Functional Analysis

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Spaces

- A *vector space* over a field K ($\mathbb{R}$ or $\mathbb{C}$) is a set X with operations vector addition and scalar multiplication satisfy properties in section 3.1.
- [1] An *inner product space* is a vector space X with inner product $\langle\cdot,\cdot\rangle : X\times X \to K$ satisfying

- $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
- $\langle x, x\rangle \geq 0$ with $\langle x, x\rangle = 0 \iff x = 0$.

[2] An inner product induces a norm on X via $\|x\| = \sqrt{\langle x, x\rangle}$.
- [1] A *normed space* is a vector space X with norm $\|\cdot\| : X \to \mathbb{R}$ satisfying

- $\|x\| \geq 0$,
- $\|x\| = 0 \iff x = 0$,
- $\|\alpha x\| = |\alpha|\cdot\|x\|$,
- $\|x+y\| \leq \|x\| + \|y\|$.

[2] Not every normed space does not comes form inner product space (e.g. l^p with $p \neq 2$). [3] Every normed space induces a metric on X via $d(x,y) = \|x-y\|$.
- [1] A *metric space* is a set X together a metric $d(\cdot,\cdot) : X\times X \to \mathbb{R}$ satisfying

- $0 \leq d(x,y) < \infty$,

- $d(x,y) = 0 \iff x = y$,
- $d(x,y) = d(y,x)$,
- $d(x,y) \leq d(x,z) + d(z,y)$.

[2] Not every metric space comes from a normed space (e.g. a metric space cannot be a vector space).
- [1] Let (X,d) and $(Y,\tilde{d})$ be metric spaces. A mapping $T : X \to Y$ is *continuous* if $\forall x, x_0 \in X$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d(x,x_0) < \delta \implies \tilde{d}(Tx,Tx_0) < \epsilon$. [2] $|\|y\| - \|x\|| \leq \|y - x\| \implies x \mapsto \|x\|$ is continuous. [3] $\langle \cdot, \cdot \rangle : X \times X \to K$ is continuous.
- [1] A set M is *dense* in a metric space X if $X = \overline{M}$. [2] X is *separable* if there exists a countable dense subset. [3] l^p with $1 \leq p < \infty$ is separable. [4] l^∞ is not separable.
- Convergent sequence is a Cauchy sequence, but the reverse may not always be true.
- [1] A metric space X is *complete* if $\forall$ Cauchy sequence in X converges in X. [2] $\mathscr{C}[a,b]$ with $d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$ is complete. [3] $\mathscr{C}[a,b]$ with $d(x,y) = \int_a^b |x(t) - y(t)| dt$ is not complete. [4] Any finite dimensional normed space is complete.
- [1] A *Hilbert space* is a complete inner product space. [2] A *Banach space* is a complete normed space.
- [1, Completion theorem] For any metric/normed/inner product space X, there exists a unique (upto isometry) complete metric/normed/inner product space Y with a dense subset $W \subset Y$ such that W is isomorphic to X. *Isomorphism* means there is a bijection $T : X \to W$ such that $d(Tx, Ty) = d(x,y)$ / $\|Tx\| = \|x\|$ / $\langle Tx, Ty \rangle = \langle x, y \rangle$. [2] Completion of $\mathscr{C}[a,b]$ with $\|x\| = (\int_a^b |x(t)|^2 dt)^{1/2}$ is $L^2[a,b]$.
- A metric space does not have a notion of basis.
- [1] B is a *Hamel basis* for a vector space for a vector space X if every $x \in X \backslash \{0\}$ can be written uniquely as a linear combination (of a finite number) of vectors from B. [2] Every vector space $X \neq \{0\}$ has a Hamel basis (e.g. Schauder basis is not a Hamel basis).
- [1] $\{e_n\}_{n=1}^\infty$ is a *Schauder basis* for normed space X if $\forall x \in X$, there is a unique set of scalars (α_n) such that $x = \sum_{n=1}^\infty \alpha_n e_n$. [2] If X has a Schauder basis, then X is separable. However, not every separable space has a Schauder basis.
- [1] A subset M of a normed space is *total* if $X = \overline{\text{span} M}$. [2] A total orthonormal family of X is sometimes called an *orthonormal basis* (i.e. one can write $x = \sum \langle x, e_n \rangle e_n$). [3] Every Hilbert space $H \neq \{0\}$ has a total orthonormal basis. [4] A total orthonormal family can be uncountable. [5] If the set is countable, then it is also Schauder basis. [6] H: separable, then every orthonormal set is countable.

LINEAR OPERATORS AND FUNCTIONALS

- [1] $T : \mathscr{D}(T) \to Y$ ($\mathscr{D}(T) \subset X, X, Y$: vector spaces over some field K) is a *linear operator* if $T(x + y) = T(x) + T(y)$ and $T(\alpha x) = \alpha T(x)$. [2] A linear operator is called a *linear functional* if $Y = K$.
- [1] The *inverse operator* exists if $Tx = 0 \implies x = 0$. [2] If T^{-1} exists, then it is also a linear operator. [3] $(ST)^{-1} = T^{-1}S^{-1}$.

- [1] T is a *bounded linear operator* if there exists $c > 0$ such that $\|Tx\| \le c\|x\|$ for all x. [2] $\|T\| = \sup_{\|x\|=1, x\in X} \|Tx\| = \sup_{x\in X\setminus\{0\}} \|Tx\|/\|x\|$. [3] T is continuous $\Longleftrightarrow$ T is bounded, of course, T is a linear operator. [4] $T : \mathscr{D}(T) \to Y$ is a bounded linear operator with Banach space Y. Then there exists an extension $\tilde{T} : \overline{\mathscr{D}(T)} \to Y$ which is also bounded and linear with $\|T\| = \|\tilde{T}\|$.

Space of operators

- $B(X,Y)$ is the space of all bounded linear operator from X to Y. It is normed space with operator norms.
- [1] The *algebraic dual space* X^* of a vector space is set of all linear functionals on X. The *dual space* X' of a vector space X is set of all bounded linear functionals on X (i.e. $X' = B(X,K)$).
- [1] If Y is a Banach space, then so does $B(X,Y)$. [2] X' is a Banach space.
- Operators and functionals defined on finite dimensional vector spaces can be understood using bases and linear algebra.
- [Riesz representation theorem] $f \in H'$ can be represented by $f(x) = \langle x, \tilde{f}\rangle$ for some $\tilde{f} \in H$ with $\|\tilde{f}\| = \|f\|$.

Orthogonal complements and Direct sums

- [1] X is a metric space and $x \in X$. $M \subset X$. Then $d(x,M) := \inf_{y\in M} d(x,y)$. [2] X is a inner product space. $M \ne \emptyset$ and M is a convex subset of X. Then there exists a unique $y \in M$ such that $d(x,M) = d(x,y)$. [3] If M is a closed subspace, then there exists $z = x - y \perp Y$ for x. [4] Y is closed subspace of a Hilbert space, then $H = Y \oplus Y^\perp$. [5] Defines *projection* $P : x \mapsto y \in Y$ and *orthogonal projection* $P^\perp : x \mapsto z \in Y^\perp$.

Generalized Fourier series

- [1] Let $\{e_k\}$ be a orthogonal sequence. Then $\sum a_k e_k$ converges $\Longleftrightarrow$ $\sum_k |a_k|^2$ converges. [2] If (e_k) is a total orthogonal sequence, then $x = \sum_k \langle x, e_k\rangle e_k$ for every $x \in H$. [3, Parseval's identity] $\|x\|^2 = \sum_k |\langle x, e_k\rangle|^2$.

Banach fixed point theorem

- [1] Let (X,d) be a metric space. $T : X \to X$ is a *contraction* if there exists $\alpha \in (0,1)$ such that $\forall x,y \in X$ $d(Tx,Ty) \le \alpha d(x,y)$. [2, Banach fixed point theorem] If $X \ne \emptyset$ is complete and T is contraction, then there exists a unique fixed point $x \in X$ $(Tx = x)$. [3] A metric space X is *compact* if every sequence has convergent subsequence. [4] A compact set is closed and bounded, but the reverse is not true in general.

FUNDAMENTAL THEOREMS

- [1, Hahn-Banach] If $f: Z \to \mathbb{R}$, $Z \subset X$, and $f(x) \le \rho(x)$ for all $x \in Z$ where ρ is sublinear functional on X, then f has an extension from Z to X satisfying $f(x) \le \rho(x)$ for all $x \in X$. A *sublinear function* is a function satisfying positive homogeniety and subadditivity. [2] For $x \in X$, $\|x\| = \sup_{f \in X', f \neq 0} |f(x)|/\|f\|$. [3] Let $T \in B(X, Y)$. Then its *adjoint operator* $T^\times : Y' \to X'$ such that $(T^\times g)x = g(Tx)$ and $\|T^\times\| = \|T\|$.
- [1] X is *algebraic reflexive* if $c: X \to X^{**}$, $x \mapsto g_x$ is an isomorphism (i.e. bijective and preserves norm). [2] If X is *reflexive*, then $c: X \to X''$, $x \mapsto g_x$ is an isomorphism. [3] If a normed space is reflexive, then it is complete. [4] A Hilbert space is reflexive.
- [Uniform boundness] Let X be a Banach and Y be a normed space. Let T_n be a sequence in $B(X, Y)$ with $\|T_n x\| \le c_x$, $\forall x \in X$. Then there exists $c > 0$ such that $\|T_n\| \le c$, $\forall n$.
- [Point convergence] *Strong convergence*: $x_n \to x \Longleftrightarrow \lim_{n\to\infty} \|x_n - x\| = 0$. *Weak convergence*: $x_n \rightharpoonup x \Longleftrightarrow \lim_{n\to\infty} f(x_n) = x$, $\forall f \in X'$.
- [Operator convergence] *Uniform convergence*: $\|T_n - T\|_{\text{op}} \to 0$. *Strong convergence*: $\|T_n x - Tx\|_{\text{op}} \to 0$, $\forall x \in X$. *Weak convergence*: $|f(T_n x) - f(Tx)| \to 0$, $\forall x \in X$ and $\forall f \in Y'$.
- [Functional convergence] *Strong convergence*: $\|f_n - f\| \to 0$. *Weak * convergence*: $|f_n(x) - f(x)| \to 0, \forall x \in X$.
- [Open mapping] Let X, y be Banach spaces. If $T \in B(X, Y)$ is surjective, then it is open mapping. Moreover, if T is bijective, then there exists bounded T^{-1}.
- [Closed graph] Let X be a Banach space. If $T: \mathscr{D}(T) \subset X \to Y$ is a closed linear operator and $\mathscr{D}(T)$ is closed, then T is bounded (and used in unbounded operator theory).

SPECTRAL THEORY

- *Resolvent* is defined by $R_\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$ if exists.
- Fernando's flow chart: For some $\lambda \in K$,

$R_\lambda(T)$ exist?

  No. Then $\lambda \in \sigma_p(T)$ (*point spectrum*), which is called *eigenvalue.*

  Yes. $\overline{\mathscr{D}(R_\lambda(T))} = X$?

    No. Then $\lambda \in \sigma_r(T)$ (*residual spectrum*).

    Yes. $R_\lambda(T)$ is bounded?

      No. $\lambda \in \sigma_c(T)$ (*continuous spectrum*).

      Yes. $\lambda \in \rho(T)$ (*resolvent set*).

    end

  end

end

- [1] $\sigma(T)$ is closed and $\rho(T)$ is open. [2] If $\lambda \in \sigma(T)$ for $T \in B(X,X)$ where X is a Banach space, then $|\lambda| \leq \|T\|$.

SPECTRAL THEORY FOR COMPACT OPERATOR

- [1] Let X, Y be normed space. $T : X \to Y$ is a *compact operator* if T is linear and for all bounded sequence (x_n) (Tx_n) has a convergent subsequence. [2] A compact operator can be approximated by finite rank operator. [3] If T_n is compact and $\|T_n - T\| \to 0$, then T is also compact. [4] $x_n \rightharpoonup x \implies Tx_n \rightharpoonup Tx$.
- [1] $\sigma_p(T)$ is countable. [2] $\lambda = 0$ is the only accumulation point of $\sigma(T)$. [3] $\sigma(T)\backslash\{0\} = \sigma_p(T)$. If X is finite, then $0 \in \sigma(T)$. [4] For $\lambda \neq 0$, the dimension of eigenspace is finite.
- [Fredholm Alternative Theorem] Let $T : X \to X$ be a compact operator on a normed space X. Let $\lambda \neq 0$. Then $T_\lambda = T - \lambda I$ satisfies the Fredholm alternative.

I. $\mathcal{N}(T_\lambda) = \{0\} \iff \mathcal{R}(T_\lambda) = X$ (uniquely solvable for all input).

II. $T_\lambda x = y$ has a solutino for a particular $y \in X$ if $f(y) = 0$, $\forall f \in \mathcal{N}(T_\lambda^\times)$ (solvable for a particular input)

SPECTRAL THEORY FOR SELF-ADJOINT OPERATOR

- [1] Let $T \in B(H)$ be a *self-adjoint operator* on a complex Hilbert space H. Then $\sigma(T) \subset \mathbb{R}$ and lies in $\sigma(T) \subset [m, M]$ where $m = \inf_{\|x\|=1}\langle Tx, x\rangle$ and $M = \sup_{\|x\|=1}\langle Tx, x\rangle$. [2] Moreover, eigenvectors corresponding to distinct eigenvalues are orthogonal. [3] $\sigma_r(T) = \emptyset$.
- [Spectral theory for compact self-adjoint operator] $Tx = \sum_j \lambda_j \langle x, v_j\rangle v_j = (\sum_j \lambda_j P_j)x$ where $P_j = \langle \cdot, v_j\rangle v_j$. Moreover, $f(T) = \sum_j f(\lambda_j)P_j$ where f is polynomial.
- [Spectral theorem for self-adjoint operator] [1] $T : H \to H$ is a self-adjoint bounded linear operator. T has a *spectral family* $\mathcal{E} = (E_\lambda)_{\lambda\in\mathbb{R}}$ where E_λ is a projection onto $\mathcal{N}(T_\lambda^+)$ (i.e. eigenspace) and $T_\lambda^+ = \frac{1}{2}((T_\lambda^2)^{1/2} + T_\lambda) = \sum_{j,\lambda_j>\lambda} \lambda_j P_j$ is the positive part of T_λ satisfying 4 conditions.

- $\lambda < \mu \implies E_\lambda \leq E_\mu$,
- $\lambda < m \implies E_\lambda = 0$,
- $\lambda \geq M \implies E_\lambda = I$,
- $\mu \to \lambda^+ \implies E_\mu x \to E_\lambda x$.

[2] T has a spectral representation $T = \int_{m-0}^{M} \lambda dE_\lambda$ and $\langle Tx, y\rangle \int_{m-0}^{M} \lambda dw(\lambda)$ where $w(\lambda) = \langle E_\lambda x, y\rangle$.
- [Hilbert-Schmidt operator]

Distribution Theory

- [1] Let $X \subset \mathbb{R}^n$ be open. Then the *support* of function $\phi : X \to \mathbb{R}$ is $\text{supp}\phi = \overline{\{x \in X : \phi(x) \neq 0\}}$. Since $\text{supp}\phi$ is closed, if $x \in X \backslash \text{supp}\phi$, then there must be $\epsilon > 0$ such that $\phi \equiv 0$ on $B_\epsilon(x)$. [2] Let $X \subset \mathbb{R}^n$. $\mathscr{C}_c^\infty(X)$ is a vector space of compactly supported smooth functions. [3] A sequence $(\phi_n) \subset \mathscr{C}_c^\infty(X)$ is *sequentially convergent* to zero if there exists a compact set $K \subset X$ such that $\text{supp}\phi_j \subset K$ for all j and all derivatives $\partial^\alpha \phi_j$ converges to zero uniformly for all $N \in \mathbb{N}$, that is, $\lim_{j\to\infty} \sup_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \phi_j(x)| = 0$. [4] The vector space $\mathscr{C}_c^\infty(X)$ endowed with topology coming from sequential convergent is the space of *test function* and written as $\mathscr{D}(X)$.
- [1] Let $X \subset \mathbb{R}^n$ be open. A *distribution* u is a continuous linear form

$$u : \mathscr{D}(X) \to \mathbb{R},$$

that is, a linear form for which $\phi_j \to 0 \implies u(\phi_j) \equiv \langle u, \phi_j \rangle \to 0$. The space of distribution is written as $\mathscr{D}'(X)$. [2] $u \in \mathscr{D}'(X)$ is *of order* p if for all compact set $K \subset X$, there exists a constant C such that $\forall \phi$ with $\text{supp}\phi \subset K$,

$$|\langle u, \phi \rangle| \leq C \cdot \sum_{|\alpha| \leq p} \sup_{x \in K} |\partial^\alpha \phi(x)|$$

- [1] A sequence $(u_j) \subset \mathscr{D}'(X)$ *converges (distribution)* to $u \in \mathscr{D}'(X)$ such that $\langle u_j, \phi \rangle \to \langle u, \phi \rangle$ for all $\phi \in \mathscr{D}(X)$. [2] Let $u \in \mathscr{D}'(X)$. Then *derivative (distribution)* $\partial_i u \in \mathscr{D}'(X)$ is defined by $\langle \partial_i u, \phi \rangle = -\langle u, \partial_i \phi \rangle$ for all $\phi \in \mathscr{D}(X)$.
- [1] Let $u \in \mathscr{D}'(X)$ and $f \in \mathscr{C}^\infty(X)$. The *product* $fu = uf \in \mathscr{D}'(X)$ is defined by $\langle fu, \phi \rangle = \langle u, f\phi \rangle \forall \phi \in \mathscr{D}(X)$. [2, duality] Let $T : \mathscr{D}(X) \to \mathscr{D}(Y)$ be continuous. Consider the *adjoint* T^* defined by $\int_Y T\phi\psi\, dy = \int_X \phi T^* \psi\, dx$ for all ϕ, ψ. It can extend T to the *dual* $T^\times : \mathscr{D}'(Y) \to \mathscr{D}'(X)$ defined by $\langle T^\times u, \phi \rangle = \langle u, T\phi \rangle$ for all $u \in \mathscr{D}'(Y), \phi \in \mathscr{D}(X)$.
- [1, convolution] $\langle u * v, \phi \rangle = \langle u, \langle v, \tau_x \phi \rangle \rangle$. [2, linear PDE] Solve $Lu = f$ using convolution and fundamental solution E of $LE = \delta$. [3, Fourier transform] $\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle$.