the pointed space $(\hat{\Gamma}, z)$. We seek to apply Proposition 2.2 to $\hat{f}$ and $\hat{g}$. Thus we will show that $\hat{f}_{*} = \hat{g}_{*}$ as maps on $\pi_1(\hat{\Gamma}, z)$. Note that any loop in $\pi_1(\hat{\Gamma}, z)$ is obtained via the image of either a loop in $\Gamma$ based at one of the $x_i$ or the image of a path in $\Gamma$ connecting some $x_i, x_j$ with $i \neq j$. We only need to check loops which are the images of paths connecting two distinct basepoints in $\Gamma$.

Let $\alpha$ be any path in $\Gamma$ connecting $x_i$ to $x_j$ with $i \neq j$. Then $\hat{\gamma} = \hat{g}_*(\hat{\alpha})^{-1}\hat{f}_*(\hat{\alpha})$ is trivial. Let $\delta \in \pi_1(\Gamma, x_i)$. Note that by assumption we have (concatenation from right to left)

$$f_{*i}(\delta) = g_{*i}(\delta), \text{ and}$$

$$f_{*j}(\alpha\delta\hat{\alpha}) = g_{*j}(\alpha\delta\hat{\alpha}).$$

These identities descend to

$$\hat{f}_*(\delta) = \hat{g}_*(\delta), \text{ and}$$

$$\hat{f}_*(\hat{\alpha}\hat{\delta}\hat{\alpha}^{-1}) = \hat{g}_*(\hat{\alpha}\hat{\delta}\hat{\alpha}^{-1}),$$

in $\pi_1(\hat{\Gamma}, z)$. Combining these two yields $\hat{\gamma} = \hat{f}_*(\hat{\delta}) = \hat{g}_*(\hat{\delta})$ so that $\hat{\gamma}$ commutes with any loops in the image of $\pi_1(\Gamma, x_i)$. Let $\epsilon_1, \epsilon_2 \in \pi_1(\Gamma, x_i)$ be two independent elements of a free basis. Then $\hat{\gamma}$ commutes with $\epsilon_1$ and $\epsilon_2$ and so $\hat{\gamma} \in \langle \epsilon_1 \rangle \cap \langle \epsilon_2 \rangle = \{1\}$.

Therefore $\hat{f}$ and $\hat{g}$ induce the same map on $\pi_1(\hat{\Gamma}, z)$ so that we can apply Proposition 2.2 to obtain a homotopy between $\hat{f}$ and $\hat{g}$ which fixes $z$. We then lift this homotopy to a homotopy from $f$ to $g$ fixing $x_1, \ldots, x_n$. \qed

2.2 Big Mapping Class Groups of Infinite Graphs

A map is proper if the inverse image of every compact set is compact.

Definition 2.4. For any (infinite) locally finite graph $\Gamma$, we define PHE($\Gamma$) as the group of proper homotopy equivalences (or PHEs). Recall that we say a map $f : \Gamma \to \Gamma$ is a proper homotopy equivalence if $f$ is proper and there exists some $g : \Gamma \to \Gamma$ that is also proper such that $gf$ and $fg$ are properly homotopic to the identity.

We define the mapping class group of $\Gamma$, Map($\Gamma$), as the group of proper homotopy classes of proper homotopy equivalences on $\Gamma$:

$$\text{Map}(\Gamma) = \text{PHE}(\Gamma) / \text{proper homotopy}.$$  

Note that when $\Gamma$ is a finite graph this definition recovers Out($F_n$) where $n = \text{rk}(\Gamma)$. Thus we see that by taking $\Gamma$ to be infinite we obtain a type of “big Out($F_n$)”. 

7
Remark 2.5. We do need to include in the definition of a proper homotopy equivalence that the inverse homotopy equivalence is also proper. That is, there are examples of homotopy equivalences which are proper but whose homotopy inverses are not proper. To illustrate, let $\Gamma$ be the graph with one end which is accumulated by loops. Label each loop by $a_1, a_2, a_3 \cdots$, which we identify with the corresponding elements in $\pi_1(\Gamma)$. Consider a map $f : \Gamma \to \Gamma$, whose induced map $f_*$ on $\pi_1(\Gamma)$ is defined by $a_1 \mapsto a_1$, and $a_i \mapsto a_{i-1}a_i$ for $i \geq 2$.

Since $f_*$ is an isomorphism, $f$ is a homotopy equivalence. Moreover, the inverse homotopy equivalence $g$ of $f$ must induce $f^{-1}_* : \pi_1(\Gamma) \to \pi_1(\Gamma)$, which is defined by $a_1 \mapsto a_1, a_2 \mapsto a_1^{-1}a_2, a_3 \mapsto a_2^{-1}a_1a_3, a_4 \mapsto a_3^{-1}a_1^{-1}a_2a_4 \cdots$.

It can be seen that $f^{-1}$ maps every loop in $\Gamma$ around $a_1$. Therefore, the preimage of $a_1$ under the homotopy equivalence $g$ is not compact, so the inverse homotopy equivalence $g$ of $f$ is not proper.

Definition 2.6. For $\phi \in \text{PHE}(\Gamma)$ we say that $\phi$ is totally supported on $K \subset \Gamma$ if $\phi(K) = K$ and $\phi|_{\Gamma \setminus K} = \text{id}$. We say that $[\phi] \in \text{Map}(\Gamma)$ is totally supported on $K$ if there is a proper homotopy representative of $[\phi]$ that is totally supported on $K$.

Remark 2.7. We will use the term support in its usual way. That is, if $\phi \in \text{PHE}(\Gamma)$, then the support of $\phi$ is the set of $x \in \Gamma$ such that $\phi(x) \neq x$. Note that for homeomorphisms (e.g. of a surface), having support $K$ is equivalent to being totally supported on $K$. This is not true for homotopy equivalences, since they are not necessarily injective.

We would like to endow $\text{Map}(\Gamma)$ with a topology that comes from the topology of $\hat{\Gamma} = \Gamma \cup \bar{E}(\Gamma)$, the end compactification of $\Gamma$.

To do so we put the compact-open topology on the set $\mathcal{C}(\hat{\Gamma})$ of continuous maps on $\hat{\Gamma}$. A proper homotopy equivalence on $\Gamma$ extends to a continuous map on $\hat{\Gamma}$, so we can embed $\text{PHE}(\Gamma)$ into $\mathcal{C}(\hat{\Gamma})$, from which $\text{PHE}(\Gamma)$ inherits the subspace topology. Algom-Kfir and Bestvina show in [1, Corollary 4.3] that the map

$$q : \text{PHE}(\Gamma) \to \text{Map}(\Gamma)$$

is an open map. Thus, $\text{Map}(\Gamma)$ inherits the quotient topology from the topology on $\text{PHE}(\Gamma)$.

With this topology, a neighborhood basis about the identity map in $\text{Map}(\Gamma)$ is given as following. For each finite subgraph $K$ in $\Gamma$, we have an open set $U_K$ labeled by $K$ as:

$$U_K = \{ [f] \in \text{Map}(\Gamma) : \exists f' \in [f] \text{ s.t. } f'|_K = \text{Id}_K, \text{ and } f' \text{ preserves each complementary component of } K \}.$$ 

Algom-Kfir and Bestvina prove that these sets are clopen subgroups [1, Proposition 4.7]. They also show [1, Proposition 4.11] that this topology makes $\text{Map}(\Gamma)$ into a Polish, i.e., separable and metrizable, group, with the underlying space homeomorphic to $\mathbb{Z}^\infty$. 

8
2.3 Pure Mapping Class Groups and Homeomorphism Groups of End Spaces

Recall as in Section 2.1, every proper homotopy equivalence of a graph \( \Gamma \) extends to a homeomorphism of the space of ends \( (E, E_{\ell}) \) of \( \Gamma \). Thus we see that \( \text{Map}(\Gamma) \) acts on the space of ends via homeomorphisms.

**Definition 2.8.** The pure mapping class group, \( P\text{Map}(\Gamma) \), is the kernel of the action of \( \text{Map}(\Gamma) \) on the space of ends of \( \Gamma \).

The pure mapping class group is a closed subgroup of \( \text{Map}(\Gamma) \) and thus is Polish with respect to the subspace topology. These groups fit into the following short exact sequence

\[
1 \rightarrow P\text{Map}(\Gamma) \rightarrow \text{Map}(\Gamma) \rightarrow \text{Homeo}(E, E_{\ell}) \rightarrow 1.
\]

In particular, if \( \Gamma \) is a tree (i.e. of rank 0), then by Theorem 2.1 every self proper homotopy equivalence of \( \Gamma \) arises from the homeomorphism of the end space \( E(\Gamma) \). By definition every element in \( P\text{Map}(\Gamma) \) induces the identity map on \( E(\Gamma) \), so we deduce that \( P\text{Map}(\Gamma) = 1 \). Also, from the short exact sequence we have \( \text{Map}(\Gamma) \cong \text{Homeo}(E(\Gamma)) \).

We record these observations as follows:

**Proposition 2.9** (Graphs of Rank 0 has trivial pure mapping class group). Let \( \Gamma \) be a locally finite, infinite tree. Then \( P\text{Map}(\Gamma) = 1 \) and \( \text{Map}(\Gamma) \cong \text{Homeo}(E(\Gamma)) \).

We equip \( \text{Homeo}(E, E_{\ell}) \) with the compact-open topology. Note that the sets of the form

\[
U_{P} = \{ f \in \text{Homeo}(E, E_{\ell}) | f(P_i) = P_i \text{ for all } i \}
\]

with \( P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_n \) a finite clopen partition of \( E \) give a neighborhood basis about the identity in \( \text{Homeo}(E, E_{\ell}) \).

**Remark 2.10.** Note that the map \( q : \text{Map}(\Gamma) \rightarrow \text{Homeo}(E, E_{\ell}) \) is an open map, so in particular a quotient map. Hence, the quotient topology on \( \text{Homeo}(E, E_{\ell}) \) induced by \( q \) coincides with the compact-open topology on \( \text{Homeo}(E, E_{\ell}) \) generated by the maps fixing partitions of end space. Indeed, consider a basic set

\[
U_K = \{ [f] : f|_K = id, \ f \text{ preserves the complementary components } \}
\]

in \( \text{Map}(\Gamma) \) for some compact set \( K \subset \Gamma \). Then by definition of \( q \), the image \( q(U_L) \) is the set of homeomorphisms on \( E \) which preserves the partition of \( (E, E_{\ell}) \) induced by \( \Gamma \setminus K \), which forms a basic set of the usual topology of \( \text{Homeo}(E, E_{\ell}) \), so \( q \) is an open map.

2.4 Stallings Folds

We will make use of the notion of Stallings folds defined and used in [18] throughout Section 4.

**Definition 2.11.** A morphism of graphs is a continuous map that sends vertices to vertices and edges to edges. An immersion is a locally injective morphism of graphs.
Definition 2.12. Let $\Gamma$ be a graph and $e_1, e_2$ two edges in $\Gamma$ which share a vertex. Form a new graph $\Gamma' = \Gamma/e_1 \sim e_2$. The natural morphism $\Gamma \to \Gamma'$ is a fold.

Folds come in two flavors depending on whether $e_1$ and $e_2$ share only a single vertex or both vertices. Type 1 folds are the folds between edges which share only one vertex and Type 2 folds are the folds between edges which share both vertices. While Type 1 folds are $\pi_1$-isomorphisms, Type 2 folds are only $\pi_1$-surjective and not injective. In fact, a Type 1 fold is a proper homotopy equivalence.

Theorem 2.13 ([18]). Let $f : \Gamma \to \Gamma'$ be a morphism between two finite graphs. Then $f$ can be factored as:

$$
\Gamma = \Gamma_0 \xrightarrow{\phi_1} \Gamma_1 \xrightarrow{\phi_2} \Gamma_2 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_n} \Gamma_n \xrightarrow{h} \Gamma'
$$

where the last map $h$ is an immersion and all the other maps $\{\phi_i\}_{i=1}^n$ are folds.

While Stallings’ theorem is for finite graphs, we will be partially folding our maps in order to obtain an immersion on a finite subgraph. When we are folding proper homotopy equivalences, we do not use Type 2 folds as they are not $\pi_1$-isomorphisms.

2.5 Coarse Structures on Groups

In this section we give definitions and some basic results about coarse structures on spaces, first introduced by Roe [15]. For more details on this section, refer to [17, Chapter 2]. In particular, we do not state the formal definition of coarsely bounded here, only the relevant equivalent definitions as worked out in [17].

Definition 2.14 ([17, Proposition 2.15]). Let $A$ be a subset of a Polish group $G$. Then we say that $A$ is coarsely bounded (CB) if one of the following equivalent conditions are satisfied.

1. (Rosendale’s Criterion) For every neighborhood $V$ of the identity in $G$, there is a finite subset $F$ and some $n \geq 1$ such that $A \subset (FV)^n$.

2. For every continuous action of $G$ on a metric space $X$ and every $x \in X$, $\text{diam}(A \cdot x) < \infty$.

Example 2.15. Any finite group equipped with the discrete topology is coarsely bounded. Similarly, any compact topological group is coarsely bounded.

Thanks to the following observation deduced from Definition 2.14, we can extend the conclusion that $\text{PMap}(\Gamma)$ is CB in itself to the fact that $\text{PMap}(\Gamma)$ is CB in $\text{Map}(\Gamma)$.

Corollary 2.16. Let $G$ be a Polish group and $H$ be a Polish subgroup. If $H$ is coarsely bounded in itself, then $H$ is coarsely bounded in $G$.

Proof. Any continuous action of $G$ on $X$ will restrict to a continuous action of $H$ on $X$, so this follows from (2) of Definition 2.14. $\square$
In the category of coarse spaces isomorphisms are given by coarse equivalences. We will not state the definition here as it can be quite technical, but note that it extends the notion of a quasi-isometry to the larger class of spaces equipped with a coarse structure. Again we refer to [17] for details on this and we collect a few facts below that will be useful to us. All of the proofs of the statements below are either contained in or given by elementary arguments using the definitions in [17, Chapter 2].

**Proposition 2.17** (Coarse boundedness is a coarse equivalence invariant). Let \( \phi : (X, E) \to (Y, F) \) be a coarse equivalence. Then \( X \) is coarsely bounded if and only if \( Y \) is coarsely bounded.

The following variant of a proposition of Rosendal tells us how coarse geometries of groups in a short exact sequence are related to one another.

**Proposition 2.18** (cf. [17, Proposition 4.37]). Suppose \( K \) is a closed normal subgroup of a Polish group \( G \) and assume that \( K \) is coarsely bounded in \( G \). Then the quotient map \( \pi : G \to G/K \) is a coarse equivalence. In particular, \( G \) is CB if and only if \( G/K \) is.

This together with Corollary 2.16, the short exact sequence in Section 2.3 and Remark 2.10 allows us to conclude:

**Corollary 2.19.** Let \( \Gamma \) be a locally finite, infinite graph with end space \( (E, E_\ell) \). If \( \text{PMap}(\Gamma) \) is CB, then \( \text{Map}(\Gamma) \) is coarsely equivalent to \( \text{Homeo}(E, E_\ell) \).

Finally, we verify that the property of being CB is closed under passing to (open) finite index subgroups and extensions.

**Proposition 2.20** (cf. [17, Proposition 5.67]). Let \( G \) be a Polish group and \( H \leq G \) be a finite index open Polish subgroup. Then \( H \) is CB if and only if \( G \) is CB.

**Proof.** Let \( [G : H] = n \) and \( G/H = \{g_1H, \ldots, g_nH\} \). Assume first that \( H \) is CB. By Corollary 2.16, \( H \) is CB in \( G \). Then for any identity neighborhood \( U \) in \( G \), there exist a finite set \( F \subset G \) and \( m > 0 \) such that \( H \subset (FU)^m \). Now, taking \( F' = \{g_1, \ldots, g_n\} \cup F \):

\[
G = \bigcup_{i=1}^n (g_iH) \subset \bigcup_{i=1}^n g_i(FU)^m \subset (F'U)^{m+1},
\]

which implies that \( G \) is CB.

Suppose that \( H \) has a continuous action on a space \( X \), say \( \sigma : H \to \text{Homeo}(X) \). We can extend this to a \( G \)-action \( \rho : G \to \text{Homeo}(X) \) by letting each of the \( g_i \) act trivially on \( X \). We claim that \( \rho \) is also continuous. For any open neighborhood \( U \) of the identity in \( \text{Homeo}(X) \), we have

\[
\rho^{-1}(U) = \bigcup_{i=1}^n g_i\sigma^{-1}(U),
\]

where \( \sigma^{-1}(U) \) is open in \( H \) because \( \sigma \) is a continuous action. Also, as \( H \) is an open subgroup of \( G \), it follows that \( \rho^{-1}(U) \) is a finite union of open sets and thus open in \( G \). Now we see that if \( H \) has an unbounded continuous action on \( X \) then so does \( G \); that is, if \( H \) is not CB then neither is \( G \).
Note that any closed finite index subgroup of a topological group is open, so we have the following.

**Corollary 2.21.** Let $\Gamma$ be a locally finite, infinite graph with a finite end space. Then $\text{PMap}(\Gamma)$ is coarsely bounded if and only if $\text{Map}(\Gamma)$ is coarsely bounded.

### 3 Elements of $\text{PMap}(\Gamma)$

In this section we call attention to different types of elements in $\text{PMap}(\Gamma)$, specifically *word maps*, *loop swaps*, and *loop shifts*. We use word maps and loop swaps in Section 4 and loop shifts in Section 7. In order to define these maps we first introduce some standard forms and notation for graphs. Additionally, we hope this section provides the reader with a better hands-on understanding of the groups $\text{PMap}(\Gamma)$.

Throughout this section we use *loop* in the graph theoretic sense, that is an edge whose initial and terminal vertices are the same. We use *based loop* to refer to an element of a fundamental group.

#### 3.1 Standard Forms of Graphs

Standard models for locally finite, infinite graphs were introduced in [4] and are used in [1]. Our arguments do not require graphs be a standard model. In particular we do not require the underlying tree to be binary, and sometimes we introduce artificial vertices of valence two. We will instead use graphs in *standard form*, defined as follows.

**Definition 3.1.** A locally finite graph, $\Gamma$, is in *standard form* if $\Gamma$ is a tree with loops attached at some of the vertices. We endow $\Gamma$ with the path metric which assigns each edge length 1.

Standard form is strictly weaker than standard model, that is, every standard model is a graph in standard form. Note that for a graph in standard form, the underlying tree is a spanning tree and it is unique. Another benefit of standard form is that it allows us to talk about the fundamental group in a very concrete way. Specifically, we can orient and enumerate the loops $\{\alpha_i\}$ for $I \subseteq \mathbb{N}$, and call the vertices to which they are incident $v_i$. For a basepoint $x_0$ in the tree, let $a_i$ be the based loop resulting from pre- and post-concatenating each loop $\alpha_i$ with the geodesic from $x_0$ to $v_i$. Then the collection $\{a_i\}_{i \in I}$ forms a basis for $\pi_1(\Gamma, x_0)$.

Section 4 focuses on graphs whose end spaces contain one end accumulated by loops. The simplest such graph is the Loch Ness Monster graph, which is the graph with exactly one end and that end is accumulated by loops. We named this graph in analogy with the Loch Ness Monster surface, which has a single end that is accumulated by genus. The next simplest class of graphs are those whose end space contains finitely many points, one of which is accumulated by loops. We call these graphs *Hungry Loch Ness Monsters*, and include Figure 2 to demonstrate why.

Let $\Gamma_N$ for $N \in \mathbb{Z}_{\geq 0}$ refer to the locally finite graph with $|E(\Gamma_N)| = 1$ and $|E(\Gamma_N)| = N + 1$. Let $\Gamma_\infty$ refer to the locally finite graph with $|E(\Gamma_\infty)| = 1$ and for which $E(\Gamma_\infty) \setminus E(\Gamma_\infty)$ has no accumulation points. We call the graph $\Gamma_\infty$ the *Millipede Monster graph* (see Figure 3) and following the above, $\Gamma_0$ is the Loch Ness Monster graph and $\Gamma_N$ for
Figure 2: A Hungry Loch Ness Monster graph, with two tongues.

$N \in \mathbb{N}$ are the Hungry Loch Ness Monster graphs. We also note that the core graph of any graph $\Gamma$ with $|E_\ell(\Gamma)| = 1$ is properly homotopic to $\Gamma_0$.

We use the vertex labeling $\{v_i\}, \{w_i\}$ from Figure 3 to introduce the following notation for any graph $\Gamma$ in standard form with $|E_\ell(\Gamma)| = 1$. Note that if $\Gamma \neq \Gamma_N$ for some $N$ then we are only labeling $\Gamma_c$ via the labeling on $\Gamma_0$. The notation $(v_i, v_j)$ is used to designate the geodesic in $\Gamma_c$ connecting $v_i$ and $v_j$. The notation $[v_i, v_j]$ designates the subgraph consisting of $(v_i, v_j)$ together with the loops $\alpha_k$ for all $k$ between $i$ and $j$, inclusive of $i$ and $j$. We can replace $v_i$ with $w_i$ and still use parenthesis to indicate the line segment in the core graph and closed brackets to include any loops which are incident to the line segment. We use $A_{i,j}$ to denote the free factor of $\pi_1(\Gamma, x_0)$ (for any basepoint) coming from $[v_i, v_j]$, that is $A_{i,j} = \langle \alpha_k \rangle_{k=1}^n$.

### 3.2 Loop Swaps

We will make ample use of a specific class of maps which swap sets of loops. We define them explicitly on the graphs $\Gamma_N$ with $N \in \mathbb{Z}_{\geq 0}$ with the path metric.

**Definition 3.2.** Given a triple $(n, m_1, m_2) \in \mathbb{N}^3$ satisfying $m_2 - m_1 \geq n$ we define the **loop swap** determined by the triple $(n, m_1, m_2)$, denoted by $L(n, m_1, m_2)$, to be the map which swaps the $n$ loops starting at $v_{m_1}$ with the $n$ loops starting at $v_{m_2}$. That is, $L(n, m_1, m_2)$ is the map that interchanges $[v_{m_1}, v_{m_1+n-1}]$ and $[v_{m_2}, v_{m_2+n-1}]$ isometrically, stretches the following edges to the following paths,

$$
(w_{m_1-1}, v_{m_1}) \mapsto (w_{m_1-1}, v_{m_2}) \\
(v_{m_1+n-1}, w_{m_1+n-1}) \mapsto (v_{m_2+n-1}, w_{m_2+n-1}) \\
(w_{m_2-1}, v_{m_2}) \mapsto (w_{m_2-1}, v_{m_1}) \\
(v_{m_2+n-1}, w_{m_2+n-1}) \mapsto (v_{m_1+n-1}, w_{m_2+n-1})
$$

and is the identity everywhere else. If the graph is the Loch Ness Monster and $m_1 = 1$ then $L(n, 1, m_2)$ is defined in the same way without the first stretch map, as there is no edge $(w_0, v_1)$ to stretch.
Figure 3: Standard forms for the Loch Ness Monster graph ($\Gamma_0$), the Hungry Loch Ness Monster graph with $N$ rays attached ($\Gamma_N$), and the Millipede Monster graph ($\Gamma_\infty$).
Remark 3.3. (1) Loop swaps are always proper homotopy equivalences.

(2) \( L(n, m_1, m_2)^2 \) is properly homotopic to the identity, so the corresponding mapping classes of loop swaps have order two.

(3) The vertices \( w_{m_1-1}, w_{m_1+n-1}, w_{m_2-1}, \) and \( w_{m_2+n-1} \) are all fixed points of \( L(n, m_1, m_2) \).

(4) If \( K = [v_{m_1}, v_{m_1+n-1}] \), then \( L(n, m_1, m_2)(K) \cap K \) is empty.

(5) If \( x_0 \in \Gamma \) is fixed by some \( L(n, m_1, m_2) \) then the induced map on \( \pi_1(\Gamma, x_0) \) is given by:

\[
L(n, m_1, m_2) : a_i \mapsto \begin{cases} 
  a_{m_2+(i-m_1)} & \text{if } m_1 \leq i < m_1 + n, \\
  a_{m_1+(i-m_2)} & \text{if } m_2 \leq i < m_2 + n, \\
  a_i & \text{otherwise.}
\end{cases}
\]

With some care one can also define loop swaps on any graph \( \Gamma \) with large enough rank, but we do not need them for the arguments in this paper.

### 3.3 Word Maps

Consider a graph \( \Gamma \) in standard form, specifically we want that \( \Gamma \) is a tree with loops attached at some of the vertices. For word maps to exist we need \( \text{rk}(\Gamma) \neq 0 \). As in Section 3.1, orient and enumerate the loops \( \{\alpha_i\}_{i \in \mathcal{A}} \) for \( \mathcal{A} \subseteq \mathbb{N} \), and call the vertices at which they are based \( v_i \). Pick a base point \( x \in \Gamma \). Now identify \( \pi_1(\Gamma, x) \) with \( \langle a_i \rangle_{i \in \mathcal{A}} \) where \( a_i \) is the based loop which traverses \( \alpha_i \). Let \( w \in \pi_1(\Gamma, x) \) and write \( w = a_{i_1}^{\pm_1} \cdots a_{i_2}^{\pm_2} \). The **word path** associated to \( w \) in \( \Gamma \) is the path which begins at \( v_{i_1} \) and traverses \( \alpha_{i_1} \) in the forward or backward orientation according to the sign of \( a_{i_1}^{\pm_1} \) in \( w \), then travels along the tree to \( v_{i_2} \) and traverses \( \alpha_{i_2} \) according to the sign and continues in this manner, ending at the base of \( v_{i_m} \).
Definition 3.4. Let $I \subset e$ be a connected subset of an edge $e \in \Gamma$ and identify $I$ with the interval $[0, 1]$, and further subdivide $I$ into $[0, \frac{1}{4}] \cup \left[ \frac{1}{4}, \frac{3}{4} \right] \cup \left[ \frac{3}{4}, 1 \right]$. Call the endpoints $I_0$ and $I_1$. We can define a word map, $\phi_{(w,I)}$, supported on $I$ as follows: the interval $\left[ \frac{1}{4}, \frac{3}{4} \right]$ is mapped to the word path associated to $w$, the interval $[0, \frac{1}{4}]$ is mapped to the path in the tree from $I_0$ to $v_i$, and the interval $[\frac{3}{4}, 1]$ is mapped to the path in the tree from $v_{im}$ to $I_1$. The word map is the identity on the rest of $\Gamma$, see Figure 5.

To see that $\phi_{(w,I)}$ is proper, note that $\phi_{(w,I)}$ is compactly supported on $I$, and $\phi_{(w,I)}(I)$ is also compact. To see that $\phi_{(w,I)}$ is a non-trivial element of $\text{Map}(\Gamma)$ observe that it induces a non-trivial automorphism of at least one of $\pi_1(\Gamma, I_0)$ and $\pi_1(\Gamma, I_1)$. In fact, the induced map will be conjugation on a free factor of $\pi_1(\Gamma, I_i)$.

In particular, if $\Gamma = \Gamma_N$ and $I \subset (v_j, v_{j+1})$, then the induced map on $\pi_1(\Gamma, v_j)$ is the partial conjugation

$$\left(\phi_{(w,I)}\right)_\ast(a_i) = \begin{cases} a_i & \text{if } i \leq j \\ w^{-1}a_iw & \text{if } i > j \end{cases}.$$  

The composition of two word maps on the same edge is again a word map. However, the composition rule $\phi_{(w_1,I)} \circ \phi_{(w_2,I)} = \phi_{(w_1w_2,I)}$ holds only if the word path corresponding to $w_2$ is disjoint from $I$. Consider post-composing $\phi_{(a_1a_4a_3,I)}$ from Figure 5 with another word map supported on $I$ to see why this is true. For the same reason, word maps supported on disjoint intervals do not necessarily commute. Instead we introduce the notion of a multi-word map, which we think of as doing multiple word maps simultaneously. It is defined as follows for disjoint $I, J \subset \Gamma$:

$$\left(\phi_{(w_1,I)\cup(w_2,J)}\right)(x) = \begin{cases} \phi_{(w_1,I)}(x) & \text{if } x \in I \\ \phi_{(w_2,J)}(x) & \text{if } x \in J \\ x & \text{else} \end{cases}.$$  

16
Word maps show up more often than one might initially expect. To see this, let $e$ be an edge between two vertices, $v_0$ and $v_1$, in a locally finite graph $\Gamma$. If $\psi \in \text{PHE}(\Gamma)$ fixes each of $v_0$ and $v_1$, then $\psi|_e$ must be (properly) homotopic to a word map $\phi(w, I)$ with $I \subset e$. Note also that any proper homotopy equivalence supported on a ray, $R$, can be realized as a word map with $I \subset R$. Further, by homotoping to fix vertices, any compactly supported proper homotopy equivalence can be realized as a multi-word map supported on the interior of finitely many edges.

Manipulation of word maps is an essential tool in Section 4, so we establish the subsequent lemmas. We say that $\{x_i\} \subseteq \Gamma$ span an $n$-pod if one of the connected components of $\Gamma - \bigcup \{x_i\}$ is a finite $n$-pod graph. Lemma 3.5 tells us how to homotope the support of a word map past a vertex of $\Gamma$, as shown in Figure 6.

![Figure 6: How to “push” a word map past a vertex $v$.](image)

**Lemma 3.5.** Let $\{x_i\}_{i=0}^{n-1}$ span a finite $n$-pod in $\Gamma$ and call the central vertex $v$. Let $I_0$ be an interval in $(v, x_0)$ and $I_i$ be an interval in $(x_i, v)$ for $1 \leq i \leq n-1$. Then for any $w \in \pi_1(\Gamma)$ we have,

$$\phi(w, I_0) \simeq \phi(w, I_1) \cup \phi(w, I_2) \cup \cdots \cup \phi(w, I_{n-1})$$

and the homotopy is proper.

**Proof.** First assume $n = 3$, as in Figure 6. Observe that $\phi(w, I_0)$ and $\phi(w, I_1) \cup \phi(w, I_2)$ induce the same maps on $\pi_1(\Gamma, x_i)$ for each of $x_0, x_1$ and $x_2$. Thus, Lemma 2.3 tells us $\phi(w, I_0) \simeq \phi(w, I_1) \cup \phi(w, I_2)$. Because the support of the homotopy is the tripod, it is proper. Now to see that the lemma holds for larger $n$, one can either blow up a valence $n$ vertex into a sequence of valence 3 vertices, or apply Lemma 2.3 directly with $n$ basepoints. \[\Box\]

**Lemma 3.6.** Let $\Gamma = \Gamma_N$ with $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. If $u \in \text{PMap}(\Gamma)$ is compactly supported, then $u$ can be homotoped to have support on the loops and rays of $\Gamma$. That is, its support will be disjoint from the spanning tree of the core graph $\Gamma_c$. Furthermore, we can write $u = u_{R_m} \circ \cdots \circ u_{R_i} \circ u_\ell$ where $u_\ell$ is supported on the loops, $u_{R_i}$ is supported on the $i$th ray, and $m \leq N$ is finite.

**Proof.** First homotope $u$ to fix the vertices, because $u$ is compactly supported, this homotopy is proper. Let $\Delta$ be a compact subgraph of $\Gamma_N$ on which $u$ is totally supported. If necessary, expand $\Delta$ so that it is connected, contains $[v_1, v_n]$ for some $n$, and contains a segment of every ray incident to $[v_1, v_n]$. Note that if $\Gamma = \Gamma_\infty$, then $\Delta$ will intersect at most finitely many of the rays, $R_1, \ldots, R_m$; otherwise, $m = N.$
Now because $u$ fixes the vertices it must be acting as simultaneous word maps on each edge of $\Delta$. Applying Lemma 3.5 we can push these word maps off of the spanning tree and onto the ray segments and loops of $\Delta$. Thus, we can choose $I_i \subset R_i$ and $J_i \subset \alpha_i$ and write

\[ u = \phi(w_1, I_1) \cup \cdots \cup \phi(w_n, I_n) \]

for some $w_i, \bar{w}_i \in A_{1,n}$. Let $u_{R_i} = \phi(w_i, I_i)$ and $u_\ell = \phi(\bar{w}_i, J_i) \cup \cdots \cup \phi(\bar{w}_n, J_n)$ . To see that $u$ splits as the desired composition, note that each interval $I_i \subset R_i$ is disjoint from every word path $w_j$ and $\bar{w}_j$.

In particular, Lemma 3.6 tells us that if $u \in \text{PMap}(\Gamma_0)$ is compactly supported, then there is a homotopy representative of $u$ which is supported only on the loops.

For any word map supported outside of the core graph of $\Gamma$, we can also compute the effect of conjugating the map by a mapping class totally supported on the core graph of $\Gamma$.

**Lemma 3.7.** Suppose $\Gamma$ is a locally finite, infinite graph with nonempty complement of the core graph. Let $\phi(w, I) \in \text{PMap}(\Gamma)$ be a word map supported outside the core graph $\Gamma_c$, and $\psi \in \text{PMap}(\Gamma)$ be totally supported on a compact subgraph of the core graph. Then

\[ \psi \circ \phi(w, I) \circ \psi^{-1} = \phi(\psi_*(w), I). \]

**Proof.** Pick a compact set $K \subset \Gamma$ so that both $\phi(w, I)$ and $\psi$ are totally supported on $K$.

In particular, we have $I \subset K$. Since $I$ is outside the core graph, we may trim $K$ a bit such that the end point $I_0$ of $I$ lies on the boundary $\partial K := K \cap \overline{\Gamma \setminus K}$. Then label the boundary points $\partial K = \{x_0 := I_0, x_1, x_2, \ldots, x_n\}$, which will serve as base points of the fundamental group of $\Gamma$.

Observe that both $\psi \circ \phi(w, I) \circ \psi^{-1}$ and $\phi(\psi_*(w), I)$ fix each $x_i \in \partial K$, by choice of $K$. Hence, by Lemma 2.3, it suffices to show that $\psi \circ \phi(w, I) \circ \psi^{-1}$ and $\phi(\psi_*(w), I)$ induce the same automorphisms on $\pi_1(\Gamma, x_i)$ for each $i \in \{0, 1, \ldots, n\}$ to conclude the proof.

For each such $i \neq 0$, and any word $\bar{w} \in \pi_1(\Gamma)$, observe that $(\phi(w, I))_* : \pi_1(\Gamma, x_i) \to \pi_1(\Gamma, x_i)$ is just the identity map, as the geodesic from $x_i$ to the core graph is disjoint from $I$, and $I$ is also disjoint from the core graph. Therefore, on $\pi_1(\Gamma, x_i)$:

\[ (\psi \circ \phi(w, I) \circ \psi^{-1})_* = \psi_* \circ \psi_*^{-1} = \text{Id} = \phi(\psi_*(w), I). \]

Now assume $i = 0$. Recall the observation made earlier in this subsection that a word map induces the conjugation by the word associated to the map. Namely, we have for each generator $a_j \in \pi_1(\Gamma, x_0)$:

\[ (\psi \circ \phi(w, I) \circ \psi^{-1})_*(a_j) = \psi_*(w^{-1} \psi_*^{-1}(a_j)w) = \psi_*(w)^{-1} a_j \psi_*(w) = (\phi(\psi_*(w), I))_*(a_j), \]

which concludes the proof.

---

### 3.4 Loop Shifts

Loop shifts are the graph equivalent of handle shifts on surfaces, which were introduced by Patel and Vlamis in [13]. Let $\Lambda$ be the graph in standard form with exactly two ends,
each of which are accumulated by loops, as in Figure 7. The simplest example of a loop shift is the right translation of \( \Lambda \) by one loop. The graph \( \Lambda \) also admits loop shifts which omit some loops from their support. It takes more care to define these loops shifts, as well as loop shifts on more general graphs.

Let \( \Gamma \) be any graph in standard form with at least two ends accumulated by loops, endowed with the path metric. Pick two distinct ends \( e_-, e_+ \in E_\ell(\Gamma) \). There is a unique oriented line, \( L \), in the spanning tree of \( \Gamma \) between these ends, and there are countably many loops of \( \Gamma \) incident to \( L \). For any infinite subset of these loops, \( \mathcal{A} \), which accumulates onto each of \( e_- \) and \( e_+ \), there is a natural well-ordered bijection between the loops of \( \mathcal{A} \) and \( \mathbb{Z} \). So, enumerate the loops \( \mathcal{A} = \{ \alpha_i \}_{i \in \mathbb{Z}} \), and call the vertices to which they are incident \( v_i \). We use parenthesis to denote the geodesic between two points in \( \Gamma \), as in Section 3.1.

**Definition 3.8.** The loop shift \( h \in \text{PMap}(\Gamma) \), associated to \( \mathcal{A} \), is constructed piecewise as follows.

1. \( h|_{L \cup \mathcal{A}} = \begin{cases} v_i & \mapsto v_{i+1} \\ \alpha_i & \mapsto \alpha_{i+1} \\ (v_i, v_{i+1}) & \mapsto (v_{i+1}, v_{i+2}) \end{cases} \)

2. Choose \( \epsilon < \frac{1}{2} \), and let \( N_\epsilon(L \cup \mathcal{A}) \) be the epsilon neighborhood of \( L \cup \mathcal{A} \); define \( h|_{\Gamma \setminus N_\epsilon(L \cup \mathcal{A})} = \text{id} \).

3. Now \( N_\epsilon(L \cup \mathcal{A}) \setminus (L \cup \mathcal{A}) \) is a disjoint union of epsilon intervals incident to \( L \). For each such interval \( I = (x, y) \) with \( y \in L \), we define \( h((x, y)) = (x, h(y)) \), as \( h(y) \) was defined in the first step.

We also refer to the two ends \( e_- \) and \( e_+ \) as \( h_- \) and \( h_+ \).

Note that different choices of \( \epsilon < \frac{1}{2} \) result in properly homotopic loop shifts, so they are the same element of \( \text{Map}(\Gamma) \).

**4 Graphs of Infinite Rank with CB Pure Mapping Class Groups**

In this section we prove the following.
**Theorem 4.1.** Let $\Gamma$ be a locally finite graph with exactly one end accumulated by loops. Then if $E(\Gamma) \setminus E_\ell(\Gamma)$ does not contain an accumulation point, $\PMap(\Gamma)$ is CB.

We will prove this theorem in steps, beginning with the simplest such graphs and increasing in complexity at each step. First, in Section 4.2, the Loch Ness Monster graph will be treated, then in Section 4.3 we consider the Hungry Loch Ness Monster graphs. Finally, when $\Gamma$ has infinite end space, the only case where $E(\Gamma) \setminus E_\ell(\Gamma)$ has no accumulation point is when $E(\Gamma)$ has countably many ends with the unique accumulation point of $E(\Gamma)$ coinciding with the point in $E_\ell(\Gamma)$. This graph, unique up to proper homotopy equivalence, is the Millipede Monster graph.

Each proof in this section emulates the proof that surfaces with self-similar end space have coarsely bounded mapping class groups, [12, Proposition 3.1] due to Mann and Rafi. The same method can be used to see that $S_\infty$, the group of bijections of a countable set (with possibly infinite support), is coarsely bounded. We present this proof here as a warm up, although this fact can also be found in [16, Example 9.14]. There the authors actually show that $S_\infty$ is Roelcke precompact, a stronger condition.

**Proposition 4.2.** The group of bijections on a countable set, $S_\infty$, is coarsely bounded.

*Proof.* We fix $\mathbb{N}$ as our countable set. We first note that the topology on $S_\infty$ has a neighborhood basis about the identity given by sets of the form $U_K = \{\sigma | \sigma(i) = i \text{ for all } i \in K\}$ where $K$ is a finite subset of $\mathbb{N}$. We will use Rosendal’s criterion to see that $S_\infty$ is CB.

Let $\mathcal{V}$ be a neighborhood of the identity. We can find a basis element $U_K \subset \mathcal{V}$ where $K$ has the form $K = \{1, \ldots, n\}$. Let

$$f = (1, n + 1)(2, n + 2) \cdots (n, 2n),$$

and set $\mathcal{F} = \{f\}$. We will show that $S_\infty = (\mathcal{F}U_K)^3$, which implies $S_\infty = (\mathcal{F}\mathcal{V})^3$.

For any $\phi \in S_\infty$, let $u$ be a finite permutation such that

$$u(\phi(i)) = i, \quad \text{for all } 1 \leq i \leq n.$$

Then $u\phi \in U_K$. Let $m = \max\{\supp(u) \cup \{2n\}\}$ and define

$$g = (n + 1, m + 1)(n + 2, m + 2) \cdots (2n, m + n).$$

By our choice of $m$, we have that $g \in U_K$. We can also check that $fguf \in U_K$. Indeed,

$$fguf(i) = fgu(i + n + m) = fg(i + n + m) = i,$$

for all $i \in \{1, \ldots, n\}$. Rearranging these inclusions, we conclude that $\phi \in (\mathcal{F}U_K)^3$. 

\[\square\]

### 4.1 Notation and a few Lemmas

Throughout this section we consider the graphs $\Gamma_N$ with $N \in \mathbb{N} \cup \{0, \infty\}$. We will refer to the graphs and labels from Figure 3 and use the notation established in Section 3. Each time we are given a compact set $K \subset \Gamma$ we first expand it to contain $[v_1, v_n]$ and we set $f$ to be the loop swap $\mathcal{L}(n, 1, n + 1)$ on the appropriate graph $\Gamma_N$. 

20
Remark 4.3. For any set $K = [v_1, v_n]$, any element in $\mathcal{U}_K$ must be the identity on each tree component of $\Gamma_N \setminus K$. So, expanding $K$ to include any of portion of the rays that it disconnects from $\Gamma$, does not change the set $\mathcal{U}_K$. This also simplifies the definition of the sets $\mathcal{U}_K$. Namely, for such $\Gamma$ and $K$, any $\phi \in \text{PMap}(\Gamma)$ is contained in $\mathcal{U}_K$ if and only if $\phi$ is homotopic to some $\phi'$ such that $\phi'|_K = \text{Id}_K$ and $\phi'(\Gamma \setminus K) \cap K = \emptyset$.

We next prove two lemmas that will be used throughout this section. Note that these lemmas hold for any general graph with only a single end accumulated by loops, not just the examples mentioned above. The first lemma shows that we can “finitely approximate” an inverse, as in the warm-up above.

Lemma 4.4 (Folding to Approximate). Let $\Gamma$ be a graph with $|E_\ell(\Gamma)| = 1$ and $K = [v_1, v_n] \subset \Gamma$. Let $\phi \in \text{PMap}(\Gamma)$, then there exists some $u \in \text{PMap}(\Gamma)$ and $K' \subset \Gamma$ with $K \subset K'$ such that the following holds.

(a) $u$ is totally supported on $K'$, i.e., $u(K') = K'$ and $u|_{\Gamma \setminus K'} = \text{Id}_{\Gamma \setminus K'}$,

(b) $u\phi \in \mathcal{U}_K$, i.e., $u\phi$ is homotopic to a map $\phi'$ such that

(i) $\phi'|_K = \text{Id}_K$,

(ii) $\phi'(\Gamma \setminus K) \cap K = \emptyset$.

Proof. Let $\psi$ be a proper homotopy inverse of $\phi$. We will make use of Stallings folds to see that we can approximate $\psi$ on $K$. After a proper homotopy, we may assume $\psi$ maps vertices to vertices. We next modify $\psi$ via a proper homotopy so that $\psi^{-1}(x)$ is a totally disconnected set for every $x \in \Gamma$. To do so, subdivide every edge that is collapsed and modify $\psi$ (via a proper homotopy) to send this new midpoint to a vertex adjacent to the original image of the edge. Note that the two endpoints of the edge will still have the same image, but the edge itself will traverse over an entire edge in the target twice. If there is a subinterval of an edge that is collapsed (as opposed to an entire edge) we can perform the same proper homotopy on the subinterval. Note that this modification does not change the fact that $\psi$ maps vertices to vertices.

The complete pre-images of vertices under $\psi$ gives us a subdivision $\Gamma_S$ of $\Gamma$. We will show that we can factor $\psi$ through finitely many folds so that the resulting map $\psi$ is injective on $K$.

Claim 1. If $\psi(v) = \psi(w)$ for some vertices $v, w$ of $\Gamma_S$, then $v$ and $w$ can be identified after finitely many folds.

Proof of Claim 1. Consider a path $\gamma$ from $v$ to $w$ in $\Gamma_S$. Since $\psi(v) = \psi(w)$, the image $\psi(\gamma)$ is a loop in $\Gamma_S$. Since $(\psi)_*$ is a $\pi_1$-isomorphism, there exists a loop $\alpha$ based at $v$ in $\Gamma_S$ such that $(\psi)_*([\alpha]) = [\psi(\gamma)]$. Thus $[\psi(\alpha^{-1} * \gamma)] = 1$, where here $\alpha^{-1} * \gamma$ is the concatenated path from $v$ to $w$, first following $\alpha^{-1}$ and then $\gamma$. The existence of a nullhomotopy of the loop $\psi(\alpha^{-1} * \gamma)$ in $\Gamma_S$ suggests that we can fold the path $\gamma$ to wrap around $\alpha$ to identify $w$ with $v$. \qed

Claim 2. If $\psi(e) = \psi(e')$ for distinct edges $e, e'$ of $\Gamma_S$, then $e$ and $e'$ can be identified after finitely many folds.
Proof of Claim 2. By the previous claim, we may assume the two edges $e$, $e'$ share a vertex in $\Gamma_S$. Since $\psi$ induces a $\pi_1$-isomorphism, the two edges cannot share both vertices, otherwise $\psi$ collapses the nontrivial loop bounded by $e, e'$. Hence, there can be only one vertex that $e, e'$ share, from which we can perform a Type 1 fold to identify $e = e'$.

Apply these two claims to every edge and vertex in $K$ as well as those that map into $K$. Let $F : \Gamma_S \to \Gamma_S'$ be the product of all of these folds, where $\Gamma_S'$ is the resulting graph. Note that since each fold is only defined on two edges, $\Gamma_S$ contains some connected compact subgraph $K'$ containing $K$ such that $F|_{\Gamma_S \setminus K'}$ is the identity map. Now $\psi$ factors as $\psi = h \circ F$ where $h : \Gamma_S' \to \Gamma_S$ is injective on $F(K)$. Since $F$ includes all the folds which map into $K$, we have that $h(\Gamma_S' \setminus F(K')) \cap K = \emptyset$. Our original map $\psi$ was a (surjective) proper homotopy equivalence, so $h$ is also surjective onto $K$. Thus, $h$ restricts to a graph isomorphism from $F(K)$ to $K$.

Next, we define a homotopy equivalence $\sigma : \Gamma_S' \to \Gamma_S$. First define $\sigma$ on $F(K) \subset \Gamma_S'$ to be $h|_{F(K)}$, which is a graph isomorphism onto $K$. Now $F(K') \setminus F(K)$ is some finite graph of fixed rank $m$. Define $\sigma$ on this set to be any homotopy equivalence onto the “next” subgraph of rank $m$, to the right of $K$, in $\Gamma_S$. That is, if $K = [v_1, v_n]$ then $\sigma$ is a homotopy equivalence to $[v_{n+1}, v_{n+m}]$. Finally, define $\sigma$ on $\Gamma_S' \setminus F(K')$ to be the identity map.

Let $u = \sigma \circ F$. We have the following diagram which commutes up to homotopy.

We can now check that $u$ has the desired properties. By construction, $u|_{\Gamma_S \setminus K'} = \text{Id}_{\Gamma_S \setminus K'}$ and $u(K') = K'$, so (a) is satisfied. Now $u \phi \simeq \sigma h$, where $\bar{h}$ is a homotopy inverse of $h$. Since $h$ does not map anything from $\Gamma_S' \setminus F(K')$ into $K$, we can pick $\bar{h}$ to map nothing from $\Gamma_S \setminus K$ into $F(K')$. Because $h|_{F(K)}$ is a graph isomorphism onto $K$, we can choose $\bar{h}$ to be the inverse graph isomorphism on $K$. Now we see that $\sigma \circ \bar{h}$ is exactly the identity on $K$, so that (b-i) is satisfied. Finally, $\bar{h}$ maps $\Gamma_S \setminus K$ into $\Gamma_S' \setminus F(K')$, and $\sigma$ maps this set back into $\Gamma_S \setminus K' \subset \Gamma_S \setminus K$, so that (b-ii) is also satisfied.

The previous lemma says that we can realize $\text{PMap}(\Gamma)$ as the closure of the compactly supported mapping classes of $\Gamma$. That is, let $\text{PMap}_c(\Gamma)$ be the subgroup of $\text{PMap}(\Gamma)$ consisting of all proper homotopy classes of proper homotopy equivalences which have a representative with compact support, then the previous lemma gives the following.

**Corollary 4.5.** Let $\Gamma$ be a graph with $|E_0(\Gamma)| = 1$. Then $\text{PMap}_c(\Gamma) = \text{PMap}(\Gamma)$.
Proof. Let $\phi \in \text{PMap}(\Gamma)$. Take a compact exhaustion $\{K_i\}$ of $\Gamma$ with $K_i$ consisting of the subgraph $[v_1, v_i]$ together with larger and larger portions of the trees extending to the other ends. We thus obtain via Lemma 4.4 a sequence $\{u_i\}$ of elements in $\text{PMap}_c(\Gamma)$ which converges to $\phi$ in $\text{PMap}(\Gamma)$.

Recall that we use the term loop to mean a single edge whose end points are the same.

**Lemma 4.6.** Let $\Gamma$ be a graph with $|E_\ell(\Gamma)| = 1$, and let $K = [v_1, v_n]$. If $u \in \text{PMap}_c(\Gamma)$ is supported only on the loops of $\Gamma$, then $u \in (FU^\infty K)^3$, where $F = \{f\}$ with $f = L(n, 1, n+1)$.

**Proof.** Without loss of generality we can assume that $u$ is totally supported on $K' = [v_1, v_m]$ where $m \geq 2n$. Let $g = L(n, n+1, m+1)$. Note that $g \in U_K$ and $(gf)(K) \cap K = \emptyset$. See Figure 8 for a schematic of the setup in the case of the Loch Ness Monster.

![Figure 8: Setup of the loop swap maps for the Loch Ness Monster graph](image)

We claim that $u = (gf)^{-1}u(gf) \in U_K$. First note that $u$ is totally supported on $[v_1, w_{m+n}]$. We seek to apply Lemma 2.3. In order to apply this lemma we will need to consider fundamental groups with basepoints in each of the complementary components of $[v_1, w_{m+n}]$. Pick a such a finite collection of basepoints and note that a basis for the fundamental group based at each one is given by pre- and post-concatenating each loop $\alpha_i$ with the unique geodesic from $v_i$ to the respective basepoint. Now, since $u$ is supported on the loops and $g$ and $f$ are loop swaps we see that $u$ symbolically induces the same map on the fundamental groups from the perspective of each of these basepoints. Therefore, we will slightly abuse notation and write $\pi_1(\Gamma)$ to refer to the fundamental group with any of these basepoints and $\{a_i\}_{i=1}^{\infty}$ to denote a free basis.

As in Section 3.1, $A_{i,j}$ for $i \leq j$ denotes the free factor of $\pi_1(\Gamma)$ generated by the basis elements $\{a_k\}_{k=i}^{j}$. We claim that $\nu_*|A_{1,n} = \text{id}|A_{1,n}$ and $\nu_*|A_{n+1,m+n} \subset A_{n+1,m+n}$. Note that $gf$ induces the following map on $\pi_1(\Gamma)$.

$$(gf)_*: a_i \mapsto \begin{cases} a_{m+i} & \text{if } 1 \leq i \leq n, \\ a_{i-n} & \text{if } n+1 \leq i \leq 2n, \\ a_{i-(m-n)} & \text{if } m+1 \leq i \leq m+n, \\ a_i & \text{otherwise.} \end{cases}$$

In particular, $(gf)_*$ acts as the permutation on the free factors $A_{1,n} \to A_{m+1,m+n} \to A_{n+1,2n} \to A_{1,n}$, by sending ordered sets of generators to ordered sets of generators. Similarly, $(gf)^{-1}_*$ acts as the permutation $A_{1,n} \to A_{n+1,2n} \to A_{m+1,m+n} \to A_{1,n}$.

23
We also have that $u_\ast(a_i) = a_i$ for all $i > m$, and that $u_\ast(a_j) \in A_{1,m}$ for all $j \leq m$. Putting everything together, we have the following equalities for $j \leq n$.

\[
\nu_\ast(a_j) = (gf)_\ast^{-1}u_\ast(gf)_\ast(a_j) = (gf)_\ast^{-1}(a_{m+j}) = (gf)_\ast^{-1}(a_{m+j}) = a_j.
\]

This shows that $\nu_\ast|_{A_{1,n}} = id|_{A_{1,n}}$ as desired.

Next we check that $\nu_\ast(a_i) \in A_{n+1,\infty}$ for $i > n$, which is equivalent to checking that $(ugf)_\ast(a_i) \in A_{1,m} \ast A_{m+n+1,\infty}$. The only generators that are mapped into $A_{n+1,m+n}$ by $u_\ast$ are exactly $a_{m+1}, \ldots, a_{m+n}$. Since we assumed that $i > n$ we have that $(gf)_\ast(a_i) \in A_{1,m} \ast A_{m+n+1,\infty}$ and we conclude that $\nu_\ast(a_i) \in A_{n+1,\infty}$ for $i > n$. Thus, we can apply Lemma 2.3 to see that $\nu \in \mathcal{U}_K$. Then by rearranging and noting that $f^{-1} = f$ we obtain $u = gfvfg \in (\mathcal{F}\mathcal{U}_K)^3$. \hfill \Box

### 4.2 The Loch Ness Monster Graph

The Loch Ness Monster graph has exactly one end, so PMap($\Gamma_0$) = Map($\Gamma_0$). The lemmas we have prepared in Section 4.1 and Section 3 are sufficient to prove that this group is coarsely bounded.

**Proposition 4.7.** Map($\Gamma_0$) is coarsely bounded.

**Proof.** We will make use of Rosendal’s criterion. Given any neighborhood of the identity in PMap($\Gamma_0$), we pass to a basis element $\mathcal{U}_K$ for some compact $K$, then expand $K$ so that $K = [v_1, v_n]$ for some $n \in \mathbb{N}$. Let $\phi \in \text{Map}(\Gamma_0)$. Find a map $u$ using Lemma 4.4, such that $u\phi \in \mathcal{U}_K$. Apply Lemma 3.6 to write $u = u_\ell$ with $u_\ell$ on supported on the rays of $\Gamma_0$. Now we apply Lemma 4.6 to show that $fguf \in \mathcal{U}_K$. Then $u = gfhfg$ for some $h \in \mathcal{U}_K$. Hence, $\phi \in (gfh^{-1}fg)\mathcal{U}_K \subseteq (\mathcal{F}\mathcal{U}_K)^3$ for $\mathcal{F} = \{f\}$, as $g \in \mathcal{U}_K$. \hfill \Box

### 4.3 The Hungry Loch Ness Monster Graphs

Next we show that for $N \in \mathbb{N}$, the group PMap($\Gamma_N$) is coarsely bounded. Recall that $\Gamma_N$ denotes a Hungry Loch Ness Monster graph, as in Figure 3. Unlike elements of PMap($\Gamma_0$), compactly supported elements of PMap($\Gamma_N$) may have support on the rays, so we start by developing a method to fit these maps into Rosendal’s criterion. On its own, Lemma 4.8 says that the subgroup of PMap($\Gamma_N$) which consists of elements which are the identity on ($\Gamma_N$)$_c$, is coarsely bounded.

**Lemma 4.8.** Let $\Gamma = \Gamma_N$ for some $N \in \mathbb{N}$, and let $K = [v_1, v_n]$. Let $\phi_{(w,I)}$ be a word map with $I \subset R$ for any ray $R \subset \Gamma$. Then we can realize $\phi_{(w,I)} \in (\mathcal{F}\mathcal{U}_K)^3$, where $\mathcal{F} = \{f, \phi_{(a_{n+1}, I)}^+\}$ and $f = L(n,1,n+1)$.

**Proof.** We first modify $\phi_{(w,I)}$ to ensure that the word $w$ is a basis element in $F_\infty = \pi_1(\Gamma_N)$. Freely reduce $w$ and let $m = \max\{i | a_i \text{ appears in } w\} \cup \{2n+1\}$. Set $h = L(1,n+1,m+1) \in \mathcal{U}_K$. Then we define $\phi' = \phi_{(a_{n+1}, I)}^+h\phi_{(w,I)}h = \phi_{(a_{n+1}w', I)}$.
where \( w' = h_*(w) \). The final equality follows from Lemma 3.7 and the composition of word maps. Note that \( w' \) does not contain any instances of \( a_{n+1} \). So, \( a_{n+1}w' \), the word defining \( \phi' \), only contains a single instance of \( a_{n+1} \), and is thus a basis element for \( F_\infty \).

Next we modify \( \phi' \) again to get a word map \( \phi'' \) whose defining word is a basis element completely contained in \( A_{n+1,\infty} \). That is, \( \phi'' \) does not hit any of the loops in \( K \). Let \( g = L(n, n + 1, m + 2) \in U_K \) and set

\[
\phi'' = f g \phi' g f = \phi((fg)_*(a_{n+1} w'), I) = \phi((a_{m+2} w''), I).
\]

where \( w'' = (fg)_*(w') = (fg)h_*(w) \). Note that \( (fg)_* \) only maps the basis elements \( a_{m+2}, \ldots, a_{m+n+1} \) into \( A_{1,n} \) so that \( w'' \in A_{n+1,\infty} \), by the choice of \( m \).

Next we choose \( \rho \in \text{PMap}(\Gamma_N) \) such that

\[
\rho_* = \begin{cases} 
    a_{m+2} \mapsto a_{m+2} w'' \\
    a_i \mapsto a_i & \text{for all } i \neq m+2.
\end{cases}
\]

Note that such a homotopy equivalence exists since \( a_{m+2} w'' \) is a basis element for \( F_\infty \) and it can be taken to be proper since \( \rho_* \) is the identity outside of the finite-rank free factor \( A_{n+1,m+n+1} \) of \( F_\infty \). This also shows that \( \rho \in U_K \).

Finally, we conjugate \( \phi'' \) to get \( \phi((a_{n+1}, I)) \), the word map in \( F \):

\[
g \rho^{-1} \phi'' g = g \rho^{-1} \phi((a_{m+2} w''), I) g = g \phi((a_{m+2}, I)) g = \phi((a_{n+1}, I)).
\]

Therefore, after substituting and rearranging we have

\[
\phi(u, I) = h_\rho \phi((a_{n+1}, I)) g f \rho g \phi((a_{n+1}, I)) g \rho^{-1} f g h \in (U_K)^5.
\]

\[\square\]

**Proposition 4.9.** For any \( N \in \mathbb{N} \), the group \( \text{PMap}(\Gamma_N) \) is coarsely bounded.

**Proof.** We will again use Rosendal’s criterion. Given any open set \( V \) the first set \( K = [v_1, v_n] \) so that \( U_K \subset V \). Let \( f = L(n, n + 1, n + 1) \), and choose intervals \( I_i \) in each ray \( R_i \). Now set \( F = \{f, \phi_{(a_{n+1}, I)}/\phi_{(a_{n+1}, I)}\} \), we will show that any \( \phi \in \text{PMap}(\Gamma_N) \) is in \( (U_K)^{4+5N} \).

First apply Lemma 4.4 to get an element \( u \in \text{PMap}(\Gamma_N) \) such that \( u \phi \in U_K \). Now use Lemma 3.6 to write \( u = u_R \circ \cdots \circ u_{R_i} \circ u_{\ell} \) where \( u_{R_i} \) is supported on \( R_i \). By Lemma 4.6 we know \( u_{\ell} \in (U_K)^3 \), for \( \mathcal{F}_0 = \{f\} \).

Each \( u_{R_i} \) has a homotopy representative as a word map \( \phi_{(w_i, I)}, \) to which we will apply Lemma 4.8. That is, \( \phi_{R_i} \in (U_K)^3 \) for \( \mathcal{F}_i = \{f, \phi_{(a_{n+1}, I)}\} \).

Because we chose \( F = \bigcup_{i=0}^N \mathcal{F}_i \), we can now write \( u \in (U_K)^{3+5N} \). Combining this with the expression \( u \phi \in U_K \) we get that \( \phi \in (U_K)^{4+5N} \). \[\square\]
Since $\Gamma_N$ with $N \in \mathbb{N}$ has finitely many ends, Corollary 2.21 deduces:

**Corollary 4.10.** For any $N \in \mathbb{N}$, the full mapping class group $\text{Map}(\Gamma_N)$ is coarsely bounded.

### 4.4 The Millipede Monster Graph

Let $\Gamma_\infty$ be the graph with infinite rank whose end space is homeomorphic to $\{ \frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$ with $E_\ell(\Gamma_\infty) = \{0\}$, as shown in Figure 3. The next proof again uses Rosendal’s criterion, but we note that this is the only case where we don’t show uniformity in the size of $\mathcal{F}$ and $n$ across different open neighborhoods of the identity. Uniformity of these constants is always present in the surface case [12].

We will also make use of loop swaps which were first defined in Section 3.2. Note that in that section we only defined these maps for $\Gamma_N$ with $N$ finite. One can extend the definition of $L(n, m_1, m_2)$ by similarly interchanging the subgraphs $[v_{m_1}, v_{m_1+n-1}]$ and $[v_{m_2}, v_{m_2+n-1}]$ isometrically and now stretching subsegments of each of the $R_i$ that are incident to these subgraphs along the spanning tree of $\Gamma_\infty$.

**Proposition 4.11.** $\text{PMap}(\Gamma_\infty)$ is coarsely bounded.

**Proof.** We will again use Rosendal’s criterion. Given any open neighborhood of the identity $V$, choose $K = [v_1, v_n]$ so that $U_K \subset V$. Let $f = L(n, 1, n + 1)$ and choose intervals $I_i$ in each ray $R_i$ for $i = 1, \ldots, n$. Set $\mathcal{F} = \{f, \phi_{(a_{n+1}, I_i)}^\pm, \phi_{(a_{n+1}, I_i)}^\pm \}$. We will show that any $\phi \in \text{PMap}(\Gamma_\infty)$ is in $(\mathcal{F}U_K)^{7+5n}$.

Once again we apply Lemma 4.4 to get an element $u \in \text{PMap}(\Gamma_\infty)$ such that $u\phi \in U_K$ and $u$ is totally supported on $K' = [v_1, v_m]$ for $m \geq n$. Use Lemma 3.6 to write $u = (u_{R_m} \circ \cdots \circ u_{R_{n+1}}) \circ u_{R_n} \circ \cdots \circ u_{R_1} \circ u_\ell$ where each $u_{R_i}$ is supported on $R_i$ and $u_\ell$ has support only on the loops of $\Gamma_\infty$. Just as before, we apply Lemma 4.6 to see that $u_\ell \in (\mathcal{F}_0U_K)^3$, for $\mathcal{F}_0 = \{f\}$, and Lemma 4.8 to see that $u_{R_i} \in (\mathcal{F}_iU_K)^5$, for $\mathcal{F}_i = \{f, \phi_{(a_{n+1}, I_i)}^\pm\}$ and $i \in \{1, \ldots, n\}$. Note that we chose $\mathcal{F} = \bigcup_{i=1}^n \mathcal{F}_i$ so that it only remains to check the following claim.

**Claim 3.** $u_{R_m} \circ \cdots \circ u_{R_{n+1}} \in (\mathcal{F}U_K)^3$.

**Proof.** Each $u_{R_i}$ is homotopic to a word map $\phi_{(w_i, I_i)}$ for $I_i$ an interval in each $R_i$. Let $M = \max\{\{j|a_j \text{ appears in } w_i\}_{i=1}^n \cup \{n\}\}$ and let $g = L(n, n + 1, M + 1)$.

Then using Lemma 3.7 we have

$$fgu_{R_i}gf = fg\phi_{(w_i, I_i)}gf$$

$$= \phi_{(fg)_i(w_i), I_i}$$

for all $i \in \{n + 1, \ldots, m\}$. Since $g$ was chosen so that $(fg)_i(w_i) \in A_{n+1, \infty}$, we have that $fgu_{R_i}gf \in U_K$ for all $i$. Finally, we note that

$$(fgu_{R_i}gf) \circ \cdots \circ (fgu_{R_{n+1}}gf) = fg \circ (u_{R_{n+1}} \circ \cdots \circ u_{R_{n+1}}) \circ gf,$$

and rearrange to get the claim.

Combining each of these allows us to conclude that $u \in (\mathcal{F}U_K)^{6+5n}$, and as $u\phi \in U_K$ we obtain that $\phi \in (\mathcal{F}U_K)^{7+5n}$. \qed
We included Proposition 4.2 as a warm up proof earlier in the section, but it is also the other key ingredient to proving the following corollary.

**Corollary 4.12.** Map(Γ∞) is coarsely bounded.

*Proof.* By Proposition 2.18, because PMap(Γ∞) is coarsely bounded, Map(Γ∞) is coarsely equivalent to Homeo (E(Γ∞), Eℓ(Γ∞)) ≅ S∞. By Proposition 4.2, S∞ is coarsely bounded. □

## 5 Graphs of Finite Positive Rank

In this section we will see that PMap(Γ) is not CB for graphs Γ of finite positive rank except when Γ = O. Recall when Γ has rank 0, Proposition 2.9 shows that PMap(Γ) is trivial, hence CB. Since any locally finite graph Γ is an Eilenberg-Maclane space, K(π1(Γ), 1), there is a natural homomorphism

\[ \Psi : \text{Map}(\Gamma) \to \text{Out}(\pi_1(\Gamma)) \]

that associates g ∈ Map(Γ) with the corresponding outer automorphism class of g∗ : π1(Γ) → π1(Γ). We refer the reader to [1, Chapter 3] for an in-depth discussion of the map Ψ and its kernel. Note Ψ is surjective when Γ has finite rank n and that π1(Γ) ≅ Fn, the free group of rank n. The restriction Ψ|PMap(Γ) to PMap(Γ) still surjects onto Out(π1(Γ)) when Γ has finite rank. This is because the fundamental group, π1(Γ) ≅ π1(Γc), only captures the finite core graph Γc of Γ, so we can choose the extension of the map Γc → Γc corresponding to a given outer automorphism to fix the ends of Γ.

In general, however, Ψ is not surjective as there are automorphisms not realized by proper homotopy equivalences, such as the automorphism induced by the inverse homotopy equivalence in the example in Remark 2.5.

### 5.1 Graphs of Rank > 1

We first deal with the generic case, when a graph has rank larger than 1.

**Lemma 5.1.** If Γ is a locally finite graph of finite rank n > 1, then Ψ : Map(Γ) → Out(Fn) is continuous.

*Proof.* Observe that UΓc is a subgroup of ker Ψ. Since Map(Γ) is a topological group and Out(Fn) is a discrete group, it is sufficient to check that ker Ψ = Ψ−1(\{[id]\}) contains an open subgroup. □

**Corollary 5.2.** If Γ is a locally finite graph of finite rank n > 1 then Map(Γ) is not CB. In particular, PMap(Γ) is not CB.

*Proof.* Note that Out(Fn) is not CB by Condition (2) of Definition 2.14. Indeed, Out(Fn) has many unbounded continuous actions on metric spaces, e.g., on its Cayley graph or on the free factor complex. We can thus precompose this action with the surjective map Ψ to obtain an unbounded continuous action of Map(Γ). Further precompose with the inclusion PMap(Γ) → Map(Γ) to deduce the latter assertion. □
5.2 Graphs of Rank 1

The technique used above fails when $\Gamma$ has rank one because the target group of $\Psi$ is now $\text{Out}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, a finite (hence CB) group. Instead we will make use of the semi-direct product description of $\text{PMap}(\Gamma)$ from [1] to see that except for one exceptional case $\text{PMap}(\Gamma)$ will also not be CB. For the sake of brevity we state definitions and results from [1, Chapter 3] in the specific case that $\Gamma$ has rank one; however, it is important to note that these definitions and results can be stated more generally.

Let $\Gamma$ have rank one and choose some $e_0 \in E(\Gamma)$. Let $\pi_1(\Gamma, e_0)$ be the group of proper homotopy classes of lines $\sigma : \mathbb{R} \to \Gamma$ with $\lim_{t \to \infty} \sigma(t) = \lim_{t \to -\infty} \sigma(t) = e_0$, with group operation given by concatenation. In [1] it is noted that given any $x_0 \in \Gamma$ there is an isomorphism $\pi_1(\Gamma, x_0) \cong \pi_1(\Gamma, e_0)$. Let $\Gamma_c^* \subset \Gamma$ be the subgraph consisting of $\Gamma_c$ together with a choice of ray in $\Gamma$ which limits to $e_0$ and intersects $\Gamma_c$ in exactly one point.

**Definition 5.3 ([1, Definition 3.3]).** The group $R$ as a set is the collection of maps $h : E(\Gamma) \to \pi_1(\Gamma_c^*, e_0)$ satisfying

(R0) $h(e_0) = 1$, and

(R1) $h$ is locally constant.

The group operation in $R$ is given by pointwise multiplication in $\pi_1(\Gamma, e_0)$.

Algorn-Kfir and Bestvina use this group to give a description of $\text{PMap}(\Gamma)$ as a semi-direct product.

**Theorem 5.4 ([1, Corollary 3.9]).** If $\Gamma$ has rank one then

$$\text{PMap}(\Gamma) \cong R \rtimes \text{PMap}(\Gamma_c^*).$$

We can now prove the following.

**Proposition 5.5.** Let $\Gamma$ be a locally finite, infinite graph of rank one. Then $\text{PMap}(\Gamma)$ is CB if and only if $|E(\Gamma)| = 1$ (that is, if $\Gamma$ is $\mathcal{O}$-).

**Proof.** First note that if $|E(\Gamma)| = 1$, then $R = 1$ so it follows that $\text{PMap}(\Gamma) \cong \mathbb{Z}/2\mathbb{Z}$, which is CB.

Next we establish that $R$ is a Polish group. By Theorem 5.4 we have that $\text{PMap}(\Gamma_c^*) \cong \text{PMap}(\Gamma)/R$. When $\Gamma$ has rank one, $\text{PMap}(\Gamma_c^*) \cong \text{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and hence is discrete. Thus we see that $R$ equipped with the subspace topology from $\text{PMap}(\Gamma)$ is open and closed and hence Polish.

Now conversely, assume $|E(\Gamma)| > 1$; we claim that $\text{PMap}(\Gamma)$ is not CB. Let $e \in E(\Gamma)$ with $e \neq e_0$ and $\phi_e : R \to \pi_1(\Gamma_c^*, e)$ be the evaluation homomorphism at $e$. Since $R$ is nontrivial, $\phi_e$ is a nontrivial homomorphism from $R$ to $\mathbb{Z}$. A classical result of Dudley [8] states that any homomorphism from a Polish group to $\mathbb{Z}$ is continuous. Thus $\phi_e$ defines a continuous nontrivial homomorphism from $R$ to $\mathbb{Z}$ showing that $R$ is not CB. Finally, because $R$ is an index 2 open subgroup of $\text{PMap}(\Gamma)$, it follows that $\text{PMap}(\Gamma)$ is also not CB by Proposition 2.20.

28
6 Length Functions

In this section we consider graphs, $\Gamma$ of rank at least one and with $E(\Gamma) \setminus E_\ell(\Gamma)$ having an accumulation point. Note that this includes the class of graphs with one end accumulated by loops and infinite end space with the one exception of the Millipede Monster graph, $\Gamma_\infty$, described in Section 4.4. We show that for any such graph $\Gamma$, its pure mapping class group is not coarsely bounded.

**Theorem 6.1.** Let $\Gamma$ be a locally finite, infinite graph with $\text{rk}(\Gamma) > 0$ and $|E(\Gamma) \setminus E_\ell(\Gamma)|$ containing an accumulation point. Then $\text{PMap}(\Gamma)$ is not CB.

We prove this by showing that these mapping class groups admit an unbounded length function.

**Definition 6.2.** A length function on a topological group $G$ is a continuous function $\ell : G \to [0, \infty)$ satisfying

(a) $\ell(g) = \ell(g^{-1})$,

(b) $\ell(\text{id}) = 0$,

(c) $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$.

**Remark 6.3.** Having an unbounded length function is in direct contradiction with Rosendal’s criterion, Definition 2.14. This is because for any $c \in \mathbb{R}^+$, the set $V := \ell^{-1}([0, c))$ is a neighborhood of the identity in $G$. However, for any finite subset $\mathcal{F}$ of $G$ and $k \geq 1$, the set $(\mathcal{F}V)^k$ consists of the elements of length no bigger than $k \cdot (c + \max_{f \in \mathcal{F}} \ell(f)) < \infty$. Hence, if $G$ admits an unbounded length function, it follows that $G \not\subset (\mathcal{F}V)^k$ for any choice of finite set $\mathcal{F}$ and $k \geq 1$, so $G$ is not CB.

In order to define the length function on $\text{PMap}(\Gamma)$ we consider the geometric realization of $\Gamma$ so that each edge is identified with $[0, 1]$ and put the path metric on $\Gamma$, namely define the distance between two points of $\Gamma$ to be the length of the shortest edge path. Consider the connected components $\{T_i\}$ of $\Gamma \setminus \Gamma_c$. Note that up to proper homotopy, the cardinality of $\{T_i\}$ is not well defined but is at most countably infinite. Because $\Gamma$ is not the Millipede Monster graph, there exists an accumulation point in $E(\Gamma) \setminus E_\ell(\Gamma) = \bigsqcup_{i \in I} E(T_i)$. Thus some $T_1$ has infinite end space. For each such $T_i$ we can define an unbounded length function on $\text{PMap}(\Gamma)$.

**Proposition 6.4.** Let $\Gamma$ be a locally finite, infinite graph with $\text{rk}(\Gamma) \geq 1$ and $E \setminus E_\ell$ having an accumulation point. Let $T$ be a connected component of $\Gamma \setminus \Gamma_c$ with infinite end space. Then the map $\ell$ defined on $\text{PMap}(\Gamma)$ as:

$$\ell : \text{PMap}(\Gamma) \to [0, \infty)$$

$$g \mapsto \min\{c | g(L) \simeq L \text{ for all geodesic lines } L \subset T \text{ with } d(L, \Gamma_c) \geq c\}$$

is an unbounded length function on $\text{PMap}(\Gamma)$. Moreover, $\ell$ is an ultranorm, namely $\ell(gh) \leq \max\{\ell(f), \ell(g)\}$.  


Proof. One can immediately see that $\ell(\text{Id}) = 0$ and $\ell(g^{-1}) = \ell(g)$. Since we give edges length 1 we have that $\ell$ takes values in $\mathbb{Z}_{\geq 0}$. Any pure proper homotopy equivalence on $\Gamma$ can be homotoped so that on $T$ it is a finite collection of disjointly supported word maps. The composition of two word maps on the same edge is also a word map on that edge, so $\ell(fg) \leq \max\{\ell(f), \ell(g)\} \leq \ell(f) + \ell(g)$. Next we show that $\ell : \text{PMap}(\Gamma) \to \mathbb{Z}_{\geq 0}$ is continuous.

Let $v$ be the vertex in $\Gamma_c$ which connects $T$ to $\Gamma_c$; that is, $T \cap \Gamma_c = \{v\}$. Let $K$ be any finite, positive rank subgraph of $\Gamma_c$ which contains $v$. Any element $u \in U_K$ is totally supported in $\Gamma \setminus T$, since $K$ disconnects $T$ from the rest of $\Gamma$, and so $\ell(u) = 0$. Now to show that $\ell$ is continuous we will show that for any $n \in \mathbb{Z}_{\geq 0}$ the preimage of $\{n\}$ is open in $\text{PMap}(\Gamma)$: Let $g \in \ell^{-1}(\{n\})$, then $gU_K \subset \ell^{-1}(\{n\})$ because for any $u \in U_K$, we have $\ell(gu) = \ell(g) = n$. Here we have equality because we know that $u$ is totally supported on $\Gamma \setminus T$, so cannot “undo” any part of $g$ and decrease its length.

Finally, because $T$ has an infinite end space, $E(T)$ contains an accumulation point. Let $I_i$ be a sequence of intervals converging to this end. Then $\ell(\phi_{w,I_i}) \to \infty$ for any word $w$. \qed

Proof of Theorem 6.1. By Proposition 6.4, $\text{PMap}(\Gamma)$ admits an unbounded length function, and Remark 6.3 implies that $\text{PMap}(\Gamma)$ is not CB. \qed

Now we would like to show that $\text{PMap}(\Gamma)$ acts on a tree. Define a pseudo-metric $\hat{d}$ on $\text{PMap}(\Gamma)$ as $\hat{d}(g,h) := \ell(g^{-1}h)$. Then $\text{PMap}(\Gamma)$ acts on itself by left multiplication isometrically: for $g, h, k \in \text{PMap}(\Gamma)$,

$$\hat{d}(kg, kh) = \ell(g^{-1}k^{-1} \cdot kh) = \ell(g^{-1}h) = \hat{d}(g,h).$$

Now we consider the quotient space $\mathcal{P} = \text{PMap}(\Gamma)/\sim$, where $g \sim h$ for $g, h \in \text{PMap}(\Gamma)$ iff $\hat{d}(g,h) = 0$, namely, identify all the elements in $\text{PMap}(\Gamma)$ that are at distance 0 from each other. Then the pseudo-metric $\hat{d}$ has the induced metric $d$ on $\mathcal{P}$, and $\text{PMap}(\Gamma)$ acts on $(\mathcal{P}, d)$ by the left multiplication isometrically.

To show $(\mathcal{P}, d)$ forms the vertex set of some tree, we first claim that $(\mathcal{P}, d)$ is 0-hyperbolic. From the “ultranorm” nature of $\ell$ as in Proposition 6.4, we observe that $d$ is an ultrametric on $\mathcal{P}$, i.e., $d(g,h) \leq \max\{d(g,k), d(h,k)\}$ for any $g, h, k \in \text{PMap}(\Gamma)$. Indeed,

$$d(g,h) = \ell(g^{-1}h) = \ell(g^{-1}k \cdot k^{-1}h) \leq \max\{\ell(g^{-1}k), \ell(k^{-1}h)\} = \max\{\ell(g^{-1}k), \ell(h^{-1}k)\} = \max\{d(g,k), d(h,k)\}.$$ 

Next we apply the following well known lemma [7, Exercise 1] to see that our metric space is 0-hyperbolic. As of yet we have not found a written proof of this lemma and so we include a proof in the appendix.

**Lemma 6.5.** Every ultrametric space is Gromov 0-hyperbolic.

**Corollary 6.6.** $\text{PMap}(\Gamma)$ acts on a simplicial tree.

**Proof.** We have seen that $\text{PMap}(\Gamma)$ acts on $(\mathcal{P}, d)$, which is an ultrametric space. By Lemma 6.5, it follows that $(\mathcal{P}, d)$ is 0-hyperbolic. Note that every 0-hyperbolic space
isometrically embeds into an $\mathbb{R}$-tree and any group action extends to an action on this $\mathbb{R}$-tree via the "Connecting the Dots" Lemma (See e.g. [5, Lemma 2.13]). More precisely, to build the $\mathbb{R}$-tree, we start with the collection of arcs $I_x = [\text{Id}], x$ of length $\ell(x)$, and then identify $I_x$ and $I_y$ along $[\text{Id}], (x,y)\text{Id}$, where $(x,y)\text{Id}$ is the Gromov product. Then the distance between the vertices of this $\mathbb{R}$-tree is a half-integer, so the vertex set is discrete. All in all, we obtain a simplicial tree.

**Proposition 6.7.** This action is continuous.

**Proof.** Let $F : \text{PMap}(\Gamma) \times (\mathcal{P}, d) \rightarrow (\mathcal{P}, d)$ be the action. Note that $(\mathcal{P}, d)$ is discrete, so we will show that $F^{-1}(\{g\})$ contains an open set for any $g \in \text{PMap}(\Gamma)$. Let $K \subset \Gamma$ be compact so that $K$ separates the $T$ used to define our length function from $\Gamma_c$. Then we claim that $U_K \times \{[g]\} \subset F^{-1}(\{g\})$. Indeed, for any $h \in U_K$ note that $h$ fixes $T$ so that $\hat{d}(hg, g) = \ell(hgg^{-1}) = \ell(h) = 0$.

Thus we see that $F(h, [g]) = [hg] = [g]$ in $\mathcal{P}$. We conclude that the action on $(\mathcal{P}, d)$ is continuous, then as the natural extension, the action on the tree is also continuous.

Note also that the action is transitive since $\text{PMap}(\Gamma)$ acts on itself transitively.

### 7 Flux Maps: Graphs with $|E_\ell| \geq 2$

Let $\Gamma$ be an infinite, locally finite graph with at least two ends accumulated by loops. We show every such graph has non-CB pure mapping class groups following ideas from [2]. We first need some background on free factors.

#### 7.1 Free Factors

**Definition 7.1.** Let $G$ be a group. Then $A < G$ is a free factor of $G$ if there exists some $P < G$ such that $G = A \ast P$.

**Lemma 7.2.** Let $C$ be a free group and $A < B < C$ with $A$ a free factor of $C$. Then $A$ is also a free factor of $B$.

**Proof.** Since $A$ is a free factor of $C$ we can realize $C$ as the fundamental group of some graph $\Gamma$ with a connected subgraph $\Delta \subset \Gamma$ such that $\pi_1(\Delta) \cong A$. Now build the cover of $\Gamma$ corresponding to $B$, denoted by $\rho : \Gamma_B \rightarrow \Gamma$. The inclusion map $i : \Delta \rightarrow \Gamma$ lifts to $\tilde{i} : \Delta \rightarrow \Gamma_B$ because $i_*(\pi_1(\Delta)) = A < B = \rho_*(\pi_1(\Gamma_B))$. Since $i$ is injective and $\rho \circ \tilde{i} = i$, the lift $\tilde{i}$ is injective. Thus, $\Delta$ homeomorphically includes as a subgraph of $\Gamma_B$, and we can realize $A$ as a free factor of $B$ as well.

**Definition 7.3.** Let $B$ be a free group and $A < B$ a free factor. Define the corank of $A$ in $B$, $\text{cork}(B, A)$, to be

$$\text{cork}(B, A) := \text{rk}(B/\langle \langle A \rangle \rangle),$$

where $\langle \langle A \rangle \rangle$ is the normal closure of $A$ in $B$. Equivalently, if we write $B = A \ast P$, then $\text{cork}(B, A) = \text{rk}(P)$. 

31
Lemma 7.4. The function $cork$ is additive. I.e., if $A < B < C$ with $A$ and $B$ both free factors of a free group $C$ then

$$cork(C, A) = cork(C, B) + cork(B, A).$$

Proof. Firstly, note that $A$ is a free factor of $B$ by Lemma 7.2 so $cork(B, A)$ is well-defined. The equality follows from the fact that the free product operation on groups is associative. Indeed, we have some $P_B, P_A$ such that

$$C = B * P_B, \text{ and}$$
$$B = A * P_A.$$ 

Thus,

$$C = (A * P_A) * P_B = A * (P_A * P_B).$$

This implies that $cork(C, A) = rk(P_A * P_B)$. If either $P_A$ or $P_B$ has infinite rank then so does $P_A * P_B$. If $P_A$ and $P_B$ both have finite rank we can apply Grushko’s Theorem [9] to see that

$$rk(P_A * P_B) = rk(P_A) + rk(P_B) = cork(B, A) + cork(C, B).$$

7.2 Constructing Flux Maps

Theorem 7.5. Let $\Gamma$ be a graph with at least two ends accumulated by loops. Then $PMap(\Gamma)$ is not CB.

We prove this by finding a flux map given any two distinct ends accumulated by loops. Let $PPHE(\Gamma)$ be the group of proper homotopy equivalences of $\Gamma$ which induce the identity map on the end space of $\Gamma$. For any partition of the ends of $\Gamma$ into two clopen sets, each of which contains an end accumulated by loops, we will define a flux map $\Phi : PMap(\Gamma) \rightarrow \mathbb{Z}$. We again will always be using standard forms of our graphs. In particular, note that any edge which is not a loop in a standard form graph is separating.

Fix a line $\gamma$ in the maximal tree of $\Gamma$ whose ends correspond to two distinct ends accumulated by loops. Pick any edge of $\gamma$, and let $x_0$ be the midpoint. Then $\Gamma \setminus \{x_0\}$ will induce our desired partition $C_L \cup C_R$ of the end space of $\Gamma$. More precisely, $\Gamma \setminus \{x_0\}$ has two components, $\Gamma_L$ and $\Gamma_R$ each of which has end spaces $C_L$ and $C_R$, respectively. Let $T_L \subset T$ be the maximal tree of $\Gamma_L$. Define for each $n \in \mathbb{Z}$:

$$\Gamma_n := \begin{cases} 
\Gamma_L \cup B_n(x_0) & \text{if } n \geq 0 \\
(\Gamma_L \setminus B_n(x_0)) \cup T_L & \text{if } n < 0
\end{cases},$$

where $B_n(x_0)$ is the open metric ball of radius $n$ about $x_0$. Note that $\Gamma_0 = \Gamma_L$.

Now for each $n \in \mathbb{Z}$, $\Gamma_n$ determines a free factor $A_n = \pi_1(\Gamma_n, x_0)$ of the infinite-rank free group $F = \pi_1(\Gamma, x_0)$. Observe that $A_n$ has infinite rank and corank within $F$. We also note that these subgraphs and corresponding free factors are totally ordered. That is, if $n, m \in \mathbb{Z}$ with $n < m$ then $\Gamma_n \subset \Gamma_m$ and $A_n \leq A_m$, which further implies that $A_n$ is a free factor of $A_m$ by Lemma 7.2.
Lemma 7.6. Let \( f \in \text{PPHE}(\Gamma) \). Then for any \( n \in \mathbb{Z} \), there exists some \( m \in \mathbb{Z} \) such that \( \Gamma_m \) contains both \( \Gamma_n \) and \( f(\Gamma_n) \).

Proof. It suffices to find \( m > n \) such that \( \Gamma_m \supset f(\Gamma_n) \). Note first that \( f(\Gamma_n) \) has end space equal to \( C_L \) since \( f \) is pure. Then there exists a common neighborhood \( M \) of \( C_L \) contained in both \( \Gamma_n \) and \( f(\Gamma_n) \). Since \( f(\Gamma_n) \setminus M \) is bounded and the collection \( \{ \Gamma_m \setminus M \}_{m>n} \) exhausts \( \Gamma \setminus M \), it follows that there exists some \( m > n \) such that \( f(\Gamma_n) \setminus M \subset \Gamma_m \setminus M \). By construction we also have \( M \subset \Gamma_n \subset \Gamma_m \), and it follows that \( f(\Gamma_n) \subset \Gamma_m \). \(\Box\)

Remark 7.7. From here on we write \( \text{cork}(A_m, A_n) \) and \( \text{cork}(A_m, f_*(A_n)) \) to implicitly include a choice of basepoint which realizes these free factors as based fundamental groups. That is, we can take \( x_0 \) as above, and then we have \( \text{cork}(A_m, A_n) = \text{cork}(\pi_1(\Gamma_m, x_0), \pi_1(\Gamma_n, x_0)) \) and \( \text{cork}(A_m, f_*(A_n)) = \text{cork}(\pi_1(\Gamma_m, f(x_0)), f_*(\pi_1(\Gamma_n, x_0))) \).

Corollary 7.8. Let \( f \in \text{PPHE}(\Gamma) \). Then for any \( n \in \mathbb{Z} \), there exists some \( m > n \) so that

(i) \( A_n \) and \( f_*(A_n) \) are free factors of \( A_m \), and

(ii) both \( \text{cork}(A_m, A_n) \) and \( \text{cork}(A_m, f_*(A_n)) \) are finite.

Proof. For (i) we apply Lemma 7.6 to find an \( m \in \mathbb{Z} \) such that both \( \Gamma_n \) and \( f(\Gamma_n) \) are contained in \( \Gamma_m \). Since \( A_n = \pi_1(\Gamma_n) \) and \( f_*(A_n) < \pi_1(f(\Gamma_n)) \), both \( A_n \) and \( f_*(A_n) \) are subgroups of \( A_m \). Moreover, as \( f_\# \) is a \( \pi_1 \)-isomorphism, both \( A_n \) and \( f_*(A_n) \) are free factors of \( \pi_1(\Gamma) \). Hence, by Lemma 7.2, we conclude \( A_n \) and \( f_*(A_n) \) are free factors of \( A_m \). In particular, the quantities \( \text{cork}(A_m, A_n) \) and \( \text{cork}(A_m, f_*(A_n)) \) are well-defined.

For (ii) we first let \( g \) be a proper homotopy inverse for \( f \). By Lemma 7.6 we can find some \( m \) such that \( \Gamma_n, f(\Gamma_n) \) and \( g(\Gamma_n) \) are contained in \( \Gamma_m \). Note that \( \text{cork}(A_m, A_n) \) is finite by definition.

Suppose, for the sake of contradiction, that \( \text{cork}(A_m, f_*(A_n)) \) is infinite. Then we have that for every integer \( i < n, A_i \) is not a subgroup of \( f_*(A_n) \). Otherwise, \( \text{cork}(A_m, f_*(A_n)) < \text{cork}(A_m, A_i) < \infty \). For each \( i < n \), there exists a basis element \( a_{ij} \) of \( A_i \) such that \( a_{ij} \not\in f_*(A_n) \). Since \( \bigcap_{i<n} A_i = \emptyset \), we can pass to an infinite sequence of distinct basis elements \( \{ a_{ij} \} \) such that \( a_{ij} \not\in f_*(A_n) \). Therefore \( g_*(a_{ij}) \not\in A_n \) for all \( ij \). Let \( \alpha_{ij} \) denote a loop representing \( a_{ij} \) in \( \Gamma_m \). Note that \( g_*(\alpha_{ij}) \not\in A_n \) implies that \( g(\alpha_{ij}) \not\in \Gamma_n \) so that \( g(\alpha_{ij}) \cap \Gamma_m \setminus \Gamma_n \neq \emptyset \). However, \( \Gamma_m \setminus \Gamma_n \) is compact and \( \Gamma_m \setminus \Gamma_n \cap \Gamma_n \) is finite, so there must exist some point \( x \in \Gamma_m \setminus \Gamma_n \) such that \( g^{-1}(x) \) is infinite, contradicting the fact that \( g \) was a proper map. Thus we conclude that \( \text{cork}(A_m, f_*(A_n)) \) is finite. \(\Box\)

Definition 7.9. Given \( f \in \text{PPHE}(\Gamma) \) we say that a pair of integers, \( (m, n) \), with \( n < m \), is admissible for \( f \) if

(i) \( A_n \) and \( f_*(A_n) \) are free factors of \( A_m \), and

(ii) both \( \text{cork}(A_m, A_n) \) and \( \text{cork}(A_m, f_*(A_n)) \) are finite.

Corollary 7.8 shows that for any \( f \in \text{PPHE}(\Gamma) \) and \( n \in \mathbb{Z} \), there exist \( m \in \mathbb{Z} \) such that \( (m, n) \) is admissible for \( f \). For a map \( f \in \text{PPHE}(\Gamma) \) and an admissible pair \( (m, n) \) for \( f \), we let

\[ \phi_{m,n}(f) := \text{cork}(A_m, A_n) - \text{cork}(A_m, f_*(A_n)) \].

33
Lemma 7.10. This quantity is independent of the choice of admissible pair \((m, n)\). That is, if \((m, n)\) and \((m', n')\) are admissible pairs for the map \(f \in \text{PPHE}(\Gamma)\) then 
\[
\phi_{m,n}(f) = \phi_{m',n'}(f).
\]

Proof. Let \(f \in \text{PPHE}(\Gamma)\). We first consider the case with admissible pairs \((m, n)\) and \((m', n)\) for \(f\) with \(m < m'\). Then by the additivity of cork we have
\[
cork(A_{m'}, A_m) = cork(A_{m'}, A_n) - cork(A_m, A_n), \quad \text{and} \quad cork(A_{m'}, A_m) = cork(A_{m'}, f_*(A_n)) - cork(A_m, f_*(A_n)).
\]

Now
\[
\phi_{m',n}(f) - \phi_{m,n}(f) = \{\text{cork}(A_{m'}, A_n) - \text{cork}(A_{m'}, f_*(A_n))\} - \{\text{cork}(A_m, A_n) - \text{cork}(A_m, f_*(A_n))\}
\]
\[
= \{\text{cork}(A_{m'}, A_n) - \text{cork}(A_m, A_n)\} - \{\text{cork}(A_{m'}, f_*(A_n)) - \text{cork}(A_m, f_*(A_n))\}
\]
\[
= \text{cork}(A_{m'}, A_m) - \text{cork}(A_{m'}, A_m) = 0.
\]

Next, let \((m, n)\) and \((m', n')\) be any two admissible pairs for \(f\), without loss of generality we assume \(m < m'\). By definition, \((m', n)\) must also be admissible for \(f\). Then we can apply the above argument to reduce to considering just the pairs \((m, n)\) and \((m', n')\). Suppose that \(n < n'\). Then we have \(A_n < A_{n'}\) and \(f_*(A_n) < f_*(A_{n'})\). Once again by additivity we have
\[
cork(A_{n'}, A_n) = \text{cork}(A_{m'}, A_n) - \text{cork}(A_{m'}, A_{n'}), \quad \text{and} \quad cork(f_*(A_{n'}), f_*(A_n)) = \text{cork}(A_{m'}, f_*(A_n)) - \text{cork}(A_{m'}, f_*(A_{n'})).
\]

Note also that the function cork is invariant with respect to group isomorphisms so that 
\[
cork(f_*(A_{n'}), f_*(A_n)) = \text{cork}(A_{m'}, A_n).\]

Thus as before we have
\[
\phi_{m',n}(f) - \phi_{m,n}(f) = \{\text{cork}(A_{m'}, A_n) - \text{cork}(A_{m'}, f_*(A_n))\} - \{\text{cork}(A_{m'}, A_{n'})) - \text{cork}(A_{m'}, f_*(A_{n'}))\}
\]
\[
= \{\text{cork}(A_{m'}, A_n) - \text{cork}(A_{m'}, A_{n'})\} - \{\text{cork}(A_{m'}, f_*(A_n)) - \text{cork}(A_{m'}, f_*(A_{n'}))\}
\]
\[
= \text{cork}(A_{n'}, A_n) - \text{cork}(f_*(A_{n'}), f_*(A_n))
\]
\[
= \text{cork}(A_{n'}, A_n) - \text{cork}(A_{n'}, A_n) = 0.
\]

This allows us to define a function
\[
\phi : \text{PPHE}(\Gamma) \to \mathbb{Z}
\]
\[
f \mapsto \phi_{m,n}(f)
\]

where \((m, n)\) is any admissible pair for \(f\).

**Proposition 7.11.** The map \(\phi\) is a homomorphism.
Proof. First note that \( \phi(\text{id}) = 0 \). Let \( f, g \in \text{PPHE}(\Gamma) \) and let \( n \in \mathbb{Z} \). By Corollary 7.8 we can find some \( m \) so that \((m, n)\) is simultaneously admissible for all three maps \( f, g \), and \( fg \). Now we have

\[
\phi_{m,n}(fg) = \text{cork}(A_m, A_n) - \text{cork}(A_m, (fg)_*(A_n)) \\
= \text{cork}(A_m, A_n) - \text{cork}(A_m, f_* (A_n)) + \text{cork}(A_m, f_* (A_n)) - \text{cork}(A_m, (fg)_*(A_n)) \\
= \phi_{m,n}(f) + \{\text{cork}(f_*^{-1}(A_m), A_n) - \text{cork}(f_*^{-1}(A_m), g_* (A_n))\} \\
= \phi(f) + \phi(g).
\]

Note that the last step follows by picking some \( m' \) such that \( f_*^{-1}(\Gamma_m) \subset \Gamma_{m'} \) and applying the same argument used to prove Lemma 7.10. \( \Box \)

**Lemma 7.12.** If \( f, g \in \text{PPHE}(\Gamma) \) are properly homotopic, then \( \phi(f) = \phi(g) \).

**Proof.** We first claim that if a map \( h \in \text{PPHE}(\Gamma) \) is properly homotopic to the identity, then \( \phi(h) = 0 \). We will prove \( \text{cork}(A_m, h_* A_n) = \text{cork}(A_m, A_n) \). Let \((m, n)\) be an admissible pair for \( h \) and \( H_t \) be a proper homotopy of \( \Gamma \) so that \( H_0 = \text{id} \) and \( H_1 = h \), and define \( \beta(t) = H_t(x_0) \). If necessary, enlarge \( \Gamma_m \) so that it contains the image of \( \beta \). The induced map \( h_* : \pi_1(\Gamma, x_0) \to \pi_1(\Gamma, h(x_0)) \) satisfies

\[
c_{\beta} \circ h_* = \text{id} : \pi_1(\Gamma, x_0) \to \pi_1(\Gamma, x_0)
\]

where \( c_\beta : \pi_1(\Gamma, h(x_0)) \to \pi_1(\Gamma, x_0) \) is conjugation by the path \( \beta \). Then

\[
\text{cork}(A_m, h_* A_n) = \text{cork}(\pi_1(\Gamma_m, h(x_0)), h_* (\pi_1(\Gamma_n, x_0))) \\
= \text{cork}(c_\beta(\pi_1(\Gamma_m, h(x_0))), c_\beta h_* (\pi_1(\Gamma_n, x_0))) \\
= \text{cork}(c_\beta(\pi_1(\Gamma_m, x_0)), c_\beta (\pi_1(\Gamma_n, x_0))) \\
= \text{cork}(\pi_1(\Gamma_m, x_0)), \pi_1(\Gamma_n, x_0)) \\
= \text{cork}(A_m, A_n),
\]

as desired. Note that we required the path \( \beta \) to be contained in \( \Gamma_m \) to write \( c_\beta(\pi_1(\Gamma_m, h(x_0))) = \pi_1(\Gamma_m, x_0) \).

Now suppose \( f \) and \( g \) are properly homotopic. Then there exists a proper homotopy inverse \( \overline{g} \) of \( g \) such that \( fg \simeq \text{id} \). By the first assertion and Proposition 7.11, we have \( \phi(f) + \phi(\overline{g}) = \phi(f \overline{g}) = 0 \), so \( \phi(f) = -\phi(\overline{g}) \). Also, by definition \( g \overline{g} \simeq \text{id} \), so \( \phi(g) + \phi(\overline{g}) = \phi(g \overline{g}) = 0 \). Hence, \( \phi(f) = -\phi(\overline{g}) = \phi(g) \), concluding the proof. \( \Box \)

Thus we obtain a well-defined homomorphism, which we call a **flux map**:

\[
\Phi : \text{PMap}(\Gamma) \to \mathbb{Z} \\
[f] \mapsto \phi(f).
\]

Finally, to see that flux maps are nontrivial, we use the loop shifts defined in Section 3.4. We say that a loop shift, \( h \), **crosses** a partition \( \mathcal{P} = C_L \sqcup C_R \) of \( E(\Gamma) \) if \( h^+ \) and \( h^- \) are contained in different partition sets.

**Proposition 7.13.** The homomorphism \( \Phi \) satisfies:

(i) \( \Phi(f) = 0 \) for all \( f \in \overline{\text{PMap}}(\Gamma) \),
(ii) \( \Phi([h]) = \pm 1 \) where \( h \) is a loop shift which crosses the partition used to define \( \Phi \).

**Proof.** (i) We first let \( g \in \text{PMap}_c(\Gamma) \). Then, after potentially modifying \( g \) by a proper homotopy, \( g \) is totally supported on some compact subset \( K \subset \Gamma \). We can then find some \( n \) such that \((n, n)\) is an admissible pair for \( g \) and \( \Gamma_n \cap K \) is a (possibly empty) tree. Thus \( g_n \) is the identity map on \( A_n \).

Next we apply a theorem of Dudley which states that any homomorphism from a Polish group to \( \mathbb{Z} \) is continuous to conclude that \( \Phi(f) = 0 \) for any \( f \in \text{PMap}_c(\Gamma) \). Note that \( \overline{\text{PMap}}_c(\Gamma) \) is a closed subgroup of \( \text{Map}(\Gamma) \) and thus Polish.

Property (ii) follows from the definition of the loop shift. Assume that \( h^- \in C_L \) and \( h^+ \in C_R \). Let \( m > 0 \) be such that \( \Gamma_m \) contains one more loop of \( \rho(A) \) than \( \Gamma_0 \). This is possible because \( h \) crosses the partition \( C_L \cup C_R \). Then we have

\[
\Phi([h]) = \phi(h) = \text{cork}(A_m, A_0) - \text{cork}(A_m, h_*(A_0)) = 1 - 0 = 1.
\]

Note that if instead \( h^- \in C_R \) and \( h^+ \in C_L \) the same argument would show that \( \Phi([h]) = -1 \). \qed

**Proof of Theorem 7.5.** By Dudley’s automatic continuity property [8] the map \( \Phi \) is continuous. Thus, we get a continuous action of \( \text{PMap}(\Gamma) \) on the metric space \( \mathbb{Z} \) with unbounded orbits. \qed

**Remark 7.14.** We could construct the flux map on any subgroup \( H \) of \( \text{Map}(\Gamma) \) that fixes the two ends accumulated by the loops but not necessarily fixes the other ends. Following the same argument, we can show that \( H \) is not CB.

The proof of Theorem 7.5 shows that we have nontrivial homomorphisms to \( \mathbb{Z} \) so that \( H^1(\text{PMap}(\Gamma); \mathbb{Z}) \neq 0 \). However, with a more delicate choice of flux maps, we can get a better lower bound on the rank of \( H^1(\text{PMap}(\Gamma); \mathbb{Z}) \).

**Proposition 7.15.** If \( n = |E_i(\Gamma)| \geq 2 \) and finite, then \( \text{rk}(H^1(\text{PMap}(\Gamma); \mathbb{Z})) \geq n - 1 \). If \( |E_i(\Gamma)| = \infty \) then \( H^1(\text{PMap}(\Gamma); \mathbb{Z}) = \bigoplus_{i=1}^{\infty} \mathbb{Z} \).

To prove this, we refine the notation for flux map as follows. If \( \mathcal{P} = C_L \cup C_R \) is a partition of the ends of \( E(\Gamma) \) into two sets such that \( C_L \cap E_i(\Gamma) \neq \emptyset \) and \( C_R \cap E_i(\Gamma) \neq \emptyset \) then we denote the resulting flux map as \( \Phi_{\mathcal{P}} \).

**Proof of Proposition 7.15.** Let \( n = |E_i(\Gamma)| \) be finite. Identify \( E_i(\Gamma) \) with the \( n \)-set \( \{0, 1, \ldots, n-1\} \). Then for \( i = 1, \ldots, n-1 \), there exists pairwise disjoint neighborhoods \( U_i \) of each of the \( i \) in \( E(\Gamma) \). Define the partition \( \mathcal{P}_i = U_i \cup (E_i(\Gamma) - U_i) \) and denote by \( h_i \) a loop shift associated to a line joining the ends \( \{0, i\} \). Then by construction each \( h_i \) crosses \( \mathcal{P}_i \) and it follows that:

\[
\Phi_{\mathcal{P}_i}(h_{ij}) = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j.
\end{cases}
\]

This implies \( \Phi_{\mathcal{P}_1}, \ldots, \Phi_{\mathcal{P}_{n-1}} \) are linearly independent in \( H^1(\text{PMap}(\Gamma); \mathbb{Z}) \), so we conclude \( \text{rk}(H^1(\text{PMap}(\Gamma); \mathbb{Z})) \geq n - 1 \).

If \( |E_i(\Gamma)| = \infty \) we similarly enumerate a collection of ends in \( E_i(\Gamma) \) as \( \{0, 1, \ldots\} \) and can find pairwise disjoint neighborhoods \( U_i \) in \( E(\Gamma) \). This is possible since end
spaces are totally disconnected. We then similarly define our partitions and see that the associated flux maps are linearly independent. This gives a lower bound on the rank of \( H^1(\text{PMap}(\Gamma); \mathbb{Z}) \). To see that the cohomology cannot be larger than countably infinite note that \( \text{PMap}(\Gamma) \) has a countable basis. Then since any homomorphism to \( \mathbb{Z} \) must be continuous there are only countably many unique homomorphisms to \( \mathbb{Z} \).

\[ \Box \]

8 Locally CB Classification

In this section we use the tools developed above to also give a full classification of which graphs have \textit{locally coarsely bounded} pure mapping class groups. Recall that because \( \text{PMap}(\Gamma) \) are topological groups, they are locally CB if a neighborhood of the identity is CB. See Figure 9 for a summary of the results and proofs of Theorem E.

\begin{center}
\begin{tikzpicture}[auto, node distance=2cm,>=latex]

  \node (start) {Start};
  \node (finite) [below of=start] {\( \Gamma = \text{locally finite, infinite graph} \)};
  \node (rank) [below of=finite] {\( \text{rank}(\Gamma) =? \)};
  \node (component) [below of=rank] {\( \text{# of components of } \Gamma \setminus \Gamma_c \text{ with an indiscrete end space?} \)};
  \node (ends) [below of=component] {\( |E_\ell(\Gamma)| =? \)};
  \node (map) [right of=ends] {\( \Phi_P|_{\mathcal{U}_K} \) (Unbounded \( \ell: \mathcal{U}_K \to \mathbb{Z}_{\geq 0} \)};
  \node (cb) [right of=map] {\( \text{PMap}(\Gamma) \) is locally CB (Proposition 8.3)};
  \node (ncb) [below of=cb] {\( \text{PMap}(\Gamma) \) is Not locally CB (Unbounded \( \ell: \mathcal{U}_K \to \mathbb{Z}_{\geq 0} \)};
  \node (open) [below of=cb] {\( \text{PMap}(\Gamma) \) is locally CB (\{id\} is open)};
  \node (notopen) [below of=ncb] {\( \text{PMap}(\Gamma) \) is Not locally CB (Flux map on \( \mathcal{U}_K \))};

  \draw [->] (start) -- (finite);
  \draw [->] (finite) -- (rank);
  \draw [->] (rank) -- (ends);
  \draw [->] (ends) -- (map);
  \draw [->] (map) -- (cb);
  \draw [->] (map) -- (ncb);
  \draw [->] (cb) -- (open);
  \draw [->] (ncb) -- (notopen);

\end{tikzpicture}
\end{center}

\begin{figure}[h]
\caption{Flowchart for classifying locally CB \( \text{PMap}(\Gamma) \).
}
\end{figure}

**Proposition 8.1.** Let \( \Gamma \) be a locally finite, infinite graph of finite rank, then \( \{\text{id}\} \) is an open set in \( \text{PMap}(\Gamma) \). In particular, \( \text{PMap}(\Gamma) \) is discrete and locally CB.

**Proof.** Take \( K = \Gamma_c \), then \( \mathcal{U}_K = \{\text{id}\} \).

**Proposition 8.2.** Let \( \Gamma \) be a locally finite, infinite graph with infinitely many ends accumulated by loops, then \( \text{PMap}(\Gamma) \) is not locally CB.

**Proof.** Let \( K \) be any compact set in \( \Gamma \). Because \( \Gamma \) has infinitely many ends accumulated by loops there is at least one component of \( \Gamma \setminus K \) with two or more ends accumulated by loops, call these ends \( e_- \) and \( e_+ \). Let \( \mathcal{P} = C_L \sqcup C_R \) be any partition of \( E(\Gamma) \) that separates \( e_- \) and \( e_+ \) and let \( \eta \) be a loop shift totally supported on \( \Gamma \setminus K \) with \( h^- = e_- \) and \( h^+ = e_+ \). Then \( \eta \in \mathcal{U}_K \) and \( \Phi_{\mathcal{P}}(\eta) = 1 \) so that \( \Phi_{\mathcal{P}}|_{\mathcal{U}_K} \) is a nontrivial homomorphism.
to \( \mathbb{Z} \). Furthermore, this restriction is continuous again by Dudley’s automatic continuity as \( \mathcal{U}_K \) is a clopen subgroup of \( \text{PMap}(\Gamma) \) (hence Polish).

Proposition 8.3. Let \( \Gamma \) be a locally finite, infinite graph with nonzero rank and \( |E_\delta(\Gamma)| < \infty \). Then \( \text{PMap}(\Gamma) \) is locally CB if and only if \( \Gamma \setminus \Gamma_c \) has only finitely many components \( T_1, \ldots, T_m \) such that \( |E(T_i)| = \infty \).

Proof. Suppose \( \Gamma \setminus \Gamma_c \) has finitely many components \( T_1, \ldots, T_m \) with infinite end spaces.

To construct \( K \) such that \( \mathcal{V}_K \) is CB, we use the following claim.

Claim. Let \( K \subset \Gamma \) be a connected subset. If \( \Gamma \setminus K \) induces a finite partition of \( E(\Gamma) \), then \( K \) is bounded.

Proof. Suppose \( K \) is bounded but \( \Gamma \setminus K \) yields an infinite partition of \( E(\Gamma) \). This means that the boundary of \( K \) has infinitely many vertices on \( K \), which contradicts to the fact that the graph is locally finite.

Now assume \( \Gamma \setminus \Gamma_c \) has finitely many components \( T_1, \ldots, T_m \), with \( |E(T_i)| = \infty \). Then there are finitely many points \( x_1, \ldots, x_m \) (some of them might be the same) in \( \partial \Gamma_c = (\Gamma \setminus \Gamma_c) \cap \Gamma_c \), such that \( \Gamma \setminus \{x_1, \ldots, x_m\} \) contains \( T_1, \ldots, T_m \) as components. Then just let \( K_1 \) be the minimal spanning tree of \( \{x_1, \ldots, x_m\} \).

Moreover, as we have finitely many ends accumulated by loops, say \( n \), consider a partition \( \mathcal{P} \) of \( E(\Gamma) \) with \( |\mathcal{P}| = n \) and each partition element exactly contains one end accumulated by loops. Then by the Claim, we can find a compact set \( K_2 \) which realizes the partition \( \mathcal{P} \). Finally, take \( K = K_1 \cup K_2 \). Then with this choice of \( K \) it follows that every element \( g \in \mathcal{U}_K \) has to fix the complementary components setwise, where each component is \( \Gamma_{N_i} \) for some \( N_i \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \), whose pure mapping class group is CB. Therefore, for each \( i \in \{1, 2, \ldots, n\} \) and for any compact set \( K' \) there exist a finite set \( \mathcal{F}_i \in \text{PMap}(\Gamma) \) and \( n_i \in \mathbb{Z}^+ \) such that \( \text{PMap}(\Gamma_{N_i}) \) can be isomorphically embedded into \( (\mathcal{F}_i \mathcal{U}_K)^{n_i} \). All in all, by taking \( \mathcal{F} = \bigcup_{i=1}^n \mathcal{F}_i \) and \( n_0 = \sum_{i=1}^n n_i \), we have that \( \mathcal{U}_K \subset (\mathcal{F} \mathcal{U}_K)^{n_0} \), proving that \( \mathcal{U}_K \) is CB, which implies \( \text{PMap}(\Gamma) \) is locally CB.

Conversely, suppose \( \Gamma \setminus \Gamma_c \) has infinitely many components with infinite end spaces. Then given any compact set \( K \), there is a component \( A \) of \( \Gamma \setminus K \) that has positive rank with \( A \setminus \Gamma_c \) with infinite end space. We can apply Proposition 6.4 to build a length function on \( \text{PMap}(\Gamma) \) which is unbounded on \( \mathcal{U}_K \). Thus no identity neighborhood is CB.

A Ultrametrics are 0-Hyperbolic

In this appendix we give a proof that every ultrametric space is 0-hyperbolic. This is a well known fact [7, Exercise 1] but as of yet we have not found a written proof in the literature. Recall that Gromov \( \delta \)-hyperbolicity can be illustrated with the Gromov product as follows: Given a metric space \( (X, d) \), define a Gromov product \( (x, y)_w := \frac{1}{2}\{d(x, w) + d(y, w) - d(x, y)\} \) for \( w, x, y \in X \). Then \( X \) is Gromov \( \delta \)-hyperbolic if and only if

\[
d(x, y)_w \geq \min\{d(x, z)_w, d(y, z)_w\} - \delta,
\]

for all \( w, x, y \in X \).

Lemma A.1. Every ultrametric space is Gromov 0-hyperbolic.
Proof. Suppose \((X, d)\) is ultrametric space. We first paraphrase what it means to be 0-hyperbolic:

\[(x, y)_w \geq \min \{ (x, z)_w, (y, z)_w \} \]
\[\Leftrightarrow d(x, w) + d(y, w) - d(x, y) \geq \min \{ d(x, w) + d(z, w) - d(x, z), d(y, w) + d(z, w) - d(y, z) \} \]
\[\Leftrightarrow d(x, w) + d(y, w) - d(x, y) \geq d(x, w) + d(z, w) - d(x, z) \quad \text{OR} \]
\[d(x, w) + d(y, w) - d(x, y) \geq d(y, w) + d(z, w) - d(y, z), \]
\[\Leftrightarrow d(x, z) + d(y, w) \geq d(x, y) + d(z, w) \quad \text{OR} \quad d(x, w) + d(y, z) \geq d(x, y) + d(z, w). \]

Now, for the sake of contradiction, suppose

\[d(x, z) + d(y, w) < d(x, y) + d(z, w) \quad \text{AND} \]
\[d(x, w) + d(y, z) < d(x, y) + d(z, w). \]

For the notational simplicity, write:

\[A_1 := d(x, z), \quad A_2 := d(y, w) \]
\[B_1 := d(y, z), \quad B_2 := d(x, w) \]
\[C_1 := d(x, y), \quad C_2 := d(z, w) \]

Refer to Figure 10. Then our assumption \((*)\) can be rewritten as:

\[A_1 + A_2 < C_1 + C_2 \quad \text{AND} \quad B_1 + B_2 < C_1 + C_2. \]

Note that the ultrametric property is equivalent to saying that every triangle in Figure 10 is isosceles, with the two identical sides no smaller than the third. From this observation, we divide into two cases in regards to the triangle \(\triangle xyz\).

**Case 1.** \(C_1 \leq A_1 = B_1.\)

We proceed as follows:

\[A_2 < C_2, \quad B_2 < C_2 \quad \therefore \text{ (i), and (ii)} \]
\[B_1 = C_2 > A_2 \quad \therefore \text{ Triangle } \triangle yzw \]
\[A_1 = C_2 > B_2 \quad \therefore \text{ Triangle } \triangle xzw \]
\[C_1 > A_2, \quad C_1 > B_2 \quad \therefore \text{ (i), and (ii)}. \]
Now in Triangle $\triangle xyw$, we have $C_1$ is the only large side and the other two $A_2, B_2$ are strictly smaller. This contradicts $\triangle xyw$ being isosceles with the two identical sides are the largest among the three sides.

**Case 2.** $C_1 = A_1 > B_1 \quad OR \quad C_1 = B_1 > A_1$.

Note that our situation is symmetric under interchanging $A_i$’s with $B_i$’s, so it suffices to consider when $C_1 = A_1 > B_1$. Now:

\[
\begin{align*}
A_2 < C_2 & \quad \therefore (i) \\
B_1 = C_2 > A_2 & \quad \therefore \text{Triangle } \triangle yzw \\
C_1 > B_2 & \quad \therefore (ii) \\
A_2 = C_1 > B_2 & \quad \therefore \text{Triangle } \triangle xyw \\
A_1 < C_2 & \quad \therefore (i) \\
B_2 = C_2 > A_1 & \quad \therefore \text{Triangle } \triangle xzw.
\end{align*}
\]

However, now we have a contradiction that $A_1 > B_1 > A_2 > B_2 > A_1$. □

**References**


