# Analysis Qualifying Exam

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## Measure Theory

**Measure spaces, measurable functions and positive measures, integration, convergence*.**

`Examples.` 1) $f_n(x) = n^{-1}\chi_{(0,n)(x)}$, 2) $f_n(x) = \chi_{(n,n+1)}(x)$, 3) $f_n(x) = n\chi_{[0,1/n]}(x)$, 4) $L^1$ convergent but oscillating.

`Definition.` *pointwise convergence*, *uniform convergence*, and *convergence in* $L^p$.

`Definition.` We say that $\{f_n\}$ of measurable complex-valued functions on $(X, \mathscr{M}, \mu)$ is *Cauchy in measure* if for every $\epsilon > 0$,

$$\mu\big(\{x : |f_n(x) - f_m(x)| \geq \epsilon\}\big) \to 0 \text{ as } m, n \to \infty,$$

and that $\{f_n\}$ *converges in measure* to $f$ for every $\epsilon > 0$,

$$\mu\big(\{x : |f_n(x) - f(x)| \geq \epsilon\}\big) \to 0 \text{ as } n \to \infty.$$

`Proposition.` If $f_n \to f$ in $L^1$, then $f_n \to f$ in measure.

`Theorem.` Suppose that $\{f_n\}$ is Cauchy in measure. 1) Then there is a measurable function $f$ such that $f_n \to f$ in measure, 2) and there is a subsequence $\{f_{n_j}\}$ that converges to f a.e. Moreover, 3) if also $f_n \to g$ in measure, then $g = f$ a.e.

`Corollary.` If $f_n \to f$ in $L^1$, there is a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ a.e.

`Theorem. (Egoroff)` Suppose that $\mu(X) < \infty$, and $\{f_n\}$ and $f$ are measurable complex-valued functions on $X$ such that $f_n \to f$ a.e. Then for every $\epsilon > 0$ there exist $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on $E^c$. Sometimes it is called *almost uniform convergence.*

`Theorem. (Generalized Egoroff)` If we replace finite space hypothesis as $\{f_n\}$ is dominated by $L^1$ function, then the result still holds.

`Theorem. (Lusin)` If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

**Construction of Lebesgue measure, properties of Lebesgue measure*.**

`Theorem.` 1) If $F:\mathbb{R}\to\mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b)-F(a)$ for all $a,b$. 2) If $G$ is another such function, we have $\mu_F = \mu_G$ iff $F-G$ is constant. 3) Conversely, if $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((0,x]) & \text{if } x>0, \\ 0 & \text{if } x-0, \\ -\mu((x,0]) & \text{if } x<0, \end{cases}$$

then $F$ is increasing and right continuous, and $\mu=\mu_F$.

`Definition.` The completion of a Borel measure $\mu_F$ is called the *Lebesgue-Stieltjes measure* associated to $F$.

`Lemma.` Let $\mu$ be a Lebesgue measure. For any $E\in\mathscr{M}_\mu$,

$$\mu(E) = \inf\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j)) : E\subset\bigcup_{j=1}^{\infty}(a_j,b_j)\right\}.$$

`Theorem.` If $E\in\mathscr{M}_\mu$, then

$$\begin{aligned}\mu(E) &= \inf\{\mu(U) : E\subset U \text{ and } U \text{ is open}\}\\ &= \sup\{\mu(K) : K\subset E \text{ and } K \text{ is compact}\}.\end{aligned}$$

`Theorem.` If $E\subset\mathbb{R}$ m the following are equivalent.

a. $E\in\mathscr{M}_\mu$.

b. $E=V\setminus N_1$ where $V$ is a $G_\delta$ set and $\mu(N_1)=0$.

c. $E=H\cup N_2$ where $H$ is an $F_\sigma$ set and $\mu(N_2)=0$.

`Proposition.` If $E\in\mathscr{M}_\mu$ and $\mu(E)<\infty$, then for every $\epsilon>0$ there is a set $A$ that is a finite union of open intervals such that $\mu(E\triangle A)<\epsilon$.

`Definition.` The Lebesgue-Stieltjes measure associatd to the function $F(x)=x$ is called *Lebesgue measure* and denote it by $m$. The domain of $m$ is called the class of *Lebesgue measurable sets*, and we denote it by $\mathscr{L}$.

`Theorem.` If $E\in\mathscr{L}$, then $E+s\in\mathscr{L}$ and $rE\in\mathscr{L}$ for all $s,r\in\mathbb{R}$. Moreover, $m(E+s)=m(E)$ and $m(rE)=|r|m(E)$.

**Fubini's Theorem.**

**Signed measures*, Radon-Nikodym theorem*.**

`Definition.` A *signed measure* on $(X,\mathscr{M})$ is a function $\nu:\mathscr{M}\to[-\infty,\infty]$ such that 1) $\nu(\emptyset)=0$, 2) $\nu$ assumes at most one of the values $\pm\infty$, 3) and $\nu$ has countable additivity.

`Definition.` A *positive measure*

`Definition.` An *extended $\mu$-integrable* function on $(X, \mathscr{M}, \mu)$ is a measurable function so that at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite.

`Proposition.` Let $\nu$ be a signed measure on $(X, \mathscr{M})$. 1) If $\{E_j\}$ is an increasing sequence on $\mathscr{M}$, then $\nu(\cup E_j) = \lim \nu(E_j)$. 2) If $\{E_j\}$ is a decreasing sequence in $\mathscr{M}$ and $|\nu(E_1)| < \infty$, then $\nu(\cap E_j) = \lim \nu(E_j)$.

`Definition.` If $\nu$ is a signed measure on $(X, \mathscr{M})$, a set $E \in \mathscr{M}$ is called *positive / negative / null* for $\nu$ is $\nu(F) \geq 0$ / $\leq 0$ / $= 0$ for all $F \in \mathscr{M}$ and $F \subset E$.

`Lemma.` 1) Any measurable subset of a positive set is positive. 2) Union of any countable family of positive sets is positive.

`Theorem. (Hahn decomposition)` If $\nu$ is a signed measure on $(X, \mathscr{M})$, there exists a positive set $P$ and a negative set $N$ for $\nu$ such that $P \cup N = X$ and $P \cap N = \emptyset$. If $P', N'$ is another such pair, then $P \triangle P'$ and $N \triangle N'$ are null for $\nu$.

`Definition.` The decomposition $X = P \cup N$ of $X$ as the disjoint union of a positive set and a negative set is called a *Hahn decomposition* for $\nu$.

`Definition.` Two signed measures $\mu$ and $\nu$ on $(X, \mathscr{M})$ are *mutually singular* if there exists $E, F \in \mathscr{M}$ such that $E \cap F = \emptyset, E \cup F = X$, $E$ is null for $\mu$, and $F$ is null for $\nu$, denoted by $\mu \perp \nu$.

`Theorem. (Jordan decomposition)` If $\nu$ is a signed measure, there exist unique positive measure $\nu^+$ and $\nu^-$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

`Definition.` The measure $\nu^+$ and $\nu^-$ are called *positive and negative variations* of $\nu$. $\nu = \nu^+ - \nu^-$ is called the `Jordan decomposition` of $\nu$. The `total variation` of $\nu$ to be the measure $|\nu| \equiv \nu^+ + \nu^-$.

`Definition.` A signed measure $\nu$ is called *finite / $\sigma$-finite* if $|\nu|$ is finite / $\sigma$-finite.

`Definition.` Let $\nu$ be a signed measure and $\mu$ be a positive measure on $(X, \mathbb{M})$. Then $\nu$ is *absolutely continuous* with respect to $\mu$ and write $\nu \ll \mu$ if $\nu(E) = 0$ for every $E \in \mathscr{M}$ for which $\mu(E) = 0$.

`Theorem.` Let $\nu$ be a finite signed measure and $\mu$ a positive measure on $(X, \mathscr{M})$. Then $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there is $\delta > 0$ such that $\mu(E) < \delta \implies |\nu(E)| < \epsilon$.

`Corollary.` If $f \in L^1(\mu)$, for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta \implies |\int_E f d\mu| < \epsilon$.

`Lemma.` Suppose that $\nu$ and $\mu$ are finite measure on $(X, \mathscr{M})$. Either 1) $\nu \perp \mu$ or 2) there exist $\epsilon > 0$ and $E \in \mathscr{M}$ such that $\mu(E) > 0$ and $\nu \geq \epsilon\mu$ on $E$.

`Theorem. (Radon-Nikodym)` Let $\nu$ be a $\sigma$-finite measure and $\mu$ a $\sigma$-finite positive measure on $(X, \mathscr{M})$. Then there exist unique $\sigma$-finite signed measure $\lambda$, $\rho$ on $(X, \mathscr{M})$ so that

$$\lambda \perp \rho, \quad \rho \ll \mu, \quad \text{and} \quad \nu = \lambda + \rho.$$

Moreover, there is an extended $\mu$-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$ and any two such functions are equal $\mu$-almost everywhere.

`Definition.` The decomposition $\nu = \lambda + \rho$ is caled the *Lebesgue decomposition* of $\nu$ with respect to $\mu$. In the case of $\nu \ll \mu$, $d\nu = f d\mu$ for some $f$, and $f$ is called the *Radon-Nikodym derivative* of $\nu$ with respect to $\mu$ denoted by $d\nu = d\nu/d\mu \cdot d\mu$.

`Proposition.` Suppose that $\nu$ is a $\sigma$-finite signed measure and $\mu, \lambda$ are $\sigma$-finite measure on $(X, \mathscr{M})$ such that $\nu \ll \mu$ and $\mu \ll \lambda$. 1) If $g \in L^1(\nu)$, then $g \cdot d\nu/d\mu \in L^1(\mu)$ and

$$\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu.$$

2) We have $\nu \ll \lambda$ and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}, \quad \lambda - \text{ almost everywhere.}$$

`Corollary.` If $\mu \ll \lambda$ and $\lambda \ll \mu$, then $d\lambda/d\mu \cdot d\mu/d\lambda = 1$ almost everywhere (with respec to $\lambda$ or $\mu$.)

**Vitali covering theorem, Lebesgue density theorem and Lebesgue differentiation theorem.**

# FUNCTIONAL ANALYSIS

**Hilbert space*, Cauchy-Schwartz*, parallelogram law*, continuous linear functionals.**

`Definition.` Let $H$ be a complex vector space with a map $\langle\cdot,\cdot\rangle : H\times H \to \mathbb{C}$ is a *complex inner product space* if 1) $\langle v,v\rangle \geq 0$ for all $v\in H$ and $\langle v,v\rangle = 0 \iff v = 0_H$, 2) $\langle v,w\rangle = \overline{\langle w,v\rangle}$, and 3) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ and $\langle \lambda u,w\rangle = \lambda\langle u,w\rangle$.

`Definition.` Let $H$ be a real vector space with a map $\langle\cdot,\cdot\rangle : H\times H\to\mathbb{R}$ is a *real inner product space* if it satisfies same three conditions.

`Theorem. (Cauchy-Schwarz)` $|\langle v,w\rangle| \leq \sqrt{\langle v,v\rangle}\cdot\sqrt{\sqrt{\langle w,w\rangle}} = \|v\|\cdot\|w\|$.

`Corollary. (Minkowski)` $\sqrt{\langle v+w,v+w\rangle} \leq \sqrt{\langle v,v\rangle}+\sqrt{\langle w,w\rangle}$, i.e. $\|v+w\|\leq\|v\|+\|w\|$.

`Definition.` If an inner product space $H$ with induced distance $d$ gives a complete metric space, then we say $H$ is a `Hilbert space`.

`Theorem. (Parallelogram)` $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$.

`Theorem. (Polarization)` $\langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2\right)$.

`Theorem.` An inner product $\langle\cdot,\cdot\rangle : H\times H\to\mathbb{C}$ is continuous and $\cdot\langle\cdot,\cdot\rangle : \mathbb{C}\times H\times H\to\mathbb{C}$ is also continuous.

`Theorem.` Every closed convex subset of a Hilbert space has a unique element of least norm.

`Definition.` If $\langle u,v\rangle = 0$, we say that $u,v$ are *orthogonal*, denoted by $u\perp v$. If $S\subset M$, then we define *orthogonal complement* $S^\perp = \{v\in M : \langle v,u\rangle = 0, \forall u\in S\}$.

`Lemma.` $S^\perp$ is always a closed subspace.

`Theorem.` Let $H$ be a Hilbert space and let $M$ be a closed subspace of $H$. Then the followings are true:

- There exists a unique nearest projection point of $x$ onto $M$, so called $x_M$.
- $P_M : H\to M$ where $x\mapsto x_M$ is linear.
- $x - P_M x\in M^\perp$.
- $H = M\oplus M^\perp$, i.e. for $x\in H$, there exists a unique $x_M\in M$ and $x_{M^\perp}\in M^\perp$ so that $x = x_M + x_{M^\perp}$.
- (Pythagorian) $\|x\|^2 = \|P_M x\|^2+\|P_{M^\perp}x\|^2$.

`Lemma.` Suppose $V$ is a 1-dimensional subspace of a Hilbert space. Then $P_v x = \langle x,s\rangle s$ where $\|s\| = 1$ and $\mathbb{C}\cdot s = V$.

`Corollary.` If $V$ is a $n$-dimensional subspace of a Hilbert space, then $V = \mathbb{C}u_1\oplus\mathbb{C}u_2\oplus\cdots\oplus\mathbb{C}u_n$ where $\|u_i\| = 1$ and $\langle u_i,u_j\rangle = \delta_{i,j}$. Moreover,

$$P_V x = \sum_{i=1}^n \langle x,u_i\rangle u_i.$$

`Definition.` A subset $S$ of a Hilbert space $H$ is called *orthonormal set* if $u,v\in S \implies$ 1) $\langle u,v\rangle = 0$ if $u\neq v$ and 2) $\langle u,u\rangle = 1$ for every $u\in S$.

`Theorem. (Bessel)` If $S$ is a orthonormal set in a Hilbert set $H$, then $\sum_{s\in S}|\langle x,s\rangle|^2\leq\|x\|^2$.

`Theorem.` Let $H$ be a Hilbert space. Then the followings are equivalent:

1) $S$ is a maximal orthonormal set in $H$.

2) linear combinations (of course, finite) of elements of $S$ are dense in $H$.

3) (Parseval) $\sum_{s\in S}|\langle x,s\rangle|^2 = \|x\|^2$ for all $x \in H$.

4) $x = \sum_{s\in S}\langle x,s\rangle s$, i.e. if $S$ is countable, then $\|x - \sum_{i=1}^n \langle x,s_i\rangle s_i\| \to 0$ as $n \to \infty$.

`Definition.` Two Hilbert spaces $H_1, H_2$ are *isomorphic* if there exists a bijective linear map $\Phi : H_1 \to H_2$ such that $\|\Phi(v_1)\|_{H_2} = \|v\|_{H_1}$ for all $v \in H_1$.
`Theorem.` Two Hilbert spaces are isomorphic if and only if they have maximal orthonormal set with the same cardinality.
`Definition.` A maximal orthonormal set is called *orthonormal basis.*
`Corollary.` A Hilbert space is separable if and only if it has at most countable orthonormal basis.

**Jensen's inequality*, Holder's inequality, Minkowski inequality, $L^p$ spaces are complete*.**
`Theorem. (Jensen's inequality)` If $(X, \mathcal{M}, \mu)$ is a measure space with finite measure, $g : X \to (a,b)$ is in $L^1(\mu)$, and $F$ is convex on $(a,b)$, then

$$F\left(\int g d\mu\right) \le \int F \circ g d\mu.$$

`Proposition.` $L^p$ is complete with $\|\cdot\|_p$ for all $1 \le p \le \infty$.
`Lemma.` Let $(V, \|\cdot\|)$ be a normed vector space. Then $V$ is complete if and only if for every sequence $\{w_i\} \subset V$ and $\sum_i \|w_i\| < \infty$ then $\lim_{n\to\infty} s\sum_{i=1}^n w_i \in V$.
`Proposition.` $(L^p, \|\cdot\|_p)$ is separable for all $1 \le p < \infty$. $(L^\infty, \|\cdot\|_\infty)$ is not separable.

**Banach-Steinhaus*, Open mapping theorem, closed graph theorem, Hahn-Banach theorem*.**
`Theorem. (Banach-Steinhaus)` Let $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ be Banach spaces. Suppose $\{F_n\}$ be a sequence of bounded linear operator from $B_1$ to $B_2$. Then either

1) $\exists M < \infty$ such that $\|F_n\|_{\text{op}} \le M$ for every $n$, or

2) $\exists x \in B_1$ such that $\sup_n \|F_n x\|_{B_2} = \infty$.

`Theorem. (Hahn-Banach, real case)` Let $(B, \|\cdot\|)$ be a Banach space and let $V$ be a closed subspace of $B$. Suppose $\varphi : V \to \mathbb{R}$ is a bounded linear function. Then there is a map $\Phi : B \to \mathbb{R}$ such that 1) $\Phi|_V = \varphi$ and 2) $\|\Phi\|_B = \|\varphi\|_V$.
`Theorem. (Hahn-Banach, complex case)` Let $B$ be a complex Banach space and let $V$ be a closed subspace of $B$. Suppose $\varphi : V \to \mathbb{C}$ is a bounded linear function. Then there exists a bounded linear map $\Phi : B \to \mathbb{C}$ such that 1) $\Phi|_V = \varphi$ and 2) $\|\Phi\|_B = \|\varphi\|_V$.

**Linear functionals on $L^p$, $p < \infty$*.**
`Theorem.` If $(X, \mathcal{M}, \mu)$ is a $\sigma$-finite and $1 \le p < \infty$, then $(L^p(X,\mu))^* = L^q(X,\mu)$. That is, for $\Phi \in (L^p)^*$, there exists unique $g \in L^q$ such that $\Phi(f) = \int f g d\mu$ for every $f \in L^p$ and $\|\Phi\|_{\text{op}} = \|g\|_q$.

`Lemma.` If $1 \le p < \infty$ and $f \in L^p$, there exists $h \in L^p$ such that $\int f h d\mu = \|f\|_q \cdot \|h\|_p$. If $f \ge 0$, $h = f^q$, otherwise $h = \overline{\operatorname{sgn} f} |f|^q$.

**Weak topology.**

**Riesz representation theorem*.**

`Definition.` A topological space is called *locally compact* if every point has a compact neighborhood.

`Proposition.` If $X$ is an LCH space, $U \subset X$ is open, and $x \in U$, there is a compact nbd $N$ of $x$ such that $N \subset U$.

`Lemma. (Urysohn)` If $X$ is an LCH space and $K \subset U \subset X$ where $K$ is compact and $U$ is open, there exists $f \in C(X, [0,1])$ such that $f = 1$ on $K$ and $f = 0$ outside a compact subset of $U$.

`Lemma. (Partition of unity)` Let $X$ be an LCH space, $K$ a compact subset of $X$, and $\{U_j\}$ and finite open cover of $K$. There is a partition of unitiy on $K$ subordinate to $\{U_j\}$ consisting of compactly supported functions.

`Definition.` Let $X$ be a LCH space. 1) $C_c(X)$ is the space of continuous functions on $X$ with compact support. 2) A linear functional $\Phi$ on $C_c(X)$ is called *positive* if $\Phi(f) \ge 0$ whenever $f \ge 0$.

`Lemma.` Let $X$ be a LCH space. If $\Phi$ is positive linear functional on $C_c(X)$, for each compact $K \subset X$ there is a constant $C_{K,\Phi}$ such that $|\Phi(f)| \le C_{K,\Phi} \|f\|_u$ for all $f \in C_c(X)$ such that $\operatorname{supp}(f) \subset K$.

`Definition.` A *Radon measure* on a LCH $X$ is a Borel measure that is 1) finite on all compact sets, 2) outer regular on all Borel sets, and 3) inner regular on all open sets. We denote that space of complex Radon measures on $X$ by $M(X)$.

`Theorem. (Riesz representation theorem)` Let $X$ be a LCH space. If $\Phi$ is a positive linear functional on $C_c(X)$, there is a unique Radon measure $\mu$ on $X$ such that $\Phi(f) = \int f d\mu$ for all $f \in C_c(X)$. Moreover, $\mu$ satisfies

$$\mu(U) = \sup\{\Phi(f) : f \in C_c(X), f \prec U\} \text{ for all open } U \subset X,$$

and

$$\mu(K) = \inf\{\Phi(f) : f \in C_c(X), f \ge \chi_K\} \text{ for all compact } K \subset X.$$

`Theorem. (Riesz-Markov)` Let $X$ be a LCH space. Then

$$\left(C_0(X), \|\cdot\|_{\sup}\right)^* = (M(X), \|\cdot\|_M = |\cdot|(X))$$

where $C_0(X)$ is a $\|\cdot\|_{\sup}$ closure of $C_c(X)$.

**Fourier transform and properties*, Inversion theorem*, Parseval's theorem*, Plancherel's Theorem*.**

`Definition.` If $f, g : \mathbb{R}^n \to \mathbb{C}$, then these convolution is defined as $f * g(x) = \int f(x-y) g(y) d\lambda^n(y)$.

`Properties.` a) (commutativity) $f * g = g * f$, b) (associativity) $\underline{(f*g)*h = f*(g*h)}$, c) $\tau_z f(x) = f(x-z)$, then $\tau_z(f*g) = \tau_z f * g = f * \tau_z g$, d) $\operatorname{supp}(f*g) \subset \overline{\{x+y : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}}$. Then if $f, g \in C_c(\mathbb{R}^n) \implies f * g \in C_c(\mathbb{R}^n)$.

`Theorem.` If $f \in L^p$ and $g \in L^q$, then $f * g$ exists for every $x$, is bounded, uniformly continuous, and $\lim_{|x| \to \infty} f * g(x) = 0$.

`Theorem.` Let $f \in L^p$ for $1 \le p \le \infty$, then $\lim_{z\to 0} \|\tau_z f - f\|_p = 0$.
`Theorem.` $f, g \in L^1$. Then $f * g$ exists almost everywhere, $f * g \in L^1$, and $\|f * g\|_1 \le \|f\|_1 \|g\|_1$.
`Theorem.` If $f \in C^1$, $\partial f/\partial x_i$ is uniformly bounded, and $g \in L^1$, then $f * g \in C^1$ and $\partial f * g = \partial f/\partial x_i * g$.
`Corollary.` If $f \in C^{|\alpha|}$, $\partial^\alpha f$ is uniformly bounded, and $g \in L^1$, then $\partial^\alpha (f * g) = \partial^\alpha f * g$ and $f * g \in C^{|\alpha|}$.
`Theorem.` Let $\phi \in L^1$ and $\int \phi d\lambda^n = a$, and let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Then a) $\lim_{t\to 0} f * \phi_t = af$ in $L^1$ if $f \in L^1$, b) $f$ is bounded and uniformly continuous, then this converges uniformly.
`Definition.` Let $f \in L^1$. The *Fourier transform* of $f$ is $\hat{f}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} d\lambda^n(x)$.
`Proposition.`

a) $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$

b) $\widehat{\tau_y f} = e^{2\pi i \xi \cdot y} \hat{f}$

c) $\widehat{f * g} = \hat{f} \cdot \hat{g}$

d) If $T \in GL_n(\mathbb{R})$, then $\widehat{f \circ T} = |\det T|^{-1} \hat{f} \circ (T^{-1})^*$

e) $\hat{f} \in C_0$ and $\|\hat{f}\|_{\sup} \le \|f\|_1$

f) If $|x|^k f(x) \in L^1$ for $k \le n$ and $|\alpha| \le n$, then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} = \widehat{(-2\pi i x)^\alpha f}$

g) If $f \in C^k, \partial^\alpha f \in L^1$, and for every $|\alpha| \le k-1$, then $\widehat{\partial^\alpha f} = (2\pi i \xi)^\alpha \hat{f}$

`Definition.` The *Schwartz class* $\{g : \mathbb{R}^n \to \mathbb{C} : g \in C^\infty, \sup_{x\in\mathbb{R}^n} (|x|^k + !)|\partial^\alpha f(x)| < \infty, \forall \alpha, k\}$. It decays always faster than any polynomials.
`Corollary.` The Fourier transform sends Schwartz class into Schwartz class.
`Lemma.` If $f, g \in L^1$, then $\int \hat{f} g = \int f \hat{g}$.
`Definition.` The *Fourier inversion* of $f$ is $\check{f} = \int f(\xi) e^{2\pi i \xi \cdot x} d\lambda^n(\xi)$.
`Theorem. (Fourier inversion)` If $f, \hat{f} \in L^1$, then there exists continuous function $g$ such that $g = f$ almost everywhere and $(\hat{f})^\vee = g$.
`Corollary.` If $f \in L^1$ and $\hat{f} \equiv 0$, then $f = 0$ in $L^1$. If $f, g \in L^1$ and $\hat{f} = \hat{g}$, then $f = g$ in $L^1$.
`Corollary.` The Fourier transform is bijective on Schwartz class.
`Theorem. (Plancherel)` If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$. Also, there exists $\mathscr{F} : L^2 \to L^2$ unitary such that $\mathscr{F}(f) = \hat{f}$ for all $f \in L^1 \cap L^2$.
`Definition.` Consider $C(S^1) = \{f \in C[0,1] : f(0) = f(1)\}$. The *Fourier series* of $f \in C(S^1), L^1(S^1), L^2(S^2)$ is $\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$ where $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$.
`Theorem.` If $f \in L^2(S^1)$, then $\|f(x) - \sum_{-k}^{k} \hat{f}(n) e^{2\pi i n x}\|_2 \to 0$ as $k \to \infty$.
`Lemma.` $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(S^1)$.
`Theorem. (Parseval)` Suppose that $f, g \in L^2(S^1)$. Then $\sum \hat{f}(n) \cdot \overline{\hat{g}(n)} = \int f \overline{g}$.
`Theorem.` There exists a residual set of $f \in C(S^1)$ such that $\sum \hat{f}(n) e^{2\pi i n x}$ does not converge at zero.
`Theorem.` $\Phi : (L^1(S^1), \|\cdot\|_1) \to (C_0(\mathbb{Z}), \|\cdot\|_{\sup})$ is a linear norm-nonincreasing, injective, and is not a surjection.

# COMPLEX ANALYSIS

**Holomorphic functions, Cauchy-Riemann equations, Cauchy's Theorem, Cauchy's integral formula, Taylor series for holomorphic functions, Liouville's theorem, Runge's Theorem*.**
`Theorem.` `(Runge)` Let $K$ be a compact set in $\mathbb{C}$ and let $f$ be a holomorphic function on an open set containing $K$. Choose a set $A$ of representatives of connected components of $S^2 - K$. Fix $\epsilon > 0$. Then there exists a rational function $R$ with poles in $A$ such that $|f(z) - R(z)| < \epsilon$.

**Isolated singularities*, Laurent series*, residue theorem*, applications to compute definite integrals.**
`Definition.` Let $U$ be an open set and let $f$ be an analytic function on $U\backslash\{z_0\}$. Then $f$ has a *removable singularity* at $z_0$ if there exists an analytic function $\tilde{f}$ on $U$ such that $\tilde{f}|_{U\backslash\{z_0\}} f$.
`Proposition.` Suppose there exists $r > 0$ such that $\sup_{|z-z_0|<r} |f(z)| \leq M < \infty$. Then $z_0$ is a removable singularity.
`Proposition.` Suppose there exists $r > 0$ such that $\{f(z) : |z - z_0| < r, z \neq z\}$ is not dense in $\mathbb{C}$. Then there exists $m \geq 0$ and a holomorphic function $g(z)$ on $\{z : |z - z_0| < r\}$ such that $g(z_0) \neq 0$ and $f(z) = g(z)/(z - z_0)^m$.
`Definition.` If $m > 0$ in the previous proposition, then it has a *pole* at $z_0$ of order $m$. If an isolated singularity is not a removable singularity or pole, it is called *essential singularity*.
`Theorem.` Suppose $c$ is an isolated singularity of $f \in \mathscr{O}(U)$ where $U$ is an open set. Then $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-c)^k$.

1) Suppose $a_k = 0$ for $k < 0$. Then $f(z) = \sum_{k=0}^{\infty} a_k(z-c)^k$ and converse is true. This means that $f$ has removable singularity at $c$.

2) Suppose $a_k = 0$ for $k < -N$ and $a_{-N} \neq 0$ where $N$ is positive integer. Then $f$ has a pole at $c$ of order $N$.

3) $f$ has essential singularity at $c \iff a_k \neq 0$ for infinitely many negative $k$.

`Theorem.` Let $f \in \mathscr{O}(U)$ and $U$ is a region. Let $Z(f) = \{z \in U : f(z) = 0$. Then $Z(f)$ is an isolated subset of $U$, unless $f \equiv 0$.
`Corollary.` 1) $Z(f)$ is countable. 2) $\cup_{n=1}^{\infty} Z(f^{(n)})$ is countable.
`Definition.` Let $f$ be an analytic function on $D(a; R_1, R_2)$. Then it can be expanded as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-a)^k, \quad a_k = \frac{1}{2\pi i}\int_{\gamma_i} \frac{f(\xi)}{(\xi - a)^{k+1}} d\xi$$

where $i = 1$ if $k < 0$ and $i = 2$ if $k \geq 0$. This expansion is called the *Laurent expansion* of $f$.
`Theorem.` `(First Residue)` Let $f \in \mathscr{O}(U)$ where $U$ is an open set. Suppose $c \in U$ is an isolated singularity of $f$. Then

$$\int_{C_r} f(z)dz = 2\pi i a_{-1},$$

where $a_{-1}$ is the coefficient corresponding to $k = -1$ in Laurent expansion of $f$ centered at $z = c$.

`Definition.` $a_{-1}$ is defined as *residue* of $f$ at $c$, denoted by $\text{Res}(f;c)$.

**Rouche's Theorem*, Maximum principle*.**

`Theorem. (Rouche)` Let $\gamma$ be a closed path homotopic to 0 in an open set $\Omega$ such that $\text{Ind}_\gamma(z) = 0$ or 1 for every $z \in \mathbb{C}\backslash\gamma^*$. Suppose $f, g \in \mathscr{O}(\Omega)$ and satisfy

$$|f(z) - g(z)| < |g(z)| \quad \text{on} \quad \gamma.$$

Then $f$ and $g$ have the same number of zeros in $\Omega_1 = \{z \in \Omega : \text{Ind}_\gamma(z) = 1\}$.

`Theorem.` Suppose $f$ is a holomorphic function on a region $\Omega$ and $f$ is non-constant. Then $|f|$ does not attain a maximum inside $\Omega$.

**Conformal mappings, examples, Schwartz lemma, isometries of the hyperbolic plane, normal families*, Montel's theorem*, Riemann mapping theorem.**

`Definition.` A family $F$ of continuous functions $f$ defined on complete metric space $X$ mapping into another complete metric space $Y$ is called *normal* if every sequence of functions in $F$ contains a subsequence which converges almost uniformly.

`Definition.` A family $F$ of functions is *equicontinuous* if for every $\epsilon > 0$ there is $\delta > 0$ $d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$ for every $f \in F$.

`Definition.` A family $F$ of functions is *uniformly bounded* if for every compact subset, there exists $M > 0$ such that $\sup_{x\in K} |f(x)| < M$ for every $f \in F$.

`Theorem. (Arzera-Ascoli)` Let $X$ be a metric space which has a countable dense subset $E$. Let $C(X)$ be a family of all continuous and complex-valued function on $X$. If $F \subset C(X)$ is equicontinuous and uniformly bounded, then for every sequence $f_n \in F$ has a subsequence which converges to a function $f \in C(X)$.

`Theorem. (Montel)` Let $\Omega$ be a region in $\mathbb{C}$ and $F \subset \mathscr{O}(\Omega)$. Suppose $F$ is uniformly bounded on each compact subset of $\Omega$. Then every sequence $(f_n)$ in $F$ has a subsequence which converges almost uniformly.

**Infinite products*, Weierstrass factorization theorem*.**

`Definition.` Let $\{u_k\}$ be a sequence of complex number. Define $p_n = \prod_{k=1}^n u_k$. Suppose $\lim_{n\to\infty} p_n$ exists and equals $p$. Then we say that $\prod_{k=1}^\infty p_k$ *converges* to $p$.

`Theorem.` $\prod_{k=1}^\infty = p$ converges to a non-zero if and only if $\sum_{k=1}^\infty \log(u_k)$ converges to a number $\lambda$. In this case $p = e^\lambda$. Moreover, if $\sum_k \log(u_k)$ converges absolutely, then its product $\prod_k u_k$ can be arbitrary rearranged.

`Definition.` Let $\{u_k\}$ be a sequence of complex-valued function on some set $S$. The product function $\prod_{k=1}^\infty u_k$ converges *uniformly* if the sequence $\{p_n\}$ of partial products converges uniformly on $S$.

`Theorem.` Suppose $\{u_k\}$ be a sequence never zero functions on $S$ which are bounded on $S$. Suppose $\sum_{k=1}^\infty \log(u_k)$ converges uniformly to $\lambda(z)$. Then $\prod_k u_k$ converges uniformly to $p(z) = e^{\lambda(z)}$.

`Theorem.` Let $\{a_k\}$ be a sequence of functions on $S$. Suppose $\sum_{k=1}^\infty |a_k|$ converges uniformly on $S$. Thus, $\prod_k (1 + a_k)$ converges uniformly.

`Theorem.` Let $E_p(z) = (1-z)\exp(\sum_{k=1}^{p} z^k/k)$. 1) The only zero of $E_p$ occurs at $z=1$ with multiplicity 1. 2) If $|z| \le 1$, then $|E_p(z)-1| \le |z|^{p+1}$.
`Definition.` Let $\{z_k\}$ be a sequence of non-zero complex numbers converging to $\infty$. Let $\{p_k\}$ be a sequence of integers such that for every $r > 0$, $\sum_{k=1}^{\infty} |r/z_k|^{p_k+1} < \infty$. Define a complex function

$$f(z) = \prod_{k=1}^{\infty} E_{p_k}\left(\frac{z}{z_k}\right)$$

and it is called *Weierstrass product.*
`Theorem. (Weierstrass)` Let $\{z_k\}$ be a sequence converging to infinity. Then there exists entire function $f(z)$ such that $\{z_k\}$ are precisely the zeros of $f$.
`Theorem.` Suppose $f$ is a meromorphic function in $\Omega$, then $g = f \cdot f_{\text{st}}$ where $f_{\text{st}}$ is a Weierstrass product function derived by poles of $f$.

**Analytic continuation, monodromy.**
**Elliptic functions.**
**Picard's theorem.**