THE WENTZEL-KRAMERS-BRILLOUIN (WKB) METHOD

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ABSTRACT

These notes are largely based on **Math 6730: Asymptotic and Perturbation Methods** course, taught by Paul Bressloff in Fall 2017, at the University of Utah. Additional examples or remarks or results from other sources are added as we see fit, mainly to facilitate our understanding. These notes are by no means accurate or applicable, and any mistakes here are of course our own. Please report any typographical errors or mathematical fallacy to us by email hkim@math.utah.edu or tan@math.utah.edu.

INTRODUCTION

The WKB method, named after Wentzel, Kramers and Brillouin, is a method for finding approximate solutions to linear differential equations with spatially varying coefficients. The origin of WKB theory dates back to 1920s where it was developed by Wentzel, Kramers and Brillouin to study time-independent Schrodinger equation. This often arises from the following problem:

$$\frac{d^2y}{dx^2} - q(\epsilon x)y = 0,$$

with the slowly varying potential energy. To handle this perturbation problem, the WKB method introduces an ansatz of the expansion term as a product of slowly varying and exponenetially rapidly varying terms.

1 INTRODUCTORY EXAMPLE

Consider the differential equation

$$\epsilon^2 y'' - q(x)y = 0, \quad x \in [0, 1], \tag{1.1}$$

where q is a smooth function. For constant $q \in \mathbb{R}$, the general solution of (1.1) is

$$y(x) = a_0 e^{-x\sqrt{q}/\epsilon} + b_0 e^{x\sqrt{q}/\epsilon}, \tag{1.2}$$

and the solution eigther blows up ($q > 0$) or oscillates ($q < 0$) rapidly on a scale of $\mathcal{O}(\epsilon)$. The hypothesis of the WKB method is that the exponential solutions (1.5) can be generalised to obtain an approximate solution of the full problem (1.1).

We expect that the solutions of (1.1) to have rapid oscillations on a scale of $\mathcal{O}(\epsilon)$ with slowly-varying amplitude and phase. Then it is natural to start with the following general WKB ansatz:

$$y(x) \sim e^{\theta(x)/\epsilon^\alpha} \left[y_0(x) + \epsilon^\alpha y_1(x) + \ldots \right], \tag{1.3}$$

for some $\alpha > 0$. Here, we assume that the solution varies exponentially with respect to the fast variation. From (1.3) we obtain

$$y' \sim \left\{ \epsilon^{-\alpha}\theta_x y_0 + y_0' + \theta_x y_1 + \ldots \right\} e^{\theta/\epsilon^\alpha}, \tag{1.4a}$$

$$y'' \sim \left\{ \epsilon^{-2\alpha}\theta_x^2 y_0 + \epsilon^{-\alpha}\left(\theta_{xx} y_0 + 2\theta_x y_0' + \theta_x^2 y_1 \right) + \ldots \right\} e^{\theta/\epsilon^\alpha}. \tag{1.4b}$$

Substituting both (1.3) and (1.4) into (1.1) and cancelling the exponential term yields

$$\epsilon^2 \left[\frac{\theta_x^2 y_0}{\epsilon^{2\alpha}} + \frac{1}{\epsilon^\alpha}\left(\theta_{xx} y_0 + 2\theta_x y_0' + \theta_x^2 y_1 \right) + \ldots \right] - q(x)\left[y_0 + \epsilon^\alpha y_1 + \ldots \right] = 0, \tag{1.5}$$

Note that Such cancellation is possible due to the linearity of the equation.

Balancing leading-order terms in (1.5) we see that $\alpha = 1$. The $\mathcal{O}(1)$ equation is the well-known **Eikonal equation**:

$$\theta_x^2 = q(x), \tag{1.6}$$

and its solutions (in one-dimensional) are

$$\theta(x) = \pm \int^x \sqrt{q(s)}\, ds. \tag{1.7}$$

To determine $y_0(x)$, we need to solve the $\mathcal{O}(\epsilon)$ equation which is the **transport equation**:

$$\theta_{xx}y_0 + 2\theta_x y_0' + \theta_x^2 y_1 = q(x)y_1, \tag{1.8}$$

The y_1 terms cancel out due to the Eikonal equation (1.6), so (1.8) reduces to

$$\theta_{xx}y_0 + 2\theta_x y_0' = 0. \tag{1.9}$$

The equation (1.9) can be easily solved since it is separable

$$\frac{y_0'}{y_0} = -\frac{\theta_{xx}}{2\theta_x},$$

and it follows its general solution

$$y_0(x) = \frac{C}{\sqrt{\theta_x}} = \pm Cq(x)^{-1/4}, \tag{1.10}$$

where C is an arbitrary nonzero constant and the last line follows from (1.7). Hence, a first-term asymptotic approximation of the general solution of (1.1) is

$$y(x) \sim q(x)^{-1/4}\left[a_0 \exp\left(-\frac{1}{\epsilon}\int^x \sqrt{q(s)}\,ds\right) + b_0 \exp\left(\frac{1}{\epsilon}\int^x \sqrt{q(s)}\,ds\right)\right], \tag{1.11}$$

where a_0, b_0 are arbitrary constants, possibly complex. It is evident that (1.11) is valid if $q(x) \neq 0$ on $[0,1]$. The x-values where $q(x) = 0$ are called **turning points** and this non-trivial issue will be addressed in the Section 2.

Example 1. Choose $q(x) = -e^{2x}$. Then the WKB approximation (1.11) is

$$y(x) \sim e^{-x/2}\left[a_0 e^{-ie^x/\epsilon} + b_0 e^{ie^x/\epsilon}\right] = e^{-x/2}\left[\alpha_0 \cos(\lambda e^x) + \beta_0 \sin(\lambda e^x)\right],$$

where $\lambda = 1/\epsilon$. With boundary conditions $y(0) = a, y(1) = b$, we obtain

$$y(x) \sim e^{-x/2}\left(\frac{b\sqrt{e}\sin\left(\lambda(e^x - 1)\right) - a\sin\left(\lambda(e^x - e)\right)}{\sin\left(\lambda(e-1)\right)}\right).$$

The exact solution of (1.1) with the given $q(x)$ can be solved as follows. Making a change of variable $\tilde{x} = e^x/\epsilon = \lambda e^x$, we obtain

$$x = \ln(\epsilon) + \ln(\tilde{x}) \implies \frac{dx}{d\tilde{x}} = \frac{1}{\tilde{x}}.$$

Setting $Y(\tilde{x}) = y(x)$ and using Chain Rule gives

$$\frac{dY}{d\tilde{x}} = \frac{dy}{dx}\cdot\frac{dx}{d\tilde{x}} = \frac{1}{\tilde{x}}Y', \quad \frac{d^2Y}{d\tilde{x}^2} = -\frac{y'}{\tilde{x}^2} + \frac{y''}{\tilde{x}^2} = -\frac{1}{\tilde{x}}Y' + \frac{y''}{\tilde{x}^2}.$$

Consequently, the equation of $Y(\tilde{x})$ is the zeroth-order Bessel's differential equation

$$\tilde{x}^2\frac{d^2Y}{d\tilde{x}^2} + \tilde{x}\frac{dY}{d\tilde{x}} + \tilde{x}^2 Y = 0,$$

and the solution of this is

$$Y(\tilde{x}) = c_0 J_0(\tilde{x}) + d_0 Y_0(\tilde{x}) = c_0 J_0(\lambda e^x) + d_0 Y_0(\lambda e^x) = y(x),$$

where $J_0(\cdot)$ and $Y_0(\cdot)$ are the zeroth-order Bessel functions of the first and second kinds respectively. Finally, solving for c_0 and d_0 using the boundary conditions yields

$$c_0 = \frac{1}{D}\left[bY_0(\lambda) - aY_0(\lambda e)\right], \quad d_0 = \frac{1}{D}\left[aJ_0(\lambda e) - bJ_0(\lambda)\right],$$

where $D = J_0(\lambda e)Y_0(\lambda) - Y_0(\lambda e)J_0(\lambda)$. One can plot the exact solution and the WKB approximation and see that their difference is almost zero, see Fig 4.1 and 4.2 in [2].

To measure the error of the WKB approximation (1.11), we look at the $\mathcal{O}(\epsilon^2)$ equation which has the form

$$\theta_{xx}y_1 + 2\theta_x y_1' + \theta_x^2 y_2 + y_0'' = q(x)y_2. \tag{1.12}$$

The y_2 terms vanish due to the Eikonal equation (1.6), so (1.12) reduces to

$$\theta_{xx}y_1 + 2\theta_x y_1' + y_0'' = 0. \tag{1.13}$$

Because the first two terms of (1.13) are similar to the transport equation (1.9), we make an ansatz $y_1(x) = y_0(x)w(x)$ and so (1.13) reduces to

$$2\theta_x y_0 w' + y_0'' = 0. \tag{1.14}$$

Suppose $q(x) > 0$ so that θ_x is a real-valued function. Substituting (1.10) into (1.14) gives

$$\frac{2C\theta_x w'}{\sqrt{\theta_x}} = -\frac{d^2}{dx^2}\left(\frac{C}{\sqrt{\theta_x}}\right) = \frac{d}{dx}\left(\frac{C\theta_{xx}}{2\theta_x^{3/2}}\right).$$

Rearranging it in terms of w',

$$w' = \frac{1}{4}\frac{d}{dx}\left(\frac{\theta_{xx}}{\theta_x^{3/2}}\right)\left(\frac{1}{\sqrt{\theta_x}}\right),$$

and performing integration by parts with respect to x yields

$$w(x) = \frac{1}{4}\int^x \left(\frac{\theta_{xx}}{\theta_x^{3/2}}\right)\left(\frac{1}{\sqrt{\theta_x}}\right) ds = d + \frac{1}{4}\left(\frac{\theta_{xx}}{\theta_x^2}\right) + \frac{1}{8}\int^x \left(\frac{\theta_{xx}^2}{\theta_x^3}\right) ds.$$

where d is an arbitrary constant. On the other hand, θ_x is a complex-valued function provided that $q(x) < 0$. Substituting $\theta_x = \pm i\sqrt{-q}$ into $w(x)$ gives the result.

$$\begin{aligned}
\theta_{xx} &= \pm\frac{i}{2}\left(\frac{-q_x}{\sqrt{-q}}\right) &&= \mp\frac{iq_x}{2\sqrt{-q}}\\
\frac{\theta_{xx}}{\theta_x^2} &= \mp\frac{iq_x}{2q\sqrt{-q}} &&= \pm\frac{iq_x}{2(-q)^{3/2}}\\
\theta_{xx}^2 &= \frac{-q_x^2}{4(-q)} &&= \frac{q_x^2}{4q}\\
\theta_x^3 &= (\pm i)^3\left(\sqrt{-q}\right)^3 &&= \mp i\,(-q)^{3/2}\\
\frac{\theta_{xx}^2}{\theta_x^3} &= \frac{q_x^2}{\mp 4iq(-q)^{3/2}} &&= \mp\frac{iq_x^2}{4(-q)^{5/2}}.
\end{aligned}$$

Finally, for small ϵ the WKB ansatz (1.3) is well-ordered provided

$$|\epsilon y_1(x)| \ll |y_0(x)|, \quad \text{or} \quad |\epsilon w(x)| \ll 1.$$

In terms of the function $q(x)$ and its first derivatives, for $x \in [x_0, x_1]$ we will have an accurate approximation if

$$\epsilon\left[|d| + \frac{1}{32}\left|\frac{q_x}{q^{3/2}}\right|\left(4 + \int_{x_0}^{x_1}\left|\frac{q_x}{q}\right| dx\right)\right] \ll 1,$$

where $|\cdot| := \|\cdot\|_\infty$ over the interval $[x_0, x_1]$. We stress that this condition holds if the interval $[x_0, x_1]$ does not contain a turning point.

Remark 1. The constants a_0, b_0 in (1.11) and d in $w(x)$ are determined from boundary conditions. However, it is very possible that these constants depend on ϵ. It is therefore necessary to make sure this dependence does not interfere with the ordering assumed in the WKB ansatz (1.3).

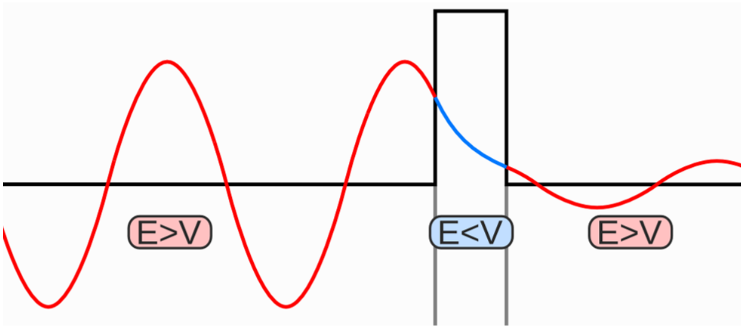

Figure 1: An example of turning points: quantum tunneling. Depending on effective potential energy, the solutions have different behavior and need to be matched (Taken from Wikipedia Commons).

2 TURNING POINTS

This section is devoted to the analysis of the effect provided that $q(x) = 0$ for some x. One example of it is the tunneling effect in quantum physics. The quantum probability in the well potential, the quantum probability differs with the potential well, see Fig. 1. Then we need to match to different solutions with transition layer.

Assume $q(x)$ is smooth and has a simple zero at x_t, that is $q(x_t) = 0$ and $q'(x_t) \neq 0$. For concreteness, we take $q'(x_t) > 0$ and so we expect solutions of (1.1) to be oscillatory for $x < x_t$ and exponential for $x > x_t$. We can apply the WKB method on the regions $\{x < x_t\}$ and $\{x > x_t\}$. More precisely, from (1.11) we have

$$y \sim \begin{cases} y_L(x, x_t) & \text{if } x < x_t, \\ y_R(x, x_t) & \text{if } x > x_t, \end{cases} \tag{2.1}$$

where

$$y_L(x, x_t) = \frac{1}{q(x)^{1/4}} \left[a_L \exp\left(-\frac{1}{\epsilon} \int_x^{x_t} \sqrt{q(s)}\, ds \right) + b_L \exp\left(\frac{1}{\epsilon} \int_x^{x_t} \sqrt{q(s)}\, ds \right) \right], \tag{2.2a}$$

$$y_R(x, x_t) = \frac{1}{q(x)^{1/4}} \left[a_R \exp\left(-\frac{1}{\epsilon} \int_{x_t}^{x} \sqrt{q(s)}\, ds \right) + b_R \exp\left(\frac{1}{\epsilon} \int_{x_t}^{x} \sqrt{q(s)}\, ds \right) \right]. \tag{2.2b}$$

An important realization is that these coefficients a_L, b_L, a_R, b_R are not all independent. In addition to the two boundary conditions at $x = 0$ and $x = 1$, we also have matching conditions in a transition layer centered at $x = x_t$.

2.1 Transition layer

Following the boundary layer analysis, we introduce the stretched coordinate

$$\tilde{x} = \frac{x - x_t}{\epsilon^\beta}, \quad \text{or equivalently} \quad x = x_t + \epsilon^\beta \tilde{x}.$$

We can reduce (1.1) by expanding the function $q(x)$ around the turning point x_t by Taylor series:

$$q(x) = q(x_t + \epsilon^\beta \tilde{x}) \sim q'(x_t)\epsilon^\beta \tilde{x},$$

since we assume x_t is a simple zero. Denote the inner solution by $Y(\tilde{x})$. Transforming (1.1) using

$$\frac{d}{dx} = \frac{1}{\epsilon^\beta} \frac{d}{d\tilde{x}},$$

gives the inner equation

$$\epsilon^{2-2\beta} Y'' - \left(\epsilon^{\beta}\tilde{x}q_t' + \dots\right) Y = 0, \tag{2.3}$$

where $q_t' := q'(x_t)$. Balancing leading-order terms in (2.3) means we require

$$2 - 2\beta = \beta \implies \beta = \frac{2}{3}.$$

Since it is not clear what the asymptotic sequence should be, we take the asymptotic expansion to be

$$Y \sim \epsilon^{\gamma} Y_0(\tilde{x}) + \dots. \tag{2.4}$$

The $\mathcal{O}(\epsilon^{2/3})$ equation is

$$Y_0'' - \tilde{x}q_t' Y_0 = 0, \quad -\infty < \tilde{x} < \infty. \tag{2.5}$$

Performing a coordinate transformation $s = (q_t')^{1/3}\tilde{x}$, (2.5) becomes **Airy's equation**:

$$\frac{d^2 Y_0}{ds^2} - sY_0 = 0, \quad -\infty < s < \infty, \tag{2.6}$$

and this can be solved either using power series expansion or Laplace transform. The general solution of (2.6) is

$$Y_0(s) = a\mathrm{Ai}(s) + b\mathrm{Bi}(s), \tag{2.7}$$

where $\mathrm{Ai}(\cdot)$ and $\mathrm{Bi}(\cdot)$ are Airy functions of the first and the second kinds respectively. It is well-known that

$$\begin{aligned}
\mathrm{Ai}(x) &= \frac{1}{3^{2/3}\pi} \sum_{k=0}^{\infty} \frac{1}{k!}\Gamma\left(\frac{k+1}{3}\right) \sin\left(\frac{2\pi}{3}(k+1)\right)\left(3^{1/3}x\right)^k \\
&= \mathrm{Ai}(0)\left(1 + \frac{1}{6}x^3 + \dots\right) + \mathrm{Ai}'(0)\left(x + \frac{1}{12}x^4 + \dots\right) \\
\mathrm{Bi}(x) &= e^{i\pi/6}\mathrm{Ai}\left(xe^{2\pi i/3}\right) + e^{-i\pi/6}\mathrm{Ai}\left(xe^{-2\pi i/3}\right) \\
&= \mathrm{Bi}(0)\left(1 + \frac{1}{6}x^3 + \dots\right) + \mathrm{Bi}'(0)\left(x + \frac{1}{12}x^4 + \dots\right),
\end{aligned}$$

where $\Gamma(\cdot)$ is the gamma function. Setting $\xi = \frac{2}{3}|x|^{3/2}$, we also have that

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\left(\xi - \frac{\pi}{4}\right) + \frac{5}{72\xi}\sin\left(\xi - \frac{\pi}{4}\right)\right] & \text{if } x \longrightarrow -\infty, \\ \frac{1}{2\sqrt{\pi}|x|^{1/4}}e^{-\xi}\left[1 - \frac{5}{72}\xi\right] & \text{if } x \longrightarrow +\infty, \end{cases} \tag{2.8a}$$

$$\mathrm{Bi}(x) \sim \begin{cases} \frac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\left(\xi + \frac{\pi}{4}\right) + \frac{5}{72\xi}\sin\left(\xi + \frac{\pi}{4}\right)\right] & \text{if } x \longrightarrow -\infty, \\ \frac{1}{\sqrt{\pi}|x|^{1/4}}e^{\xi}\left[1 + \frac{5}{72}\xi\right] & \text{if } x \longrightarrow +\infty. \end{cases} \tag{2.8b}$$

2.2 Matching

From (2.7), the general solution of (1.1) in the transition layer is

$$Y_0(\tilde{x}) = a\mathrm{Ai}\left[(q_t')^{1/3}\tilde{x}\right] + b\mathrm{Bi}\left[(q_t')^{1/3}\tilde{x}\right]. \tag{2.9}$$

We now have 6 undetermined constants from (2.2) and (2.9), but these are all connected since the inner solution (2.9) must match the outer solutions (2.2) and this

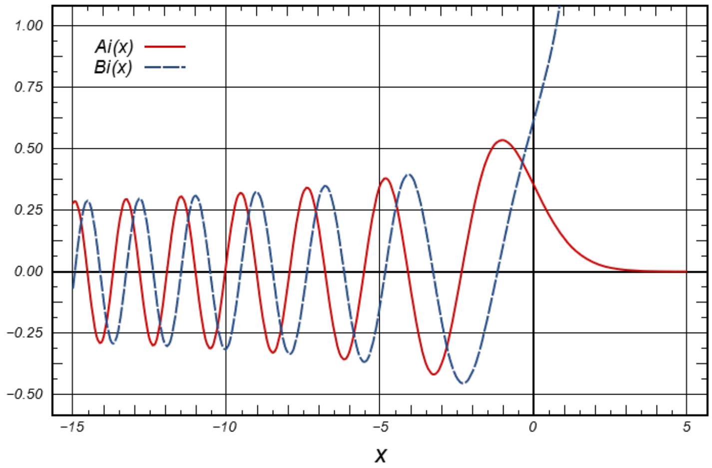

Figure 2: Plot of Airy functions, $\mathrm{Ai}(x)$ in red and $\mathrm{Bi}(x)$ in blue. This figure comes from Wikipedia Commons.

will results in two arbitrary constants in the general solution. Since the inner solution is unbounded, we introduce an intermediate variable

$$x_\eta = \frac{x - x_t}{\epsilon^\eta}, \quad 0 < \eta < \frac{2}{3},$$

where the interval for η comes from the requirement that the scaling for the intermediate variable must lie between the outer scale, $\mathcal{O}(1)$ and the inner scale, $\mathcal{O}(\epsilon^{2/3})$.

2.2.1 *Matching for $x > x_t$*

We first change the stretched variable $\tilde{x}$ to the intermediate variable x_η:

$$\tilde{x} = \frac{x - x_t}{\epsilon^\beta} = \frac{x - x_t}{\epsilon^\eta \epsilon^{\beta - \eta}} = \epsilon^{\eta - \beta} x_\eta = \epsilon^{\eta - 2/3} x_\eta.$$

Note that $x_\eta > 0$ since $x > x_t$. From (2.4) and (2.9), the inner solution $Y(\tilde{x})$ now becomes

$$\begin{aligned} Y &\sim \epsilon^\gamma Y_0 \left(\epsilon^{\eta - 2/3} x_\eta\right) + \dots \\ &= \epsilon^\gamma \left[a\mathrm{Ai}\left(\left(q_t'\right)^{1/3} \epsilon^{\eta - 2/3} x_\eta\right) + b\mathrm{Bi}\left(\left(q_t'\right)^{1/3} \epsilon^{\eta - 2/3} x_\eta\right)\right] + \dots \\ &= \epsilon^\gamma \left[a\mathrm{Ai}(r) + b\mathrm{Bi}(r)\right] + \dots \\ &\sim \epsilon^\gamma \left[\frac{a}{2\sqrt{\pi} r^{1/4}} \exp\left(-\frac{2}{3} r^{3/2}\right) + \frac{b}{\sqrt{\pi} r^{1/4}} \exp\left(\frac{2}{3} r^{3/2}\right)\right], \end{aligned} \tag{2.10}$$

where $r = q'(x_t)^{1/3} \epsilon^{\eta - 2/3} x_\eta > 0$ and the last line follows from (2.8). On the other hand, since

$$\begin{aligned} \int_{x_t}^{x} \sqrt{q(s)}\, ds &\sim \int_{x_t}^{x_t + \epsilon^\eta x_\eta} \sqrt{(s - x_t) q_t'}\, ds \\ &= \sqrt{q_t'} \left[\frac{2}{3}(s - x_t)^{3/2}\right] \Bigg|_{x_t}^{x_t + \epsilon^\eta x_\eta} \\ &= \frac{2}{3} \sqrt{q_t'} \left(\epsilon^\eta x_\eta\right)^{3/2} \\ &= \frac{2}{3} \epsilon r^{3/2}, \end{aligned}$$

and

$$q(x)^{-1/4} \sim \left[q(x_t) + (x - x_t) q_t'\right]^{-1/4} = \left[\epsilon^\eta x_\eta q_t'\right]^{-1/4} = \epsilon^{-1/6} \left(q_t'\right)^{-1/6} r^{-1/4},$$

the right outer solution becomes

$$y_R \sim \frac{\epsilon^{-1/6}}{(q_t')^{1/6} r^{1/4}} \left[a_R \exp\left(-\frac{2}{3}r^{3/2}\right) + b_R \exp\left(\frac{2}{3}r^{3/2}\right)\right]. \tag{2.11}$$

Consequently, matching (2.10) the right outer solution y_R with (2.11) the inner solution Y yields the following:

$$\gamma = -\frac{1}{6}, \quad a_R = \frac{a}{2\sqrt{\pi}} \left(q_t'\right)^{1/6}, \quad b_R = \frac{b}{\sqrt{\pi}} \left(q_t'\right)^{1/6}. \tag{2.12}$$

2.2.2 *Matching for $x < x_t$*

Because $x < x_t$, we have $x_\eta < 0$ which introduces complex numbers into the outer solution y_L. Using the asymptotic properties of Airy functions as $r \to -\infty$ (see (2.8)), the inner solution becomes

$$\begin{aligned} Y &\sim \epsilon^\gamma \left[a\mathrm{Ai}(r) + b\mathrm{Bi}(r)\right] + \dots \\ &\sim \epsilon^\gamma \left[\frac{a}{\sqrt{\pi}|r|^{1/4}} \cos\left(\frac{2}{3}|r|^{3/2} - \frac{\pi}{4}\right) + \frac{b}{\sqrt{\pi}|r|^{1/4}} \cos\left(\frac{2}{3}|r|^{3/2} + \frac{\pi}{4}\right)\right] \end{aligned}$$

Using the identity $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, a more useful form of the inner expansion Y as $r \longrightarrow -\infty$ is

$$Y \sim \frac{\epsilon^\gamma}{2\sqrt{\pi}|r|^{1/4}} \left[\left(ae^{-i\pi/4} + be^{i\pi/4}\right) e^{i\zeta} + \left(ae^{i\pi/4} + be^{-i\pi/4}\right) e^{-i\zeta}\right], \tag{2.13}$$

where $\zeta = \frac{2}{3}|r|^{3/2}$. On the other hand, since

$$\int_x^{x_t} \sqrt{q(s)}\, ds \sim \int_{x_t + \epsilon^\eta x_\eta}^{x_t} \sqrt{(s - x_t) q_t'}\, ds,$$

and this follows that

$$\int_x^{x_t} \sqrt{q(s)}\, ds \sim -\frac{2}{3}\sqrt{q_t'} \left(\epsilon^\eta x_\eta\right)^{3/2} = \frac{2}{3} i\epsilon |r|^{3/2}.$$

Furthermore,

$$\begin{aligned} q(x)^{-1/4} \sim \left[\epsilon^\eta x_\eta q_t'\right]^{-1/4} &= \epsilon^{-1/6} \left(q_t'\right)^{-1/6} |r|^{-1/4} (-1)^{-1/4}, \\ &= \epsilon^{-1/6} \left(q_t'\right)^{-1/6} |r|^{-1/4} e^{-i\pi/4}, \end{aligned}$$

the left outer solution becomes

$$y_L \sim \frac{\epsilon^{-1/6} e^{-i\pi/4}}{(q_t')^{1/6} |r|^{1/4}} \left[a_L e^{-i\zeta} + b_L e^{i\zeta}\right]. \tag{2.14}$$

Consequently, matching (2.14) the left outer solution y_L with (2.13) the inner solution Y yields the following:

$$a_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}} (ia + b), \quad b_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}} (a + ib) = i\bar{a}_L. \tag{2.15}$$

From (2.12), it follows that

$$a_L = ia_R + \frac{b_R}{2}, \quad b_L = a_R + \frac{i}{2} b_R, \tag{2.16}$$

or in matrix form,

$$\begin{bmatrix} a_L \\ b_L \end{bmatrix} = \begin{bmatrix} i & 1/2 \\ 1 & i/2 \end{bmatrix} \begin{bmatrix} a_R \\ b_R \end{bmatrix}. \tag{2.17}$$

2.2.3 *Conclusion*

Because we assume $q(t) < 0$ for $x < x_t$, this introduces complex numbers on y_L:

$$q(x)^{-1/4} = e^{-i\pi/4}|q(x)|^{-1/4}$$
$$\int_x^{x_t} \sqrt{q(s)}\, ds = i \int_x^{x_t} \sqrt{|q(s)|}\, ds$$

In conclusion, we have

$$y(x) = \begin{cases} y_L(x, x_t) & \text{if } x < x_t, \\ y_R(x, x_t) & \text{if } x > x_t, \end{cases}$$

where

$$\begin{aligned} y_L(x, x_t) &= \frac{1}{|q(x)|^{1/4}} \left[\left(ia_R + \frac{b_R}{2} \right) e^{-i\theta(x)/\epsilon} e^{-i\pi/4} + \left(a_R + \frac{ib_R}{2} \right) e^{i\theta(x)/\epsilon} e^{-i\pi/4} \right] \\ &= \frac{1}{|q(x)|^{1/4}} \left[2a_R \cos\left(\frac{1}{\epsilon}\theta(x) - \frac{\pi}{4} \right) + b_R \cos\left(\frac{1}{\epsilon}\theta(x) + \frac{\pi}{4} \right) \right] \end{aligned}$$

and

$$y_R(x, x_t) = \frac{1}{q(x)^{1/4}} \left[a_R e^{-\kappa(x)/\epsilon} + b_R e^{\kappa(x)/\epsilon} \right],$$

with

$$\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}\, ds, \quad \kappa(x) = \int_{x_t}^{x} \sqrt{|q(s)|}\, ds.$$

Example 2. Consider $q(x) = x(2 - x)$, where $-1 < x < 1$. The simple turning point is at $x_t = 0$, with $q'(0) = 2 > 0$. One can compute and show that:

$$\begin{aligned} \theta(x) &= \frac{1}{2}(1 - x)\sqrt{x(x-2)} - \frac{1}{2}\ln\left[1 - x + \sqrt{x(x-2)} \right], \quad x < 0 \\ \kappa(x) &= \frac{1}{2}(x - 1)\sqrt{x(2-x)} - \frac{1}{2}\arcsin(x - 1) + \frac{\pi}{4}, \qquad x > 0. \end{aligned}$$

2.3 The opposite case: $q_t' < 0$

In the same fashion, one can obtain by change of variables with a new variable $z = x_t - x$ (see details in Section 4.3.2 in [2])

$$\begin{bmatrix} a_L \\ b_L \end{bmatrix} = \begin{bmatrix} i/2 & 1 \\ 1/2 & i \end{bmatrix} \begin{bmatrix} a_R \\ b_R \end{bmatrix}.$$

Consequently,

$$\begin{aligned} y_L(x) &= \frac{1}{q(x)^{1/4}} \left[a_L e^{\theta(x)/\epsilon} + b_L e^{-\theta(x)/\epsilon} \right], \\ y_R(x) &= \frac{1}{|q(x)|^{1/4}} \left[2b_L \cos\left(\frac{1}{\epsilon}\kappa(x) - \frac{\pi}{4} \right) + a_L \cos\left(\frac{1}{\epsilon}\kappa(x) + \frac{\pi}{4} \right) \right]. \end{aligned}$$

3 WAVE PROPAGATION AND ENERGY METHODS

In this section, we study how to obtain an asymptotic approximation of a travelling-wave solution of the following PDE which models the string displacement

$$u_{xx} = \mu^2(x)u_{tt} + \alpha(x)u_t + \beta(x)u, \quad 0 < x < \infty, \quad t > 0, \tag{3.1a}$$
$$u(0,t) = \cos(\omega t), \tag{3.1b}$$

The terms $\alpha(x)u_t$ and βu correspond to damping and elastic support respectively. From the initial condition, we see that the string is periodically forced at the left end and so the solution will develop into a wave that propagates to the right.

Observe that there is no obvious small parameter ϵ, but we will extract one from the following observation. In the special case where $\alpha = \beta = 0$ and μ equals some constant, (3.1) reduces to the classical wave equation and we obtain the right-moving plane waves

$$u(x,t) = e^{i(wt-kx)},$$

where the wavenumber k satisfies $k = \pm\omega\mu$. For higher temporal frequencies $\omega \gg 1$, these waves have short wavelength, that is $\lambda = |2\pi/k| \ll 1$. Motivated by this, we choose $\epsilon = 1/\omega$ and construct an asymptotic approximation of the travelling-wave solution of (3.1) in the case of a high frequency. The WKB ansatz is assumed to be

$$u(x,t) \sim \exp\left[i\left(wt - \underbrace{w^{\gamma}\theta(x)}_{\text{fast oscillation}}\right)\right]\left\{u_0(x) + \underbrace{\frac{1}{w^{\gamma}}u_1(x)}_{\text{slowly-varying}} + \dots\right\}. \tag{3.2}$$

Substituting (3.2) into (3.1) we obtain

$$-\omega^{2\gamma}\theta_x^2\left(u_0 + w^{-\gamma}u_1 + \dots\right) + iw^{\gamma}\theta_x\left(\partial_x u_0 + \dots\right) + \frac{d}{dx}\left(i\omega^{\gamma}\theta_x u_0 + \dots\right)$$
$$= -\mu^2\omega^2\left(u_0 + \omega^{-\gamma}u_1 + \dots\right) - i\omega\alpha\left(u_0 + \dots\right) + \beta\left(u_0 + \dots\right).$$

Balancing the first terms on each side of this equation gives $\gamma = 1$. The $\mathcal{O}(\omega^2) = \mathcal{O}(1/\epsilon^2)$ equation is the Eikonal equation:

$$\theta_x^2 = \mu^2(x), \tag{3.3}$$

and its solutions are

$$\theta(x) = \pm\int_0^x \mu(s)\,ds. \tag{3.4}$$

We choose the positive solution as we are considering the right-moving waves. The $\mathcal{O}(\omega) = \mathcal{O}(1/\epsilon)$ equation is the transport equation:

$$-\theta_x^2 u_1 + i\theta_x\partial_x u_0 + i\left(\theta_x\partial_x u_0 + \theta_{xx}u_0\right) = -\mu^2 u_1 - i\alpha u_0. \tag{3.5}$$

The u_1 terms cancel out due to the Eikonal equation (3.3), so (3.5) reduces to

$$\theta_{xx}u_0 + 2\theta_x\partial_x u_0 = -\alpha u_0,. \tag{3.6}$$

With $\theta_x = \mu(x)$, we can rearrange (3.6) and obtain a first order ODE in u_0:

$$\partial_x u_0 + \left(\frac{\mu_x + \alpha}{2\mu}\right)u_0 = 0, \tag{3.7}$$

which can be solved using the method of integrating factor. The integrating factor is given by

$$I(x) = \exp\left(\int_0^x \left(\frac{\mu_s(s)+\alpha(s)}{2\mu(s)}\right) ds\right) = \sqrt{\mu(x)}\exp\left(\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}\,ds\right),$$

and so (3.7) can be written as

$$\frac{d}{dx}\left(I(x)u_0\right) = 0, \quad u_0 = \frac{a_0}{I(x)} = \frac{a_0}{\sqrt{\mu(x)}}\exp\left(-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}\,ds\right). \tag{3.8}$$

Finally, imposing the boundary condition at $x=0$ we obtain a first-term asymptotic expansion of the travelling-wave solution of (3.1)

$$u(x,t) \sim \sqrt{\frac{\mu(0)}{\mu(x)}}\exp\left[-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)}\,ds\right]\cos\left(\omega t - \omega\int_0^x \mu(s)\,ds\right). \tag{3.9}$$

Observe that in (3.9) the amplitude and phase of the travelling wave depend on the spatial position x. Interestingly, (3.9) is independent of $\beta(x)$!

3.1 Connection to energy methods

Energy methods are extremely powerful in the study of wave-related problems. To determine the energy equation in this case, we multiply (3.1) by u_t

$$u_t u_{xx} = \mu^2(x)u_t u_{tt} + \alpha(x)u_t^2 + \beta(x)uu_t,$$

and one can rewrite it as

$$\partial_x(u_t u_x) - \frac{1}{2}\partial_t\left(u_x^2\right) = \frac{1}{2}\mu^2(x)\partial_t\left(u_t^2\right) + \alpha(x)u_t^2 + \beta(x)\partial_t\left(u^2\right).$$

Collecting time derivative and spatial derivative terms yields

$$\underbrace{\partial_t\left[\frac{1}{2}\mu^2(x)\left(u_t^2\right) + \frac{1}{2}\beta(x)u^2 + \frac{1}{2}\left(u_x^2\right)\right]}_{:=E(x,t)} \underbrace{-\partial_x\left(u_t u_x\right)}_{:=S(x,t)} = -\underbrace{\alpha(x)u_t^2}_{:=\Phi(x,t)},$$

where $E(x,t)$ is the energy density, $S(x,t)$ is the energy flux and $\Phi(x,t)$ is the dissipation function at position x and time t. Subsequently, we can achieve the energy equation

$$\partial_t E(x,t) + \partial_x S(x,t) = -\Phi(x,t). \tag{3.10}$$

We are interested in the energy over some spatial interval of the form $[x_1(t), x_2(t)]$ with assuming $x_1(t) < x_2(t)$. It follows from Leibniz's rule

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} E(x,t)\,dx = E(x_2(t),t)\dot{x_2} - E(x_1(t),t)\dot{x_1} + \int_{x_1(t)}^{x_2(t)} \partial_t E(x,t)\,dx.$$

Imposing the energy equation (3.10) gives

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} E\,dx = E(x_2(t),t)\dot{x_2} - E(x_1(t),t)\dot{x_1} - S(x_2(t),t) + S(x_1(t),t) - \int_{x_1(t)}^{x_2(t)} \Phi(x,t)\,dx. \tag{3.11}$$

The term $E(x_j(t),t)\dot{x_j}$ is the change of energy due to the motion of the endpoint, $S(x_j(t),t)$ is the flux of energy across the endpoint due to wave motion and the last integration term on the right-hand side is the energy loss over the interval due to dissipation.

The WKB solution can be written in the more general form:

$$u(x,t) \sim A(x)\cos\left[wt - \varphi(x)\right], \quad \varphi(x) = \omega\theta(x). \tag{3.12}$$

with slowly changing amplitude $A(x)$ and rapidly changing phase $\varphi(x)$. It follows that

$$E(x,t) \sim \frac{1}{2}A^2\left(\mu^2\omega^2 + \varphi_x^2\right)\sin^2\left[\omega t - \varphi(x)\right], \tag{3.13a}$$

$$S(x,t) \sim \omega\varphi_x A^2 \sin^2\left[\omega t - \varphi(x)\right], \tag{3.13b}$$

$$\Phi(x,t) \sim \alpha\omega^2 A^2 \sin^2\left[\omega t - \varphi(x)\right]. \tag{3.13c}$$

Suppose we choose $x_i(t)$ satisfying

$$\dot{x}_i = \frac{\omega}{\varphi_x(x_i)}, \tag{3.14}$$

that is the reference frame moves as phase velocity. It is worth noting that the phase velocity of plane waves is

$$v_p = \frac{w}{\partial_x(kx)} = \frac{w}{k}.$$

A curve satisfying (3.14) in the $x - t$ plane are called **phase lines**. Now, imposing (3.14) to (3.13) gives

$$E\dot{x} - S \sim \frac{1}{2}\frac{\omega A^2}{\varphi_x}\left[\mu^2\omega^2 + \varphi_x^2\right]\sin^2\left[\omega t - \varphi(x)\right] - \omega\varphi_x A^2\sin^2\left[\omega t - \varphi(x)\right],$$

and it follows that

$$E\dot{x} - S \sim \frac{1}{2}\frac{\omega A^2}{\varphi_x}\left[\mu^2\omega^2 - \varphi_x^2\right]\sin^2\left[\omega t - \varphi(x)\right] = 0,$$

since $\theta(x) = \varphi(x)/\omega$ satisfies the Eikonal equation (3.3). Hence, if $x_2 - x_1 = \mathcal{O}(1/\omega)$ then it follows from (3.11) that $\frac{dE_{tot}}{dt} \approx 0$, that is the total energy remains constant (to the first term) between any two phase lines $x_1(t), x_2(t)$ that are $\mathcal{O}(1/\omega)$ apart. Recall the energy equation that

$$\partial_t E + \partial_x S = -\Phi.$$

Averaging the energy equation over one period in time results in

$$\partial_x\left(\int_0^{2\pi/\omega} S(x,t)\,dt\right) = -\int_0^{2\pi/\omega}\Phi(x,t)\,dt,$$

where the average of $\partial_t E$ over one period vanishes using (3.13) for E. Substituting (3.13) for S and Φ, we obtain

$$\partial_x\left(\varphi_x A^2\right) = -\alpha\omega A^2,$$

and it follows that

$$\theta_{xx}A + 2\theta_x A_x = -\alpha A.$$

This implies that $A = u_0$ since the last equation is precisely the transport equation (3.6). Physically, this means that the transport equation corresponds to the balance of energy over one period in time.

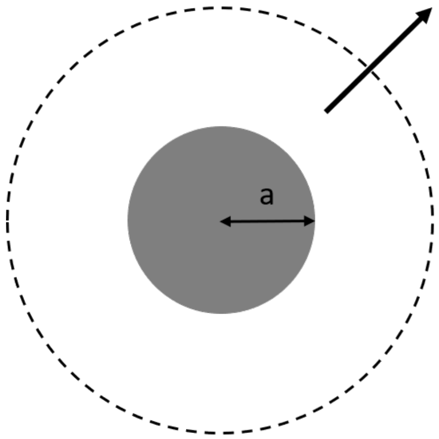

Figure 3: Instructive case of multi-dimensional wave equation. In $\mathbb{R}^2$, the wave propagates from the circle with radius a.

4 HIGHER-DIMENSIONAL WAVES – RAY METHODS

The extension of the WKB method to higher dimensions is relatively straightforward, but the equations could be difficult to solve explicitly. Consider the multi-dimensional wave equation

$$\nabla^2 u = \mu^2(\boldsymbol{x})\partial_t^2 u, \quad \boldsymbol{x} \in \mathbb{R}^n, \quad n = 2, 3. \tag{4.1}$$

We look for time-harmonic solutions $u(\boldsymbol{x}, t) = e^{-i\omega t}V(\boldsymbol{x})$ and (4.1) reduces to the Helmholtz equation

$$\nabla^2 V + \omega^2\mu^2(\boldsymbol{x})V = 0. \tag{4.2}$$

It is more instructive to have some understanding of what properties the solution has and how the WKB approximation takes advantage of them. Suppose μ is some constant and we want to solve (4.2) in the region exterior to the circle $\|\boldsymbol{x}\| = a$ in $\mathbb{R}^2$. Exploiting the geometry leads to the choice of polar coordinates

$$x = \rho\cos(\varphi), \;\; y = \rho\sin(\varphi).$$

We impose the Dirichlet boundary condition $V = f(\varphi)$ at $\rho = a$ and the **Sommerfeld radiation condition** which ensures that waves only propagate outward from the circle:

$$\sqrt{\rho}\left[\partial_\rho V - i\omega\mu V\right] = 0 \quad \text{for } \rho \longrightarrow \infty.$$

Using separation of variables, the general solution of (4.2) is given by

$$V(\rho, \varphi) = \sum_{n=-\infty}^{\infty} \alpha_n \left(\frac{H_n^{(1)}(\omega\mu\rho)}{H_n^{(1)}(\omega\mu a)}\right) e^{-in\varphi}, \tag{4.3}$$

where $H_n^{(1)}$ is the Hankel function of first kind and the α_n are determined from the boundary condition at $\rho = a$. It is known that for large values of z

$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left(i\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right).$$

Consequently, in the regime of higher frequency $\omega \gg 1$ (4.3) reduces to

$$V(\rho, \varphi) \sim f(\varphi)\sqrt{\frac{a}{\rho}} e^{i\omega\mu(\rho - a)}. \tag{4.4}$$

Thus we have a WKB-like solution for constant μ. Radial lines in this example correspond to **rays** and from (4.4) we see that along a ray (so that φ is fixed), the solution has a highly oscillatory component that is multiplied by a slowly varying amplitude $V_0 = f(\varphi)\sqrt{a/\rho}$ that decays as ρ increases.

4.1 WKB Expansion

We first specify the domain and boundary conditions. Equation (4.2) is to be solved in a region exterior to a smooth surface S, where S encloses a bounded convex domain. This means that there is a well-defined unit outward normal at every point on the surface. We impose the Dirichlet boundary condition

$$V(\boldsymbol{x}_0) = f(\boldsymbol{x}_0), \quad \boldsymbol{x}_0 \in S, \tag{4.5}$$

and focus only on outward propagating waves.

For higher frequency waves, we take a WKB ansatz of the form

$$V(\boldsymbol{x}) \sim e^{i\omega\theta(\boldsymbol{x})}\left[V_0(\boldsymbol{x}) + \frac{1}{\omega}V_1(\boldsymbol{x}) + \ldots\right]. \tag{4.6}$$

Then we have

$$\nabla V \sim \{i\omega\nabla\theta V_0 + i\nabla\theta V_1 + \nabla V_1 + \ldots\}\, e^{i\omega\theta}, \tag{4.7a}$$

$$\nabla^2 V \sim \left\{-\omega^2\nabla\theta\cdot\nabla\theta V_0 + \omega\left(-\nabla\theta\cdot\nabla\theta V_1 + 2i\nabla\theta\cdot\nabla V_0 + \nabla^2\theta V_0\right) + \ldots\right\} e^{i\omega\theta}. \tag{4.7b}$$

Substituting (4.7) into (4.2) gives

$$\omega^2\left(-\nabla\theta\cdot\nabla\theta V_0 + \mu^2 V_0\right) + \omega\left[-\nabla\theta\cdot\nabla\theta V_1 + 2i\nabla\theta\cdot\nabla V_0 + i\nabla^2\theta V_0 + \mu^2 V_1\right] + \mathcal{O}(1) = 0,$$

and rearranging we find that

$$\left(\nabla\theta\cdot\nabla\theta - \mu^2\right)V_0 + \frac{1}{\omega}\left[\left(\nabla\theta\cdot\nabla\theta - \mu^2\right)V_1 - i\nabla^2\theta V_0 - 2i\nabla\theta\cdot\nabla V_0\right] + \mathcal{O}\left(\frac{1}{\omega^2}\right) = 0.$$

The $\mathcal{O}(1)$ equation is the Eikonal equation which is now nontrivial to solve:

$$\nabla\theta\cdot\nabla\theta = \mu^2. \tag{4.8}$$

After cancelling the V_1 term using the Eikonal equation (4.8), the $\mathcal{O}(1/\omega)$ equation is the transport equation:

$$2\nabla\theta\cdot\nabla V_0 + \left(\nabla^2\theta\right)V_0 = 0. \tag{4.9}$$

Both $\pm\theta$ are solutions to the Eikonal equation and we choose positive θ since this corresponds to the outward propagating waves.

4.2 Surfaces and wave fronts

The usual method method for solving the nonlinear Eikonal equation (4.8) is to introduce **characteristic coordinates**. More precisely, we use curves that are orthogonal to the level surfaces of $\theta(\boldsymbol{x})$ which are also known as **wave fronts** or **phase fronts**.

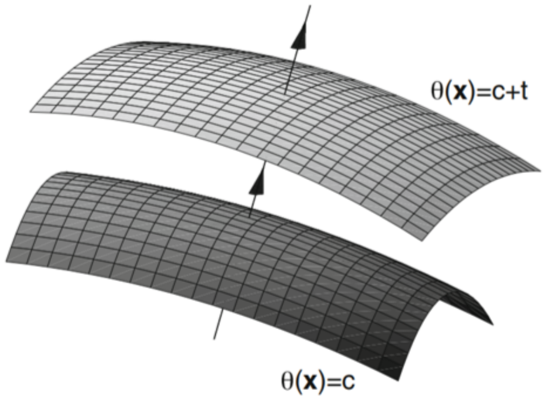

Figure 4: Schematic figure of wave fronts in $\mathbb{R}^3$ and path followed by one of the points in the wave front. This figure comes from Fig 4.11 in [2].

First, note that the WKB approximation of (4.1) has the form

$$u(\boldsymbol{x}, t) \sim e^{i(\omega\theta(\boldsymbol{x}) - \omega t)} V_0(\boldsymbol{x}).$$

We introduce the phase function

$$\Theta(\boldsymbol{x}, t) = \omega\theta(\boldsymbol{x}) - \omega t.$$

Suppose we start at $t = 0$ with the surface $S_c = \{\theta(\boldsymbol{x}) = c\}$, so that

$$\Theta(\boldsymbol{x}, 0) = \omega c.$$

As t increases, the points where $\Theta = \omega c$ change, and therefore points forming S_c move and form a new surface $S_{c+t} = \{\theta(\boldsymbol{x}) = c + t\}$. We still have

$$\Theta(\boldsymbol{x}, t) = \omega c,$$

that is S_c to S_{c+t} has same phase. The path each point takes to get from S_c to S_{c+t} is obtained from the solution of the Eikonal equation and in the WKB method these paths are called **rays**.

The evolution of the wave front generates a natural coordinate system (s, α, β) where α, β comes from parameterising the wave front and s from parameterising the rays. Note that these coordinates are not unique as there are no unique parameterisation for the surfaces and rays. It turns out that determining these coordinates is crucial in the derivation of the WKB approximation.

Example 3. Suppose we know a-priori that $\theta(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}$. In this case, the surface S_{c+t} is described by the equation $|\boldsymbol{x}|^2 = c + t$, which is just the sphere with radius $c + t$. The rays are now radial lines and so the points forming S_c move along radial lines to form the surface S_{c+t}. To this end, we use a modified version of spherical coordinates:

$$(x, y, z) = \rho(s) \left(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha\right),$$

with

$$0 \leq \alpha < \pi, \quad 0 \leq \beta \leq 2\pi, \quad 0 \leq s.$$

The function $\rho(s)$ is required to be smooth and strictly increasing. Examples are $\rho = s$, $\rho = e^s - 1$ or $\rho = \ln(1 + s)$.

An important property of the preceeding modified spherical coordinates is that (s, α, β) forms an orthogonal coordinate system. That is, under the change of variables $\boldsymbol{x} = \boldsymbol{X}(s, \alpha, \beta)$, the vector $\partial_s \boldsymbol{X}$ tangent to the ray is orthogonal to the wave front S_{c+t}. We now in the opposite case: we need to find $\theta(\boldsymbol{x})$ given conditions on the map $\boldsymbol{X}(s, \alpha, \beta)$. Observe the degree of freedom on specifiy $\boldsymbol{X}$!

4.3 Solution of the Eikonal equation

In what follows, we will assume that (s, α, β) forms an orthogonal coordinate system. This means that a ray's tangent vector $\partial_s \boldsymbol{X}$ points in the same direction as $\nabla\theta$ when $\boldsymbol{x} = \boldsymbol{X}(s, \alpha, \beta)$, or equivalently

$$\frac{\partial \boldsymbol{X}}{\partial s} = \lambda \nabla \theta, \tag{4.10}$$

where λ is a smooth positive function, to be specified later. Without loss of generality, we assume that the rays are parameterized so that $s \geq 0$. One should not confuse s with the arclength parameterization.

Along a ray,

$$\partial_s \theta(\boldsymbol{x}) = \partial_s \theta(\boldsymbol{X}) = \nabla\theta \cdot \partial_s \boldsymbol{X} = \lambda \nabla\theta \cdot \nabla\theta.$$

Therefore we can rewrite the Eikonal equation as

$$\partial_s \theta = \lambda \mu^2, \tag{4.11}$$

which can be integrated directly to yield

$$\theta(s, \alpha, \beta) = \theta(0, \alpha, \beta) + \int_0^s \lambda \mu^2 \, d\sigma, \tag{4.12}$$

assuming we can find such a coordinate system (s, α, β). This amounts to solving (4.10) which is generally nonlinear and requires the assistance of numerical method. Nonetheless, we still have the freedom of choosing the function λ.

4.4 Solution of the transport equation

It remains to find the first term V_0 of the WKB approximation (4.6). Using (4.10) we have

$$del_s V_0 = \nabla V_0 \cdot \partial_s \boldsymbol{X} = \lambda \nabla V_0 \cdot \nabla \theta.$$

Consequently we can also rewrite the transport equation (4.9) as

$$2\partial_s V_0 + \lambda \left(\nabla^2 \theta\right) V_0 = 0. \tag{4.13}$$

Using the identity

$$\partial_s \left(\frac{J}{\lambda}\right) = J \nabla^2 \theta, \tag{4.14}$$

where $J = \left|\frac{\partial(x, y, z)}{\partial(s, \alpha, \beta)}\right|$ is the Jacobian of the transformation $\boldsymbol{x} = \boldsymbol{X}(s, \alpha, \beta)$, we can rewrite (4.13) as

$$2J\partial_s V_0 + \lambda \partial_s \left(\frac{J}{\lambda}\right) V_0 = 0.$$

This follows that

$$\partial_s \left(\frac{1}{\lambda} J V_0^2\right) = 0,$$

and its general solution is

$$V_0(\boldsymbol{x}) = a_0 \sqrt{\frac{\lambda(\boldsymbol{x})}{J(\boldsymbol{x})}}. \tag{4.15}$$

Imposing the boundary condition $V_0(\boldsymbol{x}_0) = f(\boldsymbol{x}_0)$, we obtain

$$V_0(\boldsymbol{x}) = f(\boldsymbol{x}_0)\sqrt{\frac{\lambda(\boldsymbol{x})J(\boldsymbol{x}_0)}{\lambda(\boldsymbol{x}_0)J(\boldsymbol{x})}}, \tag{4.16}$$

and this is true provided $\theta(0,\alpha,\beta) = 0$ in (4.12). (Otherwise we will get an exponential term!)

We now prove the identity (4.14) in 2D but this easily extends to 3D. Main idea comes from Exercise 4.43 in [2]. The transformation in 2D is $\boldsymbol{x} = \boldsymbol{X}(s,\alpha)$ and its Jacobian is

$$J = \left|\frac{\partial(x,y)}{\partial(s,\alpha)}\right| = \partial_s x \partial_\alpha y - \partial_\alpha x \partial_s y.$$

Using the ray equation (4.10)

$$\partial_s J = \partial_s(\lambda\partial_x\theta)\partial_\alpha y + \partial_s x \partial_\alpha(\lambda\partial_y\theta) - \partial_\alpha(\lambda\partial_x\theta)\partial_s y - \partial_\alpha x \partial_s(\lambda\partial_y\theta),$$

and the chain rule yields

$$\begin{aligned}\partial_s J = {} & \partial_\alpha y[\partial_s x\partial_x + \partial_s y\partial_y](\lambda\partial_x\theta) - \partial_\alpha x[\partial_s x\partial_x + \partial_s y\partial_y](\lambda\partial_y\theta) \\ & - \partial_s y[\partial_\alpha x\partial_x + \partial_\alpha y\partial_y](\lambda\partial_x\theta) + \partial_s x[\partial_\alpha x\partial_x + \partial_\alpha y\partial_y](\lambda\partial_y\theta).\end{aligned}$$

Rearranging it

$$\partial_s J = J\partial_x(\lambda\partial_x\theta) + J\partial_y(\lambda\partial_y\theta),$$

and this follows that

$$\partial_s J = J\nabla\cdot(\lambda\nabla\theta). \tag{4.17}$$

For a smooth function $q(\boldsymbol{x})$, one can achieve from the chain rule

$$\partial_s(qJ) = \nabla q \cdot \partial_s x J + q\partial_s J.$$

Imposing the ray equation (4.10) and (4.17) gives

$$\partial_s(qJ) = [\nabla q\cdot(\lambda\nabla\theta) + q\nabla\cdot(\lambda\nabla\theta)]\, J,$$

and it follows that

$$\partial_s(qJ) = J\nabla\cdot(q\lambda\nabla\theta). \tag{4.18}$$

Finally, setting $q = 1/\lambda$ leads the equation to the identity (4.14).

4.5 Ray equation

Set $\boldsymbol{X} = (X_1, X_2, X_3)$, divide the ray equation (4.10) by λ and differentiating the resulting equation yields

$$\frac{\partial}{\partial s}\left[\frac{1}{\lambda}\frac{\partial X_i}{\partial s}\right] = \frac{\partial}{\partial s}\left(\frac{\partial\theta(\boldsymbol{x})}{\partial x_i}\right),$$

and applying the chain rule

$$\frac{\partial}{\partial s}\left[\frac{1}{\lambda}\frac{\partial X_i}{\partial s}\right] = \sum_{j=1}^{3}\frac{\partial x_i}{\partial s}\frac{\partial}{\partial x_j}\frac{\partial\theta(\boldsymbol{x})}{\partial x_i}$$

Imposing the ray equation again

$$\frac{\partial}{\partial s}\left[\frac{1}{\lambda}\frac{\partial X_i}{\partial s}\right] = \left(\frac{\partial}{\partial x_i}\nabla\theta\right)\cdot(\lambda\nabla\theta),$$

and it follows that

$$\frac{\partial}{\partial s}\left[\frac{1}{\lambda}\frac{\partial X_i}{\partial s}\right] = \frac{1}{2}\lambda\frac{\partial}{\partial x_i}\left(\nabla\theta\cdot\nabla\theta\right).$$

Invoking the Eikonal equation (4.8) gives

$$\frac{\partial}{\partial s}\left[\frac{1}{\lambda}\frac{\partial X_i}{\partial s}\right] = \frac{1}{2}\lambda\frac{\partial}{\partial x_i}\mu^2.$$

In vector form, this equals

$$\frac{\partial}{\partial s}\left(\frac{1}{\lambda}\frac{\partial}{\partial s}\boldsymbol{X}\right) = \lambda\mu\nabla\mu. \tag{4.19}$$

We require two boundary conditions as (4.19) is a second order equation in s. Each ray starts on the initial surface S. Given any point on $\boldsymbol{x}_0 \in S$, its ray satisfies

$$\boldsymbol{X}|_{s=0} = \boldsymbol{x}_0. \tag{4.20}$$

The second boundary condition is typically

$$\left.\frac{\partial \boldsymbol{X}}{\partial s}\right|_{s=0} = \lambda_0\mu_0\boldsymbol{n}_0, \tag{4.21}$$

where $\boldsymbol{n}_0$ is the unit outward normal at $\boldsymbol{x}_0$, $\lambda_0 = \lambda(0,\alpha,\beta)$ and $\mu_0 = \mu(0,\alpha,\beta)$.

4.6 Choice of λ

Taking the dot product of the ray equation against $\partial_s\boldsymbol{X}$, we have

$$\frac{\partial \boldsymbol{X}}{\partial s}\cdot\frac{\partial \boldsymbol{X}}{\partial s} = \lambda^2\nabla\theta\cdot\nabla\theta = \lambda^2\mu^2.$$

Let ℓ be the arc length along a ray. Then

$$\ell = \int_0^s \|\partial_s X\|\,ds = \int_0^s \lambda\mu ds'.$$

Hence, if we choose $\lambda = 1/\mu$ then s equals the arclength. Another common choice is $\lambda = 1$.

4.7 Summary for $\lambda = 1/\mu$

The ray equation (4.10) becomes

$$\frac{\partial}{\partial s}\left(\mu\frac{\partial}{\partial s}\boldsymbol{X}\right) = \nabla\mu(\boldsymbol{X}), \tag{4.22}$$

with $\boldsymbol{X}|_{s=0} = \boldsymbol{x}_0 \in S$ and $\partial_s\boldsymbol{X}|_{s=0} = \boldsymbol{n}_0$. Once this is solved, the phase function becomes

$$\Theta(\boldsymbol{X}) = \int_0^s \mu(\boldsymbol{X})\,d\sigma, \tag{4.23}$$

and the amplitude is

$$V_0(\boldsymbol{x}) = f(\boldsymbol{x}_0)\sqrt{\frac{\mu(\boldsymbol{x}_0)J(\boldsymbol{x}_0)}{\mu(\boldsymbol{x})J(\boldsymbol{x})}}. \tag{4.24}$$

Finally, the WKB approximation is

$$u(\boldsymbol{x},t) \sim f(\boldsymbol{x}_0)\sqrt{\frac{\mu(\boldsymbol{x}_0)J(\boldsymbol{x}_0)}{\mu(\boldsymbol{x})J(\boldsymbol{x})}}\exp\left[i\omega\left(-t+\int_0^s \mu(\boldsymbol{X}(\sigma))\,d\sigma\right)\right], \tag{4.25}$$

where s is the value for which the solution of satisfies $\boldsymbol{X}(s) = \boldsymbol{x}$.

Example 4. Suppose μ is some constant μ_0. The ray equation (4.7) becomes

$$\frac{\partial^2 \boldsymbol{X}}{\partial s^2} = 0 \implies \boldsymbol{X}(s) = \boldsymbol{x}_0 + s\boldsymbol{n}_0.$$

Also,

$$\theta = \mu_0 \int_0^s d\sigma = \mu s.$$

Thus, given a point $\boldsymbol{x}$ on the ray $s = \boldsymbol{n}_0 \cdot (\boldsymbol{x} - \boldsymbol{x}_0)$ the WKB approximation is

$$u(\boldsymbol{x}, t) \sim f(\boldsymbol{x}_0) \sqrt{\frac{J(\boldsymbol{x}_0)}{J(\boldsymbol{x})}} \exp\left[i\left(\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}_0) - \omega t\right)\right],$$

where $\boldsymbol{k} = \mu_0 \omega \boldsymbol{n}_0$. In $\mathbb{R}^2$, when the boundary surface is a circle of radius a, one finds that the Jacobian in polar coordinates is just ρ and

$$u(\boldsymbol{x}, t) \sim f(\boldsymbol{x}_0) \frac{\sqrt{a}\rho}{e}^{i\omega(\mu(\rho - a) - t)}.$$

4.8 Breakdown of the WKB solution

- Fails at turning points $\boldsymbol{x}$ where $\mu(\boldsymbol{x}) = 0$. But this can be handled analogously to 1D case (boundary layer).
- A more likely complication arises when $J = 0$. Points where this occur are called caustics - arise when two or more rays intersect. Breakdown of the characteristic coordinates (s, α, β). If a ray passes through a caustic, one picks up an additional factor in WKB solution of the form $e^{im\pi/2}$, where m depends on the rank of J at the caustic.
- A less obvious breakdown occurs when $\boldsymbol{X}(s) = \boldsymbol{x}$ has no solution. This happens with shadow regions - ray splitting.

5 EXERCISES

Problem 1. *Use the WKB method to find an approximation of the following problem on* $x \in [0, 1]$*:*

$$\epsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

Proof. The WKB ansatz gives

$$y(x) \sim e^{\theta(x)/\epsilon^\alpha} \left[y_0(x) + \epsilon^\alpha y_1(x) + \cdots\right].$$

To apply it into the given equation, find

$$y' \sim e^{\theta(x)/\epsilon^\alpha} \left[\epsilon^{-\alpha}\theta' y_0 + (y_0' + \theta' y_1) + \epsilon^\alpha y_1' + \cdots\right],$$

and

$$y'' \sim e^{\theta(x)/\epsilon^\alpha} \left[\epsilon^{-2\alpha}(\theta')^2 y_0 + \epsilon^{-\alpha}[(\theta' y_0)' + \theta'(y_0' + \theta' y_1)] + \cdots\right].$$

To balance the given equation, we require $\alpha = 1$. Substituting them into the given equation and collecting $\mathcal{O}(\epsilon^{-1})$ gives the Eikonal equation

$$(\theta')^2 + 2\theta' = 0. \tag{5.1}$$

Balancing $\mathcal{O}(1)$ terms yields

$$(\theta' y_0)' + \theta'(y_0' + \theta' y_1) + 2(y_0' + \theta' y_1) + 2y_0 = 0,$$

and it follows that

$$(\theta' + 1)y_0' + y_0 = 0. \tag{5.2}$$

Then one can achieve the first term of its WKB expansion

$$y(x) \sim A_1 e^{-x} + A_2 e^{x - \frac{2x}{\epsilon}}. \tag{5.3}$$

Imposing the left-hand boundary condition gives

$$A_1 + A_2 = 0,$$

and the right-hand boundary condition

$$A_1 e^{-1} + A_2 e^{1 - \frac{2}{\epsilon}} = 1.$$

Then one can achieve

$$A_1 = \frac{1}{e^{-1} - e^{1-2/\epsilon}}, \quad A_2 = -A_1.$$

Substituting it into the general solution yields

$$y_W(x) \sim A_1(\epsilon)\left(e^{-x} - e^{x(1-2/\epsilon)}\right). \tag{5.4}$$

It is known that the match asymptotic expansion of given problem is

$$y_M(x) \sim e\left(e^{-x} - e^{-2x/\epsilon}\right). \tag{5.5}$$

One clear difference is that y_W satisfies both boundary conditions but

$$y_M(1) = 1 - e^{1-2/\epsilon},$$

It implies that y_M does not satisfies the right-hand boundary condition exactly but $\mathcal{O}(1)$ order. Moreover, for sufficiently small $\epsilon > 0$,

$$A_1(\epsilon) \sim e, \quad 1 - \frac{2}{\epsilon} \sim -\frac{2}{\epsilon},$$

and it implies that y_W and y_M are similar when ϵ is sufficiently small. ■

Problem 2. *Consider seismic waves propagating through the upper mantle of the earth from a source on the earth's surface (see Fig. 5). We want to use a WKB approximation in $\mathbb{R}^3$ to solve the equation*

$$\nabla^2 v + w^2 \mu^2(r) v = 0,$$

where μ has spherical symmetry. Take $\lambda = 1/\mu$.

1. *Use the ray equation to show the vector*

$$\mathbf{p} = \mathbf{r} \times (\mu \partial_s \mathbf{r}),$$

is independent of s. Hence, show that $r\mu \sin(\chi)$ is constant along a ray, where χ is the angle between $\mathbf{r}$ and $\partial_s \mathbf{r}$.

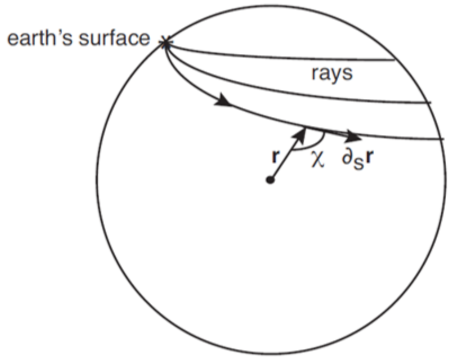

Figure 5: Rays representing waves propagating inside the earth from a source on the surface of the earth.

Proof. It suffices to show that the partial derivative of $\mathbf{p}$ with respect to s is zero. Applying product rule gives

$$\partial_s \mathbf{p} = \partial_s \mathbf{r} \times (\mu \partial_s \mathbf{r}) + \mathbf{r} \times \partial_s(\mu \partial_s \mathbf{r}),$$

and one can rewrite it by ray equation

$$\partial_s \mathbf{p} = \mu(\partial_s \mathbf{r} \times \partial_s \mathbf{r}) + \mathbf{r} \times \nabla \mu(r).$$

Using the fact that $\nabla \mu = \mu' \mathbf{r}/r$ gives

$$\partial_s \mathbf{p} = \mu(\partial_s \mathbf{r} \times \partial_s \mathbf{r}) + \frac{\mu'(r)}{r}(\mathbf{r} \times \mathbf{r}) = 0,$$

because the product of parallel vectors is zero. This implies that $\mathbf{p}$ is independent of s. Then it follows that $\|\mathbf{p}\|$ is also independent of s. One can obtain

$$\|\mathbf{p}\| = \|\mathbf{r}\| \cdot \|\mu \partial_s \mathbf{r}\| \sin \chi = r\mu \|\partial_s \mathbf{r}\| \sin \chi. \tag{5.6}$$

Taking the dot product of the ray equation with respect to $\partial_s \mathbf{r}$ gives

$$\partial_s \mathbf{r} \cdot \partial_s \mathbf{r} = \lambda^2 \mu^2 = 1. \tag{5.7}$$

Combining (5.6) and (5.7) yields

$$\|\mathbf{p}\| = r\mu \sin \chi,$$

and it shows that $r\mu \sin \chi$ is constant along a ray. ■

2. *Part 1 implies that each ray lies in a plane containing the origin of the sphere. Let ρ, φ be polar coordinates of this plane. It follows that for a polar curve $\rho = \rho(\varphi)$, the angle χ satisfies*

$$\sin(\chi) = \frac{\rho}{\sqrt{\rho^2 + (\partial_\varphi \rho)^2}}.$$

Assuming $\partial_\varphi \neq 0$, show that

$$\varphi = \varphi_0 + \int_{\rho_0}^{\rho} \frac{dr}{r\sqrt{\mu^2 r^2 - \kappa^2}},$$

where $\rho_0, \varphi_0, \kappa$ are constants.

Proof. On the polar coordinate, it follows that

$$\mathbf{r} = \rho(\cos\varphi, \sin\varphi), \quad \partial_s \mathbf{r} \parallel \partial_\varphi \mathbf{r} = \partial_\varphi \rho(\cos\varphi, \sin\varphi) + \rho(-\sin\varphi, \cos\varphi),$$

and

$$\|\mathbf{r}\| = \rho, \quad \|\partial_\varphi \mathbf{r}\| = \sqrt{\rho^2 + (\partial_\varphi \rho)^2}.$$

This follows that

$$\sin\chi = \frac{\mathbf{r} \times \partial_\varphi \mathbf{r}}{\|\mathbf{r}\| \cdot \|\partial_\varphi \mathbf{r}\|} = \frac{\rho}{\sqrt{\rho^2 + (\partial_\varphi \rho)^2}}. \tag{5.8}$$

From part (a), $r\mu \sin\chi$ is constant along a ray and this gives that

$$r\mu \sin\chi = \frac{\rho^2 \mu}{\sqrt{\rho^2 + (\partial_\varphi \rho)^2}} = \kappa, \tag{5.9}$$

where κ is an constant. Then one can solve it for $\partial_\varphi \rho$

$$\partial_\varphi \rho = \frac{\rho}{\kappa}\sqrt{\rho^2\mu^2 - \kappa^2},$$

and performing chain rule yields that

$$\partial_\rho \varphi(\rho) = \frac{\kappa}{\rho\sqrt{\rho^2\mu^2 - \kappa^2}}. \tag{5.10}$$

Integrating it with respect to ρ gives that

$$\varphi = \varphi_0 + \kappa \int_{\rho_0}^{\rho} \frac{dr}{r\sqrt{r^2\mu^2 - \kappa^2}}. \tag{5.11}$$

■

3. *Using the definition of arc length, show that for a polar curve*

$$ds = \sqrt{\rho^2 + (\partial_\varphi \rho)^2} d\varphi.$$

Combining this result with part 2, show that the solution of the Eikonal equation is given by

$$\theta = \frac{1}{\kappa}\int_{\varphi_0}^{\varphi} \mu^2 \rho^2 d\varphi.$$

Proof. Since the choice of λ, then $\|\partial_s \mathbf{r}\| ds = 1 \cdot ds$. Moreover, $\|\partial_\varphi \mathbf{r}\| d\varphi = \sqrt{\rho^2 + (\partial_\varphi \rho)^2} d\varphi$. By the definition of arc length, one can obtain that

$$dl = 1 \cdot ds = \sqrt{\rho^2 + (\partial_\varphi \rho)^2} \cdot d\varphi. \tag{5.12}$$

In case of choosing $\lambda = 1$, then we have $\|\partial_s \mathbf{r}\| ds = \mu \cdot ds$ and it follows that

$$dl = \mu ds = \sqrt{\rho^2 + (\partial_\varphi \rho)^2} \cdot d\varphi.$$

Recall the Eikonal equation

$$\partial_s \theta = \lambda \mu^2 = \mu.$$

Performing integration with the change of variables in (5.12) yields

$$\theta = \int_{\varphi_0}^{\varphi} \mu \sqrt{\rho^2 + (\partial_\varphi \rho)^2} d\varphi,$$

and one can obtain θ by imposing (5.9)

$$\theta = \int_{\varphi_0}^{\varphi} \mu \cdot \frac{\rho^2 \mu}{\kappa} d\varphi = \frac{1}{\kappa} \int_{\varphi_0}^{\varphi} \mu^2 \rho^2 d\varphi. \quad (5.13)$$

■

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