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These notes are largely based on Math 6730: Asymptotic and Perturbation Methods course, taught by Paul Bressloff in Fall 2017, at the University of Utah. Additional examples or remarks or results from other sources are added as we see fit, mainly to facilitate our understanding. These notes are by no means accurate or applicable, and any mistakes here are of course our own. Please report any typographical errors or mathematical fallacy to us by email hkim@math.utah.edu or tan@math.utah.edu.
1 INTRODUCTORY EXAMPLE

As in the previous chapter, we will introduce the ideas underlying the method by a simple example. Consider the initial value problem

\[ y'' + \epsilon y' + y = 0 \quad \text{for } t > 0 \]  
\[ y(0) = 0, \quad y'(0) = 1 \]  

which models a linear oscillator with weak damping. This reduces to the linear oscillator model when \( \epsilon = 0 \).

1.1 Regular expansion

We do not expect boundary layers since \((1.1)\) is not a singular problem. This suggests that the solution might have a regular asymptotic expansion, that is we try a regular expansion

\[ y(t) \sim y_0(t) + \epsilon y_1(t) + \cdots, \quad \text{as } \epsilon \to 0 \]  

Substituting \((1.2)\) into \((1.1)\) and collecting terms in equal powers of \( \epsilon \) yields

\[ y_0'' + y_0 = 0, \quad y_n'' + y_n = -y_{n-1} \]  

for \( n \geq 1 \), with initial conditions

\[ y_0(0) = 0, \quad y_0'(0) = 1, \quad y_n(0) = y_n'(0) = 0, \quad n \geq 1. \]

Solving the \( O(1) \) and \( O(\epsilon) \) equations we obtain

\[ y(t) \sim \sin(t) - \frac{1}{2} \epsilon t \sin(t), \]  

but this is problematic since the correction term \( y_1(t) \) contains a secular term \( t \sin(t) \) which blows up as \( t \to \infty \). Consequently, the asymptotic expansion is valid for only small values of \( t \), since \( \epsilon y_1(t) \sim y_0(t) \) when \( \epsilon t \sim 1 \). The problem is that regular perturbation theory does not capture the correct behaviour of the exact solution. Indeed, \((1.1)\) is a constant-coefficient linear ODE and it can be solved exactly:

\[ y(t) = \frac{1}{\sqrt{1 - \epsilon^2/4}} e^{-\epsilon t/2} \sin \left( t \sqrt{1 - \epsilon^2/4} \right) \]

It is clear that the exact solution decays but the first term in our regular asymptotic approximation \((1.3)\) does not. Also, we will pick up the secular terms if we naively expand the exponential function around \( t = 0 \), since

\[ y(t) \approx \left( 1 - \frac{\epsilon t}{2} + \frac{\epsilon^2 t^2}{8} + \ldots \right) \sin(t). \]

1.2 Multiple-scale expansion

In fact, there are two time-scales in the exact solution:

1. The slowly decaying exponential component which varies on a time-scale of \( O(1/\epsilon) \);
2. The fast oscillating component which varies on a time-scale of \( O(1) \).
To identify or separate these time-scales, we introduce the variables

\[
t_1 = t, \quad t_2 = \epsilon^\alpha t, \quad \alpha > 0,
\]

where \( t_2 \) is called the slow time-scale because it does not affect the asymptotic expansion until \( \epsilon^\alpha t \sim 1 \). We treat these two time-scales as independent variables and consequently the original time derivative becomes

\[
\frac{d}{dt} \rightarrow \frac{dt_1}{dt} \frac{\partial}{\partial t_1} + \frac{dt_2}{dt} \frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \epsilon^\alpha \frac{\partial}{\partial t_2}.
\]

Substituting (1.5) into (1.1) yields the transformed problem

\[
\left[ \frac{\partial^2}{\partial t_1^2} + 2\epsilon^\alpha \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} + \epsilon^{2\alpha} \frac{\partial^2}{\partial t_2^2} \right] y + \epsilon \left( \frac{\partial}{\partial t_1} + \epsilon^\alpha \frac{\partial}{\partial t_2} \right) y + y = 0,
\]

\[
y(t_1, t_2) \bigg|_{t_1=t_2=0} = 0, \quad \left( \frac{\partial}{\partial t_1} + \epsilon^\alpha \frac{\partial}{\partial t_2} \right) y(t_1, t_2) \bigg|_{t_1=t_2=0} = 1.
\]

Unlike the original problem, additional constraints are needed for (1.6) to have a unique solution, and it is precisely this degree of freedom that allows us to eliminate the secular terms!

We now introduce an asymptotic expansion

\[
y \sim y_0(t_1, t_2) + \epsilon y_1(t_1, t_2) + \cdots.
\]

Substituting (1.7) into (1.6) yields

\[
\left[ \frac{\partial^2}{\partial t_1^2} + 2\epsilon^\alpha \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} + \epsilon^{2\alpha} \frac{\partial^2}{\partial t_2^2} \right] [y_0 + \epsilon y_1 + \cdots]
\]

\[
+ \epsilon \left( \frac{\partial}{\partial t_1} + \epsilon^\alpha \frac{\partial}{\partial t_2} \right) (y_0 + \cdots) + (y_0 + \epsilon y_1 + \cdots) = 0.
\]

The \( O(1) \) problem becomes

\[
\left( \frac{\partial^2}{\partial t_1^2} + 1 \right) y_0 = 0,
\]

with boundary conditions \( y_0(0, 0) = 0, \frac{\partial}{\partial t_1} y_0(0, 0) = 1 \), and its general solution is

\[
y_0(t_1, t_2) = a_0(t_2) \sin(t_1) + b_0(t_2) \cos(t_1),
\]

Figure 1: Comparison between the regular asymptotic approximation (1.4) and the exact solution (1.4) for \( \epsilon = 0.1 \).
where \( a_0(0) = 1, b_0(0) = 0 \). Note that \( y_0(t_1, t_2) \) consists of purely harmonic components with slowly varying amplitude. We now need to determine \( a \) in the slow time-scale \( t_2 \). Observe that for \( a > 1 \) the \( \mathcal{O}(e) \) equation is

\[
\left( \partial_{t_1}^2 + 1 \right) y_1 = -\partial_{t_1} y_0,
\]

and the inhomogeneous term \( \partial_{t_1} y_0 \) will generate secular terms, since it belongs to the kernel of homogeneous linear operator \( (\partial_{t_1}^2 + 1) \). More importantly, there is no way to generate non-trivial solution that will cancel the secular term. This can be prevented by choosing \( a = 1 \).

The \( \mathcal{O}(e) \) equation is

\[
\left( \partial_{t_1}^2 + 1 \right) y_1 = -2\partial_{t_1} \partial_{t_2} y_0 - \partial_{t_1} y_0,
\]

with boundary conditions \( y_1(0, 0) = 0 \) and \( \partial_{t_1} y_1(0, 0) + \partial_{t_2} y_0(0, 0) = 0 \). Substituting \( y_0 \) gives

\[
\left( \partial_{t_1}^2 + 1 \right) y_1 = (2b'_0 + b_0) \sin(t_1) - (2a'_0 + a_0) \cos(t_1).
\]

The general solution of the \( \mathcal{O}(e) \) problem is

\[
y_1(t_1, t_2) = a_1(t_2) \sin(t_1) + b_1(t_2) \cos(t_1)
= \frac{1}{2} (2b'_0 + b_0) t_1 \cos(t_1) - \left( \frac{1}{2} (2a'_0 + a_0) + \frac{1\text{ secular}}{2} \right) t_1 \sin(t_1),
\]

with \( a_1(0) = b'_0(0), b_1(0) = 0 \). We can choose the functions \( a_0, b_0 \) to remove the secular terms, which results in

\[
2b_0' + b_0 = 0 \implies b_0(t_2) = \beta_0 e^{-t_2/2} = 0,
\]

since \( b_0(0) = 0 \) and

\[
2a_0' + a_0 = 0 \implies a_0(t_2) = \alpha_0 e^{-t_2/2} = e^{-t_2/2},
\]

because \( a_0(0) = 0 \). Hence, a first term approximation of the solution \( y(t) \) of (1.1) is

\[
y \sim e^{-t/2} \sin(t).
\]

One can prove that this asymptotic expansion is uniformly valid for \( 0 \leq t \leq \mathcal{O}(1/e) \).

1.3 Discussion

1. Many problems have the \( \mathcal{O}(1) \) equation as

\[
y_0'' + \omega^2 y_0 = 0.
\]

and the general solution is

\[
y_0(t) = a \cos(\omega t) + b \sin(\omega t).
\]

If the original problem is nonlinear and the \( \mathcal{O}(1) \) equation is as above, then it is usually more convenient to use a complex representation of \( y_0 \), that is

\[
y(t) = Ae^{i\omega t} + Ae^{-i\omega t} = B \cos(\omega t + \theta).
\]

These complex representations make identify the secular terms much easier.
2. Often, higher-order equations have the form

\[ y'' + \omega^2 y = f(t). \]

A secular term arises if \( f(t) \) contains a solution of the \( O(1) \) problem, for example \( \cos(\omega t) \) or \( \sin(\omega t) \). We can avoid secular terms by requiring the \( t^2 \)-dependent coefficients of \( \cos(\omega t_1) \) and \( \sin(\omega t_1) \) to vanish. For example, there are no secular terms if \( f(t) = \sin(\omega t) \cos(\omega t) = \sin(2\omega t)/2, \)

but there is a secular term if \( f(t) = \cos^3(\omega t) = \frac{1}{4}(3\cos(\omega t) + \cos(3\omega t)) \).

3. The time scales should be modified depending on the problem. Some possibilities include:

a) Several time-scales: for example, \( t_1 = t/\epsilon, t_2 = t, t_3 = \epsilon t, \ldots \).

b) More complex \( \epsilon \)-dependency:

\[ t_1 = (1 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \ldots) \quad t, \quad t_2 = \epsilon t. \]

This is called the Lindstedt’s method or the method of strained coordinates.

c) Correct scaling may not be obvious, so we might start off with

\[ t_1 = e^\alpha t, \quad t_2 = e^\beta t, \quad \alpha < \beta. \]

d) Nonlinear time-dependence:

\[ t_1 = f(t, \epsilon), \quad t_2 = \epsilon t. \]

2 FORCED MOTION NEAR RESONANCE

In this section, we consider an extension of the introductory example: a damped nonlinear oscillator that is forced at a frequency near resonance. As an example, we will study the damped Duffing equation

\[ y'' + \epsilon \lambda y' + y + \epsilon \kappa y^3 = \epsilon \cos((1 + \epsilon \omega)t) \quad \text{for } t > 0 \quad (2.1a) \]

\[ y(0) = 0, \quad y'(0) = 0. \quad (2.1b) \]

The damping term, nonlinear correction term and forcing term are small. Also, \( \omega, \lambda, \kappa \) are constants with \( \lambda \) and \( \kappa \) nonnegative. We expect the solution to be small due to the small forcing and zero initial conditions.

Consider the simpler equation

\[ y'' + y = \epsilon \cos(\Omega t), \quad y(0) = y'(0) = 0. \quad (2.2) \]

with \( \Omega \neq \pm 1 \). The unique solution is

\[ y(t) = \frac{\epsilon}{1 - \Omega^2} \left[ \cos(\Omega t) - \cos(t) \right] \quad (2.3) \]

and the solution blows up as expected when the driving frequency \( \Omega \approx 1 \). To understand the situation, suppose \( \Omega = 1 + \epsilon \omega \). The particular solution of (2.2) is given by

\[ y_p(t) = \begin{cases} 
-\frac{1}{\omega(2+\epsilon \omega)} \cos((1 + \epsilon \omega)t) & \text{if } \omega \neq 0, -2/\epsilon, \\
\frac{1}{2} \epsilon t \sin(t) & \text{otherwise}. 
\end{cases} \quad (2.4) \]
In both cases a relatively small, order $O(\varepsilon)$, forcing results in at least an $O(1)$ solution. Moreover, the behaviour of the solution depends on $\omega$, which is typical of a forcing system.

We take $t_1 = t$ and $t_2 = \epsilon t$, although we should take $t_2 = \epsilon^a t, a > 0$ in general to allow for some flexibility. The forced Duffing equation becomes

$$
\left[ \partial_t^2 + 2\varepsilon \partial_t + \varepsilon^2 \partial_t^2 \right] y + \varepsilon \lambda \left[ \partial_t + \varepsilon \partial_t^2 \right] y + y + \varepsilon y^3 = \varepsilon \cos (t_1 + \epsilon \omega t_1).
$$

Although we expect the leading-order term in the expansion to be $O(\varepsilon)$, the solution can become larger near a resonant frequency. Because it is not clear what amplitude the solution actually reaches, we guess a general asymptotic expansion of the form

$$
y \sim \varepsilon^\beta y_0(t_1, t_2) + \varepsilon^{\gamma} y_1(t_1, t_2) + \cdots, \quad \beta < \gamma.
$$

We also assume that $\beta < 1$ due to the resonance effect. Substituting (2.6) into (2.5) gives

$$
\left[ \varepsilon^\beta \partial_t^2 y_0 + 2\varepsilon^{1+\beta} \partial_t \partial_t y_0 + \varepsilon^{2\beta} \partial_t^2 y_0 + \cdots \right] + \left[ \varepsilon^\gamma \partial_t y_0 + \varepsilon^{\gamma+1} y_0 + \cdots \right] + \left[ \varepsilon^{1+\beta} \partial_t y_0 + \varepsilon^{1+\beta+1} y_0 + \cdots \right] + \left[ \varepsilon^{1+3\beta} y_0 + \cdots \right] = \varepsilon \cos (t_1 + \epsilon \omega t_1).
$$

The $O(\varepsilon^\beta)$ problem is

$$
\left( \partial_t^2 + 1 \right) y_0 = 0,
$$

with $y_0(0,0) = \partial_t y_0(0,0) = 0$, and its general solution is

$$
y_0 = A(t_2) \cos(t_1 + \theta(t_2)), \quad A(0) = 0.
$$

We need to determine $\beta$ and $\gamma$ before proceed any further. The terms (2) concern with the preceding solution $y_0$ and the term (3) is the forcing term. For the most complete approximation, the problem for the second term $y_1$ in the expansion (2.6), which comes from the terms (1), must deal with both (2) and (3). This is possible if we choose $\gamma = 1$ and $\beta = 0$. Collecting $O(\varepsilon)$ terms gives

$$
\left( \partial_t^2 + 1 \right) y_1 = -2\partial_t \partial_t y_0 - \lambda \partial_t y_0 - \kappa y_0^3 + \cos(t_1 + \omega t_2),
$$

and it follows that

$$
\left( \partial_t^2 + 1 \right) y_1 = \left[ 2A' + \lambda A \right] \sin(t_1 + \theta) + 2\theta' A \cos(t_1 + \theta)
$$

$$
- \frac{\kappa}{4} A^3 \left[ 3 \cos(t_1 + \theta) + \cos \left( 3(t_1 + \theta) \right) \right] + \cos(t_1 + \omega t_2).
$$

Note that

$$
\cos(t_1 + \omega t_2) = \cos(t_1 + \theta) \cos(\theta - \omega t_2) + \sin(t_1 + \theta) \sin(\theta - \omega t_2).
$$

Thus, we can remove the secular terms $\sin(t_1 + \theta)$ and $\cos(t_1 + \theta)$ by requiring

$$
2A' + \lambda A = -\sin(\theta - \omega t_2), \quad (2.7a)
$$

$$
2\theta' A - \frac{3\kappa}{4} A^3 = -\cos(\theta - \omega t_2). \quad (2.7b)
$$

From $A(0) = 0$ and assuming $A'(0) > 0$, it follows that $\theta(0) = -\pi/2$. 
Figure 2: Nullcline for $\phi_\tau$. (a) $F(r, \beta)$ as a function of $r$ with varying $\beta$. Parameter are given by $\gamma = 0.75$ and $\beta = -\beta_c, \beta_c/2, \beta_c, 1.5 \beta_c$, respectively.

It remains to solve (2.7) with initial conditions $A(0) = 0, \theta(0) = -\pi/2$ to find the amplitude function $A(t_2)$ and phase function $\theta(t_2)$. For the analytic simplicity, changing variables as $r = \sqrt{\kappa} A/2$ and $\phi = \theta - \omega t_2$ gives

$$
\begin{cases}
2r' = -\lambda r - \frac{\gamma}{2} \sin \phi, \\
2\phi' = \beta + 3r^2 - \frac{\gamma}{2} \cos \phi.
\end{cases}
$$

(2.8)

where $\gamma = \sqrt{\kappa}$ and $\beta = -2\omega$. We now analyze the rewritten amplitude equation (2.8). The nullcline for $r_\tau$ is $r = -\gamma \sin \theta/2\lambda$. Similarly, nullcline for $\phi_\tau$ is given by $\cos \theta = 2r(\beta + 3r^2)/\gamma \equiv F(r, \beta)$, see Fig. 2:

- If $\beta > 0$, there is unique $r$ for each $\theta$, see the blue line.
- If $0 > \beta > \beta_c$ where $\min_r F(r, \beta_c) = -1$ (and it turns out that $\beta_c^2 = -81\gamma^2/16$), then there are two values of $r$ for each $\cos \theta$ in some interval $(z, 0)$ for some $z \in [0, 1]$. See the red line.
- If $\beta < \beta_c$, then two values of $r$ exist for all $\cos \theta$ between $-1$ and 0. See the purple line.

For $0 > \beta > \beta_c$, then the non-trivial fixed point (FB) stability of (2.8) with varying the nullcline $r_\tau$ for $\lambda \geq 0$ is the following, see Fig. 3:

- For small $\lambda$, only one stable fixed point, see the curve $A$ intersecting with the red line.
- If $\lambda = \lambda_1 C$, there is a SN bifurcation, that is, saddle and a stable FP. See the curve $B$ and $B'$ intersecting with the red line.
- At $\lambda = \lambda_2 C$, there is a second SN bifurcation in which saddle and other stable FP (from $A$) annihilate leaning the stable FP (from $B$). See the curve $D$ and $E$ intersecting with the red line.
This section is taken from Chapter 1.2 in [1] and Chapter 7.1 in [3]. Consider a general model of a nonlinear oscillator
\[
\frac{d\mathbf{u}}{dt} = f(\mathbf{u}), \quad \mathbf{u} = (u_1, \ldots, u_M), \quad \text{with } M \geq 2.
\] (3.1)

For example, \(u_1\) might represent the membrane potential of the neuron (treated as a point processor) and \(u_2, \ldots, u_M\) represent various ionic channel gating variables. Suppose there exists a stable periodic solution \(\mathbf{U}(t) = \mathbf{U}(t + \Delta_0)\), where \(\omega_0 = 2\pi/\Delta_0\) is the natural frequency of the oscillator. In phase space, the solution is an isolated attractive trajectory called a limit cycle. The dynamics on the limit cycle can be described by a uniformly rotating phase, that is
\[
\frac{d\phi}{dt} = \omega_0 \quad \text{and} \quad \mathbf{U}(t) = g(\phi(t)),
\] (3.2)

with \(g\) a \(2\pi\)-periodic function. The phase \(\phi\) should be viewed as a coordinate along the limit cycle, such that it grows monotonically in the direction of the motion and gains \(2\pi\) during each rotation. Note that the phase is neutrally stable with respect to perturbations along the limit cycle - this reflects the time-shift invariance of an autonomous dynamical system. On the limit cycle, the time shift \(\Delta t\) is equivalent to the phase shift \(\Delta \phi = \omega_0 \Delta t\). Now, suppose that a small external periodic input is applied to the oscillator such that
\[
\frac{d\mathbf{u}}{dt} = f(\mathbf{u}) + \epsilon P(\mathbf{u}, t),
\] (3.3)

where \(P(\mathbf{u}, t) = P(\mathbf{u}, t + \Delta)\) with \(\omega = 2\pi/\Delta\) the forcing frequency. If the amplitude \(\epsilon\) is sufficiently small and the cycle is stable, then the resulting deviations transverse to the limit cycle are small so that the main effect of the perturbation is a phase-shift along the limit cycle. This suggests a description of the perturbed dynamics with the phase variable only. Therefore, we need to extend the definition of phase to a neighbourhood of the limit cycle.

### 3.1 Isochrones

Roughly speaking, the idea is to define the phase variable in such a way that it rotates uniformly on the limit cycle as well as its neighbourhood. Suppose that
we observe the unperturbed system stroboscopically at time intervals of length $\Delta_0$. This leads to a Poincaré mapping

$$u(t) \rightarrow u(t + \Delta_0) \equiv G(u(t)).$$

The map $G$ has all points on the limit cycle as fixed points. Choose a point $U^*$ on the limit cycle and consider all points in a neighbourhood of $U^*$ in $\mathbb{R}^M$ that are attracted to it under the action of $\Phi$. They form an $(M-1)$-dimensional hypersurface, called an isochrone, crossing the limit cycle at $U^*$. A unique isochrone can be drawn through each point on the limit cycle so we can parameterise the isochrones by the phase $\phi$, that is $I = I(\phi)$. Finally, we extend the definition of phase to the vicinity of the limit cycle by taking all points $u \in I(\phi)$ to have the same phase, $\Phi(u) = \phi$, which then rotates at the natural frequency $\omega_0$ (in the unperturbed case).

**Example 1.** Consider the following complex amplitude equation that arises for a limit cycle oscillator close to a Hopf bifurcation:

$$\frac{dA}{dt} = (1 + i\eta)A - (1 + i\alpha)|A|^2, \quad A \in \mathbb{C}.$$ In polar coordinates $A = Re^{i\theta}$, we have

$$\frac{dR}{dt} = R(1 - R^2), \quad \frac{d\theta}{dt} = \eta - \alpha R^2.$$ Observe that the origin is unstable and the unit circle is a stable limit cycle. The solution for arbitrary initial data $R(0) = R_0, \theta(0) = \theta_0$ is

$$R(t) = \left[1 + \frac{1 - R_0^2}{R_0} \right] e^{-2it}^{-1/2}, \quad \theta(t) = \theta_0 + \omega_0 t - \frac{\alpha}{2} \log \left[R_0^2 + (1 - R_0^2) e^{-2it} \right],$$

where $\omega_0 = \eta - \alpha$ is the natural frequency of the stable limit cycle at $R = 1$. Strobing the solution at time $t = n\Delta_0$, we see that

$$\lim_{n \to \infty} \theta(n\Delta_0) = \theta_0 - \alpha \ln R_0.$$ Hence, we can define a phase on the whole plane as

$$\Phi(R, \theta) = \theta - \alpha \ln R$$

and the isochrones are the lines of constant phase $\Phi$, which are logarithmic spirals on the $(R, \theta)$ plane. We verify that this phase rotates uniformly:

$$\frac{d\Phi}{dt} = \frac{d\theta}{dt} - \frac{\alpha dR}{R \frac{dt}{dt}} = \eta - \alpha R^2 - \alpha (1 - R^2) = \eta - \alpha = \omega_0.$$ It seems like the angle variable $\theta$ can be taken to be the phase variable $\Phi$ since it rotates with a constant angular velocity $\omega_0$. However, if the initial amplitude deviates from unity, an additional phase shift occurs due to the term proportional to $\alpha$ in the $\theta$-equation. It can be seen from $\theta(t)$ and $R(t)$ that the additional phase shift is $-\alpha \ln R_0$.

### 3.2 Phase equation

For an unperturbed oscillator in the vicinity of the limit cycle, we have from (3.1) and (3.2)

$$\omega_0 = \frac{d\Phi(u)}{dt} = \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} \frac{du_k}{dt} = \sum_{k=1}^{M} \frac{\partial \Phi}{\partial u_k} f_k(u).$$
Now consider the perturbed system (3.3) but with the “unperturbed” definition of
the phase:

$$\frac{d\Phi(u)}{dt} = \sum_{k=1}^{M} \frac{\partial\Phi}{\partial u_k} \left( f_k(u) + \epsilon p_k(u, t) \right) = \omega_0 + \epsilon \sum_{k=1}^{M} \frac{\partial\Phi}{\partial u_k} p_k(u, t).$$

Because the sum is $O(\epsilon)$ and the deviations of $u$ from the limit cycle $U$ are small, to a first approximation, we can neglect these deviations and calculate the sum on the limit cycle. Consequently,

$$\frac{d\Phi(u)}{dt} = \omega_0 + \epsilon \sum_{k=1}^{M} \frac{\partial\Phi(U)}{\partial u_k} p_k(U, t).$$

Finally, since points on the limit cycle are in one-to-one correspondence with the phase $\theta$, we obtain the closed phase equation

$$\frac{d\phi}{dt} = \omega_0 + \epsilon Q(\phi, t), \tag{3.4}$$

where

$$Q(\phi, t) = \sum_{k=1}^{M} \frac{\partial\Phi(U(\phi))}{\partial u_k} p_k(U(\phi), t) \tag{3.5}$$

is a $2\pi$-periodic function of $\phi$ and a $\Delta$-periodic function of $t$. The phase equation (3.4) describes the dynamics of the phase of a periodic oscillator in the presence of a small periodic external force and $Q(\phi, t)$ contains all the information of the dynamical system. This is known as the phase reduction method.

**Example 2.** Returning to Example 1, the system in Cartesian coordinate is

$$\frac{dx}{dt} = x - \eta y - \left( x^2 + y^2 \right) (x - \eta y) + \epsilon \cos(\omega t)$$

$$\frac{dy}{dt} = y + \eta y - \left( x^2 + y^2 \right) (y + \alpha x)$$

where we periodically force the nonlinear oscillator in the $x$-direction. The isochrone is given by

$$\Phi = \arctan \left( \frac{y}{x} \right) - \frac{\alpha}{2} \ln(x^2 + y^2),$$

and differentiating with respect to $x$ yields

$$\frac{\partial\Phi}{\partial x} = -\frac{y}{x^2 + y^2} - \frac{\alpha x}{x^2 + y^2}.$$

On the limit cycle $u_0 = (x_0, y_0) = (\cos \phi, \sin \phi)$, we have

$$\frac{\partial\Phi}{\partial x}(u_0(\phi)) = -\sin \phi - \alpha \cos \phi.$$

It follows that the corresponding phase equation is

$$\frac{d\phi}{dt} = \omega_0 - \epsilon (\alpha \cos \phi + \sin \phi) \cos(\omega t).$$
3.3 Phase resetting curves

In neuroscience, the function $Q(\phi, t)$ can be related to an easily measurable property of a neural oscillator, namely its phase resetting curves (PRC). Let us denote this by a $2\pi$-periodic function $R(\phi)$. For a neural oscillator, the PRC is found experimentally by perturbing the oscillator with an impulse at different times in its cycle and measuring the resulting phase shift from the unperturbed oscillator. Suppose we perturb $u_1$, it follows from (3.4) that

$$\frac{d\phi}{dt} = \omega_0 + \epsilon \left( \frac{\partial \Phi(U(\phi))}{\partial u_1} \right) \delta(t - t_0).$$

Integrating over a small interval around $t_0$, we see that the impulse induces a phase shift $\Delta \phi = \epsilon R(\phi_0)$, where

$$R(\phi) = \frac{\partial \Phi(U(\phi))}{\partial u_1} \quad \text{and} \quad \phi_0 = \phi(t_0).$$

Given the phase resetting curve $R(\phi)$, a general time-dependent voltage perturbation $\epsilon P(t)$ is determined by the phase equation

$$\frac{d\phi}{dt} = \omega_0 + \epsilon R(\phi) P(t) = \omega_0 + \epsilon Q(\phi, t).$$

We can also express the PRC in terms of the firing times of a neuron. Let $T^n$ be the $n$th firing time of the neuron. Consider the phase $\phi = 0$. In the absence of perturbation, we have $\phi(t) = 2\pi nt / \Delta_0$ so the firing times are $T^n = n\Delta_0$. On the other hand, a small perturbation applied at the point $\phi$ on the limit cycle at time $t \in (T^n, T^{n+1})$, induces a phase shift that changes the next firing time. Depending on the type of neurons, the impulse either advance or delay the onset of the next spike. Oscillators with a strictly positive PRC $R(\phi)$ are called type I, whereas those for which the PRC has a negative regime are called type II.

3.4 Averaging theory

In the zero-order approximation, that is $\epsilon = 0$, the phase equation (3.4) gives rise to $\phi(t) = \phi_0 + \omega_0 t$. Since $Q(\phi, t)$ is $2\pi$-periodic in $\phi$ and $\Delta$-periodic in $t$, we expand $Q(\phi, t)$ as a double Fourier series

$$Q(\phi, t) = \sum_{l,k} a_{l,k} e^{il\phi + il\omega t}$$

$$= \sum_{l,k} a_{l,k} e^{il\phi_0} e^{i(k\omega_0 + l\omega)t},$$

where $\omega = 2\pi / \Delta$. Thus $Q$ contains fast oscillating terms (compared to the time scale $1/\epsilon$) together with slowly varying terms, the latter satisfy the resonance condition

$$k\omega_0 + l\omega \approx 0.$$

Substituting this double Fourier series into the phase equation (3.4), we see that the fast oscillating terms lead to $O(\epsilon)$ phase deviations, while the resonant terms can lead to large variations of the phase and are mostly important for the dynamics. Thus we have to average the forcing term $Q$ keeping only the resonant terms. We now identify the resonant terms using the resonance condition above:

1. The simplest case is $\omega \approx \omega_0$ for which the resonant terms satisfy $l = -k$. This results in an averaged forcing

$$Q(\phi, t) \approx \sum_k a_{-k,k} e^{i(k\phi - \omega t)} = q(\phi - \omega t)$$
and the phase equation becomes
\[ \frac{d\phi}{dt} = \omega_0 + \epsilon q(\phi - \omega t). \]

Introducing the phase difference \( \psi = \phi - \omega t \) between the oscillator and external input, we obtain
\[ \frac{d\psi}{dt} = -\Delta \omega + \epsilon q(\psi), \]
where \( \Delta \omega = \omega - \omega_0 \) is the degree of frequency detuning.

2. The other case is \( \omega \approx m\omega_0 / n \), where \( m \) and \( n \) are coprime. The forcing term becomes
\[ Q(\phi, t) \approx \sum_k a_{nk} e^{i(m\phi - n\omega t)} = \hat{q}(m\phi - n\omega t) \]
and the phase equation has the form
\[ \frac{d\phi}{dt} = \omega_0 + \epsilon \hat{q}(m\phi - n\omega t). \]

Introducing the phase difference \( \psi = m\phi - n\omega t \), we obtain
\[ \frac{d\psi}{dt} = m\omega_0 - n\omega + \epsilon m\hat{q}(\psi), \]
where the frequency detuning is \( \Delta \omega = n\omega - n\omega_0 \) instead.

The above analysis is an application of the averaging theorem. Assuming \( \Delta \omega = \omega - \omega_0 = \mathcal{O}(\epsilon) \) and setting \( \psi = \phi - \omega t \), we have
\[ \frac{d\psi}{dt} = -\Delta \omega + \epsilon Q(\psi + \omega t, t) = \mathcal{O}(\epsilon). \]

Define
\[ q(\psi) = \lim_{T \to 0} \frac{1}{T} \int_0^T Q(\psi + \omega t, t) \, dt, \]
and consider the averaged equation
\[ \frac{d\bar{\psi}}{dt} = -\Delta \omega + \epsilon q(\bar{\psi}), \]
where \( q \) only contains the resonant terms of \( Q \) as above. The averaging theorem guarantees that there exists a change of variable \( \psi = \bar{\psi} + \epsilon w(\phi, t) \) that maps solutions of the full equation to those of the averaged equation to leading order in \( \epsilon \).

In general, one can only establish that a solution of the full equation is \( \epsilon \)-close to a corresponding solution of the average equation for times of \( \mathcal{O}(1/\epsilon) \), that is
\[ \sup_{t \in I} |\psi(t) - \bar{\psi}(t)| \leq C\epsilon. \]

### 3.5 Phase-locking and synchronisation

We now discuss the solutions of the averaged phase equation
\[ \frac{d\psi}{dt} = -\Delta \omega + \epsilon q(\psi). \]  

(3.6)

Suppose that the 2\( \pi \)-periodic function \( q(\psi) \) has a unique maximum \( q_{\text{max}} \) and a unique minimum \( q_{\text{min}} \). The fixed points \( \psi^* \) of (3.6) satisfy \( \epsilon q(\psi^*) = \Delta \omega \).
1. **Synchronisation regime**

If the degree of detuning is sufficiently small, in the sense that

\[ \epsilon \eta_{\min} < \Delta \omega < \epsilon \eta_{\max}, \]

then there exists at least one pair of stable/unstable fixed points \( (\psi_s, \psi_u) \). (This follows from the fact that \( q(\psi) \) is \( 2\pi \)-periodic and continuous so it has to cross any horizontal line an even number of times.) The system then evolves to the solution \( \phi(t) = \omega t + \psi_s \) and this is the *phase-locked synchronise state*. The oscillator is also said to be *frequency entrained*, meaning that the frequency of the oscillator coincides with that of the external force.

2. **Drift regime**

Increasing \( |\Delta \omega| \) means that \( \psi_s, \psi_u \) coalesce at a saddle point, beyond which there are no fixed points. This results in a saddle-node bifurcation and phase-locking disappears. If \( |\Delta \omega| \) is large, then \( \psi \) never changes sign and the oscillation frequency differs from the forcing frequency. The phase \( \psi(t) \) rotates through \( 2\pi \) with period

\[ T_\psi = \int_0^{2\pi} \frac{d\psi}{\epsilon q(\psi) - \Delta \omega}. \]

The mean frequency of rotation is thus \( \Omega = \omega + \Omega_\phi \), where \( \Omega_\phi = 2\pi / T_\psi \) is the *beat frequency*. For a fixed \( \epsilon \), suppose that \( \Delta \omega \) is close to one of the bifurcation point \( \Delta \omega_{\max} := \epsilon \eta_{\max} \). The integral in \( T_\psi \) is dominated by a small region around \( \psi_{\max} \) and expanding \( q(\psi) \) around \( \psi_{\max} \) yields

\[ \Omega_\phi \approx 2\pi \left( \int_{\psi_{\max}}^{\psi_{\min}} \frac{d\psi}{\epsilon q''(\psi_{\max}) \psi'' - (\Delta \omega - \Delta \omega_{\max})} \right)^{-1}, \]

and it follows that

\[ \Omega_\phi \approx \sqrt{\epsilon q''(\psi_{\max}) (\Delta \omega - \Delta \omega_{\max})} = O(\sqrt{\epsilon}). \]

### 3.6 Phase Reduction for Networks of Coupled Oscillators

We extend the analysis to a network of \( N \) coupled oscillators. Let \( \mathbf{u}_i \in \mathbb{R}^M, i = 1, \ldots, N \) denote the state of the \( i \)th oscillator. The general model can be written as

\[ \frac{d\mathbf{u}_i}{dt} = f(\mathbf{u}_i) + \epsilon \sum_{j=1}^{N} a_{ij} H(\mathbf{u}_j), \quad i = 1, \ldots, N, \]

(3.7)

where the first term represents the local autonomous dynamics and the second term describes the interaction between oscillators. In a similar fashion to a single periodically forced oscillator, we can write down the phase equation:

\[ \frac{d\phi_i(\mathbf{u}_i)}{dt} = \omega_0 + \epsilon \left( \frac{\partial \phi_i}{\partial \mathbf{u}_i} \right) \cdot \left( \sum_{j=1}^{N} a_{ij} H(\mathbf{u}_j) \right), \quad i = 1, \ldots, N. \]

(3.8)

Since the limit cycle is uniquely defined by phase,

\[ \frac{d\phi_i}{dt} = \omega_0 + \epsilon \sum_{j=1}^{N} a_{ij} Q_i(\phi_i, \phi_j), \quad i = 1, \ldots, N, \]

(3.9)

where

\[ Q_i(\phi_i, \phi_j) = \frac{\partial \phi_i}{\partial \mathbf{u}_i} (U(\phi_i)) \cdot H(U(\phi_j)). \]

(3.10)
The final step is to use the method of averaging to obtain the phase-difference equation. Introducing \( \psi_i = \phi_i - \omega_0 t \), we obtain
\[
\frac{d\psi_i}{dt} = \epsilon \sum_{j=1}^{N} a_{ij} Q_j (\psi_i + \omega_0 t, \psi_j + \omega_0 t).
\]
Upon averaging over one period, we obtain
\[
\frac{d\psi_i}{dt} = \epsilon \sum_{j=1}^{N} a_{ij} h(\psi_j - \psi_i), \quad (3.11)
\]
where
\[
h(\psi_j - \psi_i) = \frac{1}{\Delta_0} \int_0^{\Delta_0} R(\psi_i + \omega_0 t) \cdot \left( \sum_{j=1}^{N} H(U(\psi_j + \omega_0 t)) \right) dt
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} R(\phi + \psi_i) \cdot \left( \sum_{j=1}^{N} H(U(\phi)) \right) d\phi,
\]
with \( \phi = \psi_j + \omega_0 t \). Here, \( R \) is the phase resetting curve.

**Phase-locked solutions**

We define a one-to-one phase-locked solutions to be
\[
\psi_i(t) = \psi_i + t\Delta w + \bar{\psi}_i, \quad (3.12)
\]
where \( \bar{\psi}_j \) is constant. Taking time derivative on (3.12) and imposing (3.11) yields
\[
\Delta \omega = \epsilon \sum_{j=1}^{N} a_{ij} h(\bar{\psi}_j - \bar{\psi}_i), \quad i = 1, \ldots, N. \quad (3.13)
\]
Since we have \( N \) equations in \( N \) unknowns \( \Delta \omega \) and \( \Delta \psi_j \), then one can find the phase-locked solutions (We only care about phase difference.)

**Stability**

In order to determine local stability, we set
\[
\psi_i(t) = \bar{\psi}_j + t\Delta w + \Delta \psi_i(t). \quad (3.14)
\]
To linearize it, taking time derivative on (3.14) and imposing the phase-locked solutions (3.13) gives
\[
\frac{d\Delta \psi_i}{dt} = \epsilon \sum_{j=1}^{N} \tilde{H}_{ij} (\Phi) \Delta \psi_j, \quad (3.15)
\]
where \( \Phi = (\bar{\psi}_1, \ldots, \bar{\psi}_N) \) and
\[
\tilde{H}_{ij} (\Phi) = a_{ij} h(\bar{\psi}_j - \bar{\psi}_i) - \delta_{ij} \sum_{k} a_{ik} h(\bar{\psi}_k - \bar{\psi}_i). \quad (3.16)
\]

**Pair of identical oscillators**

For example, we assume that \( N = 2 \) and symmetric coupling, that is \( a_{12} = a_{21} \) and \( a_{11} = a_{22} = 0 \) (no self-interaction). Let \( \psi = \psi_2 - \psi_1 \). Then (3.14) turns out to be
\[
\frac{d\psi}{dt} = cH^-(\psi),
\]
where $H^{-1}(\psi) = h(-\psi) - h(\psi)$. Imposing the assumption on (3.13) implies that the phase-locked states are given by zeros of the odd function, $H^{-1}(\psi) = 0$. Furthermore, it is stable if
\[
\epsilon \frac{dH^{-1}(\psi)}{d\psi} < 0.
\]
By symmetry and periodicity, the in-phase solution $\psi = 0$ and anti-phase solution $\psi = \pi$ are guaranteed to exist.

4 Partial Differential Equations

In this section, we apply the method of multiple scales to the linear wave equation and the nonlinear Klein-Gordon equation.

4.1 Elastic string with weak damping

Consider the one-dimensional wave equation with weak damping:
\[
\begin{align*}
\partial_x^2 u &= \partial_t^2 u + \epsilon \partial_t u, & 0 < x < 1, & t > 0, \\
\partial_x u &= 0, & \text{at } x = 0 \text{ and } x = 1, \\
\partial_x u(x, 0) &= g(x), & \partial_t u(x, 0) &= 0.
\end{align*}
\]

Similar to the weakly damped oscillator, we introduce two separate time scales $t_1 = t, t_2 = \epsilon t$. In this case, (4.1) becomes
\[
\begin{align*}
\partial_{x_1}^2 u &= \left[ \partial_{t_1}^2 + 2\epsilon \partial_{t_1} \partial_{t_2} + \epsilon^2 \partial_{t_2}^2 \right] u + \epsilon \left[ \partial_{t_1} + \epsilon \partial_{t_2} \right] u, \\
\partial_{x_1} u &= 0, & \text{at } x = 0 \text{ and } x = 1, \\
\partial_{x_1} u(x, 0) &= g(x), & \left[ \partial_{t_1} + \epsilon \partial_{t_2} \right] u \bigg|_{t_2=0} &= 0.
\end{align*}
\]

As before, the solution of (4.2) is not unique and we will use this degree of freedom to eliminate the secular terms.

We try a regular asymptotic expansion of the form
\[
u \sim u_0(x, t_1, t_2) + \epsilon u_1(x, t_1, t_2) + \ldots \text{ as } \epsilon \to 0.
\]

Collecting $O(1)$ terms gives
\[
\begin{align*}
\partial_{x_1}^2 u_0 &= \partial_{t_1}^2 u_0, \\
u_0(x, 0, 0) &= g(x), & \partial_{t_1} u_0(x, 0, 0) &= 0.
\end{align*}
\]

Separation of variables yields the general solution
\[
u_0(x, t_1, t_2) = \sum_{n=1}^{\infty} \left[ a_n(t_2) \sin(\lambda_n t_1) + b_n(t_2) \cos(\lambda_n t_1) \right] \sin(\lambda_n x),
\]

where $\lambda_n = n\pi$. The initial conditions will be imposed once we determine $a_n(t_2)$ and $b_n(t_2)$. The $O(\epsilon)$ equation is
\[
\partial_{x_1}^2 u_1 = \partial_{t_1}^2 u_1 + 2\partial_{t_1} \partial_{t_2} u_0 + \partial_{t_1} u_0
\]
and it follows that
\[
\partial_{x_1}^2 u_1 = \partial_{t_1}^2 u_1 + \sum_{n=1}^{\infty} A_n(t_1, t_2) \sin(\lambda_n x),
\]
where
\[ A_n = (2a_n' + a_n) \lambda_n \cos(\lambda_n t_1) - (2b_n' + b_n) \lambda_n \sin(\lambda_n t_1). \]

Given the zero boundary conditions in (4.1), it is appropriate to introduce the Fourier expansion
\[ u_1 = \sum_{n=1}^{\infty} V_n(t_1, t_2) \sin(\lambda_n x). \]

Substituting this into (4.7) together with the expression of \( A_n \), we obtain
\[ \partial_t^2 V_n + \lambda_n^2 V_n = - (2a_n' + a_n) \lambda_n \cos(\lambda_n t_1) + (2b_n' + b_n) \lambda_n \sin(\lambda_n t_1). \]

The secular terms are eliminated provided
\[ 2a_n' + a_n = 0, \quad 2b_n' + b_n = 0, \]
and these have general solutions of the form
\[ a_n(t_2) = a_n(0) e^{-t_2/2}, \quad b_n(t_2) = b_n(0) e^{-t_2/2}. \]

Finally, a first term approximation of the solution of (4.1) is
\[ u(x, t) \sim \sum_{n=1}^{\infty} \left[ a_n(0) e^{-\epsilon t/2} \sin(\lambda_n t) + b_n(0) e^{-\epsilon t/2} \cos(\lambda_n t) \right] \sin(\lambda_n x), \quad \lambda_n = n\pi. \] (4.7)

Applying the initial condition in (4.4), we find that \( a_n(0) = 0 \) and
\[ b_n(0) = 2 \int_0^1 g(x) \sin(\lambda_n x) \, dx. \]

### 4.2 Nonlinear wave equation

Consider the nonlinear Klein-Gordon equation
\[ \begin{align*}
\partial_x^2 u &= \partial_t^2 u + u + \epsilon u^3, \quad -\infty < x < \infty, \quad t > 0, \\
u(x, 0) &= F(x), \quad \partial_t u(x, 0) = G(x).
\end{align*} \] (4.8a)

It describes the motion of a string on an elastic foundation as well as the waves in a cold electron plasma.

As usual, let us consider (4.8) with \( \epsilon = 0 \):
\[ \begin{align*}
\partial_x^2 u &= \partial_t^2 u + u, \quad -\infty < x < \infty, \quad t > 0, \\
u(x, 0) &= F(x), \quad \partial_t u(x, 0) = G(x).
\end{align*} \] (4.9a)

We guess a solution of the form \( \exp[i(kx - \omega t)] \). This yields the dispersion relation
\[ -k^2 = -\omega^2 + 1 \implies \omega = \pm \sqrt{1 + k^2} = \pm \omega(k). \]

We may solve (4.9) using the spatial Fourier transform
\[ \hat{u}(k, t) = \int_{-\infty}^{\infty} u(x, t) e^{-ikx} \, dx. \]

This produces an ODE for \( \hat{u}(k, t) \):
\[ \begin{align*}
-\omega^2 \hat{u} &= \partial_t \hat{u} + \hat{u}, \\
\hat{u}(k, 0) &= \hat{F}(k), \quad \partial_t \hat{u}(k, 0) = \hat{G}(k).
\end{align*} \] (4.10a)
Solving (4.10) and applying the inverse Fourier transform we obtain the general solution of (4.9):

\[ u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i[kx - \omega(k)t]} \, dk + \int_{-\infty}^{\infty} B(k) e^{i[kx + \omega(k)t]} \, dk. \]  

(4.11)

where \( A(k) \) and \( B(k) \) are determined from the initial conditions in (4.10). This shows that the solution of (4.9) can be written as the superposition of the plane wave solutions \( u_{\pm}(x,t) = \exp \left( i[kx + \omega(k)t] \right) \). We would like to investigate how the nonlinearity affects a right-moving plane wave \( u(x,t) = \cos(kx - \omega t) \), where \( k > 0 \) and \( \omega = \sqrt{1 + k^2} \).

A regular asymptotic expansion of the form

\[ u(x,t) \sim w_0(kx - \omega t) + \epsilon w_1(x,t) + \ldots \]

will lead to secular terms, and thus we use multiple scales to find an asymptotic approximation of the solution of (4.8). We take three independent variables

\[ \theta = kx - \omega t, \quad x_2 = \varepsilon x, \quad t_2 = \varepsilon t. \]

The spatial and time derivatives become

\[ \frac{\partial}{\partial x} \rightarrow k \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial t} \rightarrow -\omega \frac{\partial}{\partial \theta} + \epsilon \frac{\partial}{\partial t_2}. \]

Consequently, the nonlinear Klein-Gordon equation becomes

\[ \left[ k \partial_\theta + \epsilon \partial_{x_2} \right]^2 u = \left[ -\partial_\theta + \epsilon \partial_{t_2} \right]^2 u + u + \epsilon u^3. \]

and rearranging it gives

\[ \left[ \partial_\theta^2 - 2\epsilon \left( k \partial_{x_2} + \omega \partial_{t_2} \right) \partial_\theta + O(\epsilon^2) \right] u + u + \epsilon u^3 = 0, \]  

(4.12)

where we use the dispersion relation \(-k^2 = -\omega^2 + 1\). We assume a regular asymptotic expansion of the form

\[ u(x,t) \sim u_0(\theta,x_2,t_2) + \epsilon u_1(\theta,x_2,t_2) + \ldots. \]

Collecting \( O(1) \) terms gives

\[ \left( \partial_\theta^2 + 1 \right) u_0 = 0, \]

and the general solution of this problem is

\[ u_0 = A(x_2,t_2) \cos \left( \theta + \phi(x_2,t_2) \right). \]

The \( O(\epsilon) \) equation becomes

\[ \left( \partial_\theta^2 + 1 \right) u_1 = 2 \left( k \partial_{x_2} + \omega \partial_{t_2} \right) \partial_\theta u_0 - u_0^3, \]

and substituting the general solution of \( u_0 \) yields

\[ \left( \partial_\theta^2 + 1 \right) u_1 = -2 \left[ (k \partial_{x_2} + \omega \partial_{t_2}) A \right] \sin(\theta + \phi) - \frac{1}{4} A^3 \cos \left( 3(\theta + \phi) \right) \]

\[ - 2 \left[ (k \partial_{x_2} + \omega \partial_{t_2}) \phi + \frac{3}{8} A^2 \right] A \cos(\theta + \phi). \]

The secular terms are eliminated provided

\[ (k \partial_{x_2} + \omega \partial_{t_2}) A = 0, \quad (k \partial_{x_2} + \omega \partial_{t_2}) \phi + \frac{3}{8} A^2 = 0. \]  

(4.13)
These constitute the **amplitude-phase equations** and can be solved using characteristic coordinates. Specifically, let

\[ r = \omega x_2 + kt_2 \quad \text{and} \quad s = \omega x_2 - kt_2. \]

With this (4.13) simplifies to

\[ \partial_r A = 0, \quad \partial_r \phi = -\frac{3}{16\omega k} A^2, \]

and solving this yields

\[ A = A(s), \quad \phi = -\frac{3}{16\omega k} A^2 \cdot r + \phi_0(s). \]

Hence, a first term approximation of the solution of (4.8) is

\[ u \sim A(wx_2 - kt_2) \cos \left[(kx - \omega t) - \frac{3}{16\omega k}(wx_2 + kt_2)A^2 + \phi_0(wx_2 - kt_2)\right]. \]  

(4.14)

We can now attempt to answer our main question: how does the nonlinearity affects the plane wave solution? Consider the plane wave initial conditions

\[ u(x, 0) = \alpha \cos(kx), \quad \partial_t u(x, 0) = \alpha \omega \sin(kx). \]

In multiple scale expansion, these translates to

\[ u_0(\theta, x_2, 0) = \alpha \cos(\theta), \quad \partial_\theta u_0(\theta, x_2, 0) = -\alpha \sin(\theta). \]

Imposing these initial conditions on (4.14) we obtain

\[ A(wx_2) = \alpha, \quad \phi_0(wx_2) = \frac{3}{16k} A^2 x_2. \]

Thus, a first term approximation of the solution of (4.8) in this case is

\[ u(x, t) \sim \alpha \cos \left(kx - \left(1 + \frac{3\alpha^2}{16\omega^2}\right) \omega t\right) \sim \alpha \cos(kx - \omega t). \]

We see that the nonlinearity increases the phase velocity since it increases from \( c = \omega / k \) to \( \omega / k \).

### 5 Pattern Formation and Amplitude Equations

#### 5.1 Neural Field Equations on a Ring

Consider a population of neurons distributed on the circle \( S^1 = [0, \pi] \):

\[ \frac{\partial a}{\partial t} = -a(\theta, t) + \frac{1}{\pi} \int_0^\pi w(\theta - \theta') f(a(\theta', t)) \, d\theta', \]  

(5.1a)

\[ f(a) = \frac{1}{1 + \exp(-\eta(a - k))}, \]  

(5.1b)

where \( a(\theta, t) \) denotes the activity at time \( t \) of a local population of cells at position \( \theta \in [0, \pi] \), \( w(\theta - \theta') \) is the strength of synaptic weights between cells at \( \theta' \) and \( \theta \) and the firing rate function \( f \) is a sigmoid function. Assuming \( w \) is an even \( \pi \)-periodic function, it can be expanded as a Fourier series:

\[ w(\theta) = W_0 + 2 \sum_{n \geq 1} W_n \cos(2n\pi \theta), \quad W_n \in \mathbb{R}. \]  

(5.2)
Suppose there exists a uniform equilibrium solution \( \bar{a} \) of (5.1), satisfying

\[
a = f(\bar{a}) \int_0^\pi \frac{w(\theta - \theta')}{\pi} d\theta' = f(\bar{a})W_0.
\] (5.3)

The stability of the equilibrium solution is determined by setting \( a(\theta,t) = \bar{a} + a(\theta)e^{\lambda t} \) and linearising (5.1) about \( \bar{a} \). Expanding \( f \) around \( \bar{a} \) by Taylor series yields

\[
f(\bar{a} + a(\theta)e^{\lambda t}) \approx f(\bar{a}) + f'(\bar{a})a(\theta)e^{\lambda t},
\]

and we obtain the eigenvalue equation

\[
\lambda a(\theta) = -a(\theta) + \frac{f'(\bar{a})}{\pi} \int_0^\pi w(\theta - \theta')a(\theta') d\theta' =: L[a](\theta).
\] (5.4)

Since the linear operator \( L \) is compact on \( L^2(S^1) \), it has a discrete spectrum with eigenvalues

\[
\lambda_n = -1 + f'(\bar{a})W_n, \quad n \in \mathbb{Z},
\]

and corresponding eigenfunctions

\[
a_n(\theta) = z_n e^{2in\theta} + z_n^* e^{-2in\theta}.
\]

The eigenvalue expression reveals the bifurcation parameter \( \mu = f'(\bar{a}) \). For sufficiently small \( \mu \), corresponding to a low activity state, \( \lambda_n < 0 \) for all \( n \) and the fixed point is stable. As \( \mu \) increases beyond a critical value \( \mu_c \), the fixed point becomes unstable due to excitation of the eigenfunctions associated with the largest Fourier component of \( w(\theta) \). Suppose that \( W_1 = \max_m W_m \). Then \( \lambda_n > 0 \) for all \( n \) if and only if

\[
1 < \mu W_n \leq \mu W_1 \implies \mu > \frac{1}{W_1} = \mu_c.
\]

Consequently, for \( \mu > \mu_c \), the excited modes will be

\[
a(\theta) = ze^{2i\theta} + ze^{-2i\theta} = 2|z| \cos(2(\theta - \theta_0)),
\]

where \( z = |z|e^{-2i\theta_0} \). We expect this mode to grow and stop at a maximum amplitude as \( \mu \) approaches \( \mu_c \), mainly because of the saturation of \( f \).

### 5.2 Derivation of amplitude equation using the Fredholm alternative

Unfortunately, the linear stability analysis breaks down for large amplitude of the activity profile. Suppose the system is just above the bifurcation point, that is

\[
\mu - \mu_c = \varepsilon \Delta \mu, \quad 0 < \varepsilon \ll 1
\] (5.5)

If \( \Delta \mu = O(1) \), then \( \mu - \mu_c = O(\varepsilon) \) and we can carry out a perturbation expansion in powers of \( \varepsilon \). We first Taylor expand the nonlinear function \( f \) around \( a = \bar{a} \):

\[
f(a) - f(\bar{a}) = \mu(a - \bar{a}) + g_2(a - \bar{a})^2 + g_3(a - \bar{a})^3 + O(a - \bar{a})^4.
\] (5.6)

Assume a perturbation expansion of the form

\[
a = \bar{a} + \sqrt{\varepsilon} a_1 + \varepsilon a_2 + \varepsilon^{3/2} a_3 + \ldots
\] (5.7)
The dominant temporal behaviour just beyond bifurcation is the slow growth of the excited mode $e^{\lambda t}$ and this motivates the introduction of a slow time scale $\tau = \epsilon t$. Substituting (5.5), (5.6) and (5.7) into (5.1) yields
\[
\left[ \partial_t + \epsilon \partial_\tau \right] \left[ \partial + \sqrt{\epsilon} a_1 + \epsilon a_2 + \epsilon^{3/2} a_3 + \ldots \right] = - \left[ \partial + \sqrt{\epsilon} a_1 + \epsilon a_2 + \epsilon^{3/2} a_3 + \ldots \right] \\
+ \frac{1}{\pi} \int_0^\pi w(\theta - \theta') f(\hat{a}) \, d\theta' \\
+ \frac{1}{\pi} \int_0^\pi w(\theta - \theta') \left( \mu_c + \epsilon \Delta \mu \right) \left[ \sqrt{\epsilon} a_1 + \epsilon a_2 + \epsilon^{3/2} a_3 + \ldots \right] \, d\theta' \\
+ \frac{1}{\pi} \int_0^\pi w(\theta - \theta') g_2 \left[ \sqrt{\epsilon} a_1 + \epsilon a_2 + \ldots \right]^2 \, d\theta' \\
+ \frac{1}{\pi} \int_0^\pi w(\theta - \theta') g_3 \left[ \sqrt{\epsilon} a_1 + \epsilon a_2 + \ldots \right]^3 \, d\theta'
\]

Define the linear operator $\hat{L}$:
\[
\hat{L}[a](\theta) = -a(\theta) + \frac{\mu_c}{\pi} \int_0^\pi w(\theta - \theta') a(\theta') \, d\theta' = -a(\theta) + \mu_c w * a(\theta).
\]

Collecting terms with equal powers of $\epsilon$ then leads to a hierarchy of equations of the form:
\[
a = W_0 f(a), \\
\hat{L}a_1 = 0, \\
\hat{L}a_2 = V_2 := -g_2 w * a_1^2, \\
\hat{L}a_3 = V_3 := \frac{\partial a_1}{\partial \tau} - \Delta \mu w * a_1 - 2g_2 w * (a_1 a_2) - g_3 w * a_3^3.
\]

The $O(1)$ equation determines the fixed point $\bar{a}$. The $O(\sqrt{\epsilon})$ equation has solutions of the form
\[
a_1 = z(\tau) e^{2i\theta} + z^*(\tau) e^{-2i\theta}.
\]

A dynamical equation for $z(\tau)$ can be obtained by deriving solvability conditions for the higher-order equations using Fredholm alternative. These equations have the general form
\[
\hat{L}a_n = V_n(a, a_1, \ldots, a_{n-1}), \quad n \geq 2.
\]

For any two periodic functions $U, V$, define the inner product
\[
\langle U, V \rangle = \frac{1}{\pi} \int_0^\pi U^*(\theta) V(\theta) \, d\theta.
\]

Using integration by parts, it is easy to see that $\hat{L}$ is self-adjoint with respect to this particular inner product and since $\hat{L}\bar{a} = 0$ for $\bar{a} = e^{\pm 2i\theta}$, we have
\[
\langle \bar{a}, \hat{L}a_n \rangle = \langle \hat{L}a_n, \bar{a} \rangle = 0.
\]

Since $\hat{L}a_n = V_n$, it follows from the Fredholm alternative that the set of solvability conditions are
\[
\langle \bar{a}, V_n \rangle = 0 \quad \text{for} \ n \geq 2.
\]

The $O(\epsilon)$ solvability condition $\langle \bar{a}, V_2 \rangle = 0$ is automatically satisfied. The $O(\epsilon^{3/2})$ solvability condition can be expanded into
\[
\langle \bar{a}, \partial_\tau a_1 - \Delta \mu w * a_1 \rangle = g_3 \langle \bar{a}, w * a_1^2 \rangle + 2g_2 \langle \bar{a}, w * (a_1 a_2) \rangle. \quad (5.8)
\]
where \( \xi \). The next step is to determine \( a_1 \). First, we have
\[
\langle e^{2i\theta}, \partial_\tau a_1 \rangle = \frac{1}{\pi} \int_0^{\pi} e^{-2i\theta} \left( \frac{dz}{d\tau} e^{2i\theta} + \frac{dz}{d\tau} e^{-2i\theta} \right) d\theta = \frac{dz}{d\tau}
\] (5.9)

To deal with the convolution terms, observe that since \( w \) is even, for any function \( b(\theta) \) we have
\[
\langle e^{2i\theta}, w * b \rangle = \langle w * e^{2i\theta}, b \rangle = \frac{1}{\pi} \int_0^{\pi} \left( \frac{1}{\pi} \int_0^{\pi} w(\theta - \theta') e^{-2i\theta'} d\theta' \right) b(\theta) d\theta,
\]
and it follows that
\[
\langle e^{2i\theta}, w * b \rangle = \frac{1}{\pi} \int_0^{\pi} W_1 e^{-2i\theta} b(\theta) d\theta = W_1 \langle e^{2i\theta}, b \rangle.
\]
Set \( W_1 = \mu_c^{-1} \). From the identity above for \( a_1 \) we then have
\[
\langle e^{2i\theta}, \Delta_{\mu} w * a_1 \rangle = \Delta_{\mu} W_1 \langle e^{2i\theta}, a_1 \rangle = \frac{\Delta_{\mu}}{\mu_c} \int_0^{\pi} \int_0^{\pi} e^{-2i\theta} \left( ze^{2i\theta} + z^* e^{-2i\theta} \right) d\theta,
\]
and it follows that
\[
\langle e^{2i\theta}, \Delta_{\mu} w * a_1 \rangle = \mu_c^{-1} \Delta_{\mu} z.
\] (5.10)

In the same fashion, one can obtain
\[
\langle e^{2i\theta}, w * a_1^3 \rangle = W_1 \langle e^{2i\theta}, a_1^3 \rangle = \frac{1}{\mu_c} \int_0^{\pi} e^{-2i\theta} \left( ze^{2i\theta} + z^* e^{-2i\theta} \right)^3 d\theta,
\]
and it turns out that
\[
\langle e^{2i\theta}, w * a_1^3 \rangle = 3\mu_c^{-1} |z|^2.
\] (5.11)

The next step is to determine \( a_2 \). From the \( O(\varepsilon) \) equation we have
\[
-\tilde{a}_2 = a_2 - \frac{\mu_c}{\pi} \int_0^{\pi} w(\theta - \theta') a_2(\theta') d\theta' = \frac{g_2}{\pi} \int_0^{\pi} w(\theta - \theta') \frac{a_2}{a_1^2}(\theta') d\theta'.
\]
Imposing the Fourier series of \( w \) and the identity of \( a_2 \) yields
\[
-\tilde{a}_2 = g_2 \left[ 2|z|^2 W_0 + z^2 W_2 e^{4i\theta} + (z^*)^2 W_2 e^{-4i\theta} \right].
\] (5.12)

Let
\[
a_2(\theta) = A_+ e^{4i\theta} + A_- e^{-4i\theta} + A_0 + \zeta a_1(\theta).
\] (5.13)
The constant \( \zeta \) remains undetermined at this order of perturbation but does not appear in the amplitude equation for \( z(\tau) \). Substituting (5.13) into (5.12) yields
\[
A_+ = \frac{g_2^2 z^2 W_2}{1 - \mu_c W_2}, \quad A_- = \frac{g_2 (z^*)^2 W_2}{1 - \mu_c W_2}, \quad A_0 = \frac{2g_2 |z|^2 W_0}{1 - \mu_c W_0}.
\] (5.14)

Consequently,
\[
\langle e^{2i\theta}, w * (a_1 a_2) \rangle = W_1 \langle e^{2i\theta}, a_1 a_2 \rangle
\]
and it follows that
\[
\langle e^{2i\theta}, w * (a_1 a_2) \rangle = z|z|^2 g_2 \mu_c^{-1} \left[ \frac{W_2}{1 - \mu_c W_2} + \frac{2W_0}{1 - \mu_c W_0} \right].
\] (5.15)

Finally, substituting (5.9), (5.10), (5.11) and (5.15) into the \( O(\varepsilon^{3/2}) \) solvability condition (5.8), we obtain the Stuart-Landau equation
\[
\frac{dz}{d\tau} = z(\Delta_{\mu} - \Lambda |z|^2),
\] (5.16)
where
\[
\Lambda = -3g_3 - 2g_2^2 \left[ \frac{W_2}{1 - \mu_c W_2} + \frac{2W_0}{1 - \mu_c W_0} \right].
\] (5.17)

Note that we also absorbed a factor of \( \mu_c \) into \( \tau \).
Problem 1. Find a first-term expansion of the solution of the following problems using two time scales.

1. \( y'' + \epsilon(y')^3 + y = 0, y(0) = 0, y'(0) = 1. \)

**Proof.** We introduce a slow scale \( \tau = \epsilon t \) and an asymptotic expansion

\[
y \sim y_0(t, \tau) + \epsilon y_1(t, \tau) + \ldots.
\]

The original problem becomes

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial}{\partial t} \frac{\partial}{\partial \tau} + \epsilon^2 \frac{\partial^2}{\partial \tau^2} &\left( y_0 + \epsilon y_1 + \ldots \right) \\
+ \epsilon \left[ \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau} \right] \left( y_0 + \epsilon y_1 + \ldots \right)^3 &+ \left( y_0 + \epsilon y_1 + \ldots \right) = 0,
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
\left( y_0 + \epsilon y_1 + \ldots \right)(0, 0) &= 0, \\
\left[ \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau} \right] \left( y_0 + \epsilon y_1 + \ldots \right)(0, 0) &= 1.
\end{align*}
\]

The \( \mathcal{O}(1) \) problem is

\[
\left( \frac{\partial^2}{\partial t^2} + 1 \right) y_0 = 0, \quad y_0(0, 0) = \frac{\partial}{\partial t} y_0(0, 0) = 0, \tag{6.1}
\]

and its general solution is

\[
y_0(t, \tau) = A(\tau)e^{it} + A^*(\tau)e^{-it}, \tag{6.2}
\]

where \( A(\tau) \) is complex function of \( \tau \). The \( \mathcal{O}(\epsilon) \) equation is

\[
\left( \frac{\partial^2}{\partial t^2} + 1 \right) y_1 = -2 \frac{\partial}{\partial t} \frac{\partial}{\partial \tau} y_0 - \left( \frac{\partial}{\partial \tau} y_0 \right)^3 \\
= F(\tau)e^{it} + F^*(\tau)e^{-it} + i\left[ A^3 e^{3it} + (A^*)^3 e^{-3it} \right].
\]

The secular terms are eliminated provided \( F(\tau) = 0 \). Writing \( A(\tau) = R(\tau)e^{i\theta(\tau)} \), \( F(\tau) \) becomes

\[
2 \left( R e^{i\theta} + iR\theta e^{i\theta} \right) + 3Re^{i\theta} R^2 = 0,
\]

or

\[
2 \left( R + iR\theta \right) + 3R^3 = 0.
\]

Consequently, we have

\[
\theta(\tau) = \theta_0 \implies \theta(\tau) = \theta_0
\]

and

\[
2R + 3R^3 = 0 \implies R(\tau) = \frac{1}{\sqrt{3} + C}.
\]

Therefore, (6.2) becomes

\[
y_0(t, \tau) = R(\tau)e^{i(t + \theta_0)} + R(\tau)e^{-i(t + \theta_0)} \\
= 2R(\tau) \cos(t + \theta_0).
\]
We now impose the initial conditions from (6.1):
\[ y_0(0,0) = 0 \implies 2R(0)\cos(\theta_0) = 0 \]
\[ \partial_\tau y_0(0,0) = 1 \implies -2R(0)\sin(\theta_0) = 1, \]
which means
\[ R(0)e^{i\theta_0} = -i = \frac{1}{\sqrt{C}} e^{i\theta_0} \implies C = 4 \quad \text{and} \quad \theta_0 = \frac{3\pi}{2}. \]
Hence, a first-term approximation of the solution of the original problem is
\[ y \sim \frac{2\cos(t + 3\pi/2)}{\sqrt{3\epsilon t + 4}} \sim \frac{2\sin(t)}{\sqrt{3\epsilon t + 4}}. \]

2. \( \epsilon y'' + \epsilon \kappa y' + y + \epsilon y^3 = 0, \ y(0) = 0, \ y'(0) = 1, \kappa > 0. \)

**Proof.** The equation appears to have a boundary layer, but it does not in this case since \( \epsilon \) appears on \( y' \) as well. Let \( T = t/\sqrt{\epsilon} \) and \( Y(T) = y(t) = y(\sqrt{\epsilon}T) \), then
\[
\frac{d}{dT} = \frac{1}{\sqrt{\epsilon}} \frac{d}{dt}
\]
and the original problem becomes
\[
\partial^2_T Y + \sqrt{\epsilon} \kappa \partial_T Y + Y + \epsilon Y^3 = 0 \quad (6.3a)
\]
\[
Y(0) = 0, \quad \partial_T Y(0) = \sqrt{\epsilon}. \quad (6.3b)
\]
Since one of the boundary conditions is of \( O(\sqrt{\epsilon}) \), we take the slow scale to be \( \tau = \sqrt{\epsilon}T = t \) and the fast scale to be \( T = t/\sqrt{\epsilon} \). Assuming an asymptotic expansion of the form
\[
Y \sim Y_0(T, \tau) + \sqrt{\epsilon} Y_1(T, \tau) + \ldots. \quad (6.4)
\]
Substituting (6.4) into (6.3) we obtain
\[
\left[ \partial^2_T + 2\sqrt{\epsilon} \partial_T \partial_\tau + \epsilon \partial^2_\tau \right] Y + \sqrt{\epsilon} \kappa \left[ \partial_T + \sqrt{\epsilon} \partial_\tau \right] Y + Y + \epsilon Y^3 = 0,
\]
with boundary conditions
\[
Y(0, 0) = 0, \quad \left[ \partial_T + \sqrt{\epsilon} \partial_\tau \right] Y(0, 0) = \sqrt{\epsilon}.
\]
The \( O(1) \) problem is
\[
\left( \partial^2_T + 1 \right) Y_0 = 0, \quad Y_0(0,0) = \partial_T Y_0(0,0) = 0,
\]
and its general solution is
\[
Y_0(T, \tau) = A(\tau) e^{iT} + A^*(\tau) e^{-iT}.
\]
The \( O(\sqrt{\epsilon}) \) equation is
\[
\left( \partial^2_T + 1 \right) Y_1 = -2\partial_T \partial_\tau Y_0 - \kappa \partial_T Y_0
\]
\[
= -2i \left[ A_\tau e^{it} - A^*_\tau e^{-it} \right] - \kappa i \left[ A e^{it} - A^* e^{-it} \right]
\]
\[
= -i \left[ 2A_\tau + \kappa A \right] e^{it} + i \left[ 2A^*_\tau + \kappa A^* \right] e^{-it}
\]
\[
= F(\tau) e^{it} + F^*(\tau) e^{-it}.
\]
The secular terms are eliminated provided \( F(\tau) = 0 \), that is
\[
2A_\tau + \kappa A = 0 \implies A(\tau) = A(0)e^{-\kappa t/2}.
\]
It can be easily seen from the initial conditions of the \( O(1) \) problem that \( A(0) = 0 \), and so \( Y_0 \equiv 0 \). Before we proceed any further, note that
\[
Y_1(T, \tau) = B(\tau)e^{iT} + B^*(\tau)e^{-iT}.
\]
The \( O(\epsilon) \) equation is
\[
\left( \frac{\partial^2}{\partial T^2} + 1 \right) Y_2 = -2\partial_T \partial_\tau Y_1 - \partial_\tau^2 Y_0 - \kappa \left( \partial_T Y_1 + \partial_\tau Y_0 \right) - Y_0^3
\]
\[
= -2\partial_T \partial_\tau Y_1 - \kappa \partial_\tau Y_1.
\]
This has the same structure as the \( O(\sqrt{\epsilon}) \) equation and it should be clear then that the secular terms are eliminated provided
\[
2B_\tau + \kappa B = 0 \implies B(\tau) = B(0)e^{-\kappa t/2}.
\]
Imposing the initial condition \( Y_1(0,0) = 0 \) and \( (\partial_T Y_1 + \partial_\tau Y_0)(0,0) = \partial_\tau Y_1(0,0) = 1 \), we obtain
\[
B(0) + B^*(0) = 0
\]
\[
iB(0) - B^*(0) = 1,
\]
which gives \( B(0) = -i/2 \). Hence, the \( O(\sqrt{\epsilon}) \) solution is
\[
Y_1(T, \tau) = B(0)e^{-\kappa t/2}e^{iT} + B^*(0)e^{-\kappa t/2}e^{-iT}
\]
\[
= e^{-\kappa t/2} \left[ -\frac{i}{2} e^{iT} + \frac{i}{2} e^{-iT} \right]
\]
\[
= e^{-\kappa t/2} \sin(T)
\]
and a first-term approximation of the solution of the original problem is
\[
y(t) = Y(T) \sim e^{-\kappa t/2} \sin \left( \frac{t}{\sqrt{\epsilon}} \right).
\]

**Problem 2.** In the study of Josephson junctions, the following problem appears
\[
\phi'' + \epsilon \left( 1 + \gamma \cos \phi \right) \phi' + \sin \phi = \epsilon \alpha,
\]  \hspace{1cm} (6.5)
with \( \phi(0) = 0 \), \( \phi'(0) = 0 \) and \( \gamma > 0 \). Use the method of multiple scales to find a first-term approximation of \( \phi(t) \).

**Proof.** Applying the regular perturbation with expanding \( \phi \sim \phi_0(x) + \epsilon \phi_1(x) \)
\[
\phi''_0 + \sin \phi_0 = 0, \quad \phi_0(0) = 0, \quad \phi'_0(0) = 0.
\]  \hspace{1cm} (6.6)
It follows that \( \phi_0(x) = 0 \). Setting multiscales \( t = t \) and \( \tau = \epsilon t \) gives
\[
\frac{d}{dt} \to \partial_t + \epsilon \partial_\tau,
\]
and substituting into the given equations
\[ (\partial_x + \epsilon \partial_t)^2 \phi + \epsilon (1 + \gamma \cos \phi) \partial_x \phi + \sin \phi = \epsilon \alpha, \] (6.7)
with the conditions \( \phi(0,0) = 0 \) and \( (\partial_x + \epsilon \partial_t)\phi(0,0) = 0 \). Expanding \( \phi(t, \tau) \sim \epsilon (\phi_0 + \epsilon \phi_1 + \cdots) \) and collecting \( \mathcal{O}(\epsilon) \) yields
\[ \partial_t^2 \phi_0 + \phi_0 = \alpha. \]

It follows that
\[ \phi_0(t, \tau) = \alpha [1 + A(\tau) \cos t + B(\tau) \sin t]. \] (6.8)

One can achieve one more equation to determine \( A, B \) by balancing \( \mathcal{O}(\epsilon^2) \)
\[ \partial_t^2 \phi_1 + \phi_1 = -[2 \partial_t \partial_t \phi_0 + (1 + \gamma) \partial_t \phi_0]. \]

Using the fact that
\[ \partial_t \phi_0 = -A \sin t + B \cos t, \quad \partial_t \partial_t \phi_0 = -A' \sin t + B' \cos t, \]
one can remove secular term with
\[ 2A' + (1 + \gamma)A = 0, \quad 2B' + (1 + \gamma)B = 0. \]

It follows that
\[ A(\tau) = A_0 e^{-(1+\gamma)\tau/2}, \quad B(\tau) = B_0 e^{-(1+\gamma)\tau/2}. \] (6.9)

Substituting it into \( \phi_0 \) yields
\[ \phi(t, \tau) = \alpha \left[ 1 + (A_0 \cos t + B_0 \sin t) e^{-\frac{1+\gamma}{2} \tau} \right]. \]

Imposing the initial conditions \( \phi_0(0,0) = 0 \) and \( \partial_t \phi_0(0,0) = 0 \) gives
\[ \alpha [1 + A_0] = 0, \quad \alpha B_0 = 0, \]
and it follows \( A_0 = -1 \) and \( B_0 = 0 \). Therefore, we have
\[ \phi(t) \sim \epsilon \alpha \left[ 1 - e^{-\frac{1+\gamma}{2} \tau} \cos t \right]. \] (6.10)

### Problem 3
Consider the equation
\[ \ddot{x} + \dot{x} = -\epsilon (x^2 - x), \quad 0 < \epsilon \ll 1. \] (6.11)

Use the method of multiple scales to show that
\[ x_0(t, \tau) = A(\tau) + B(\tau) e^{-t}, \]
with \( \tau = \epsilon t \), and identify any resonant terms at \( \mathcal{O}(\epsilon) \). Show that the non-resonance condition is \( \partial_t A = A - A^2 \) and describe the asymptotic behaviour of solutions.

**Proof.** With the slow scale \( \tau = \epsilon t \) and assuming an asymptotic expansion of the form
\[ x(t, \tau) \sim x_0(t, \tau) + \epsilon x_1(t, \tau) + \ldots, \]
the differential equation (6.11) becomes
\[
\left[\partial_t^2 + 2\epsilon \partial_t \partial_\tau + \epsilon^2 \partial_\tau^2\right] x + \left[\partial_t + \epsilon \partial_\tau\right] x = -\epsilon \left[ x^2 - x \right] = -\epsilon \left[ x_0^2 - x_0 \right] + O(\epsilon^2).
\]

The $O(1)$ equation is
\[
\partial_t^2 x_0 + \partial_t x_0 = 0,
\]
and its general solution is
\[
x_0(t, \tau) = A(\tau) + B(\tau) e^{-t}.
\]

The $O(\epsilon)$ equation is
\[
\partial_t^2 x_1 + \partial_t x_1 = -2 \partial_t \partial_\tau x_0 - \partial_\tau x_0 - (x_0^2 - x_0)
\]
\[
= -\left[A^2 - A + A_1\right] - e^{-t} \left[B_1 - 2B_1 + 2AB - B\right] - B^2 e^{-2t}
\]
\[
= F(\tau) + G(\tau) e^{-t} + H(\tau) e^{-2t}.
\]

Since the first two terms belong to the kernel of the homogeneous operator, the corresponding particular solution has the form $F(\tau)$ and $G(\tau) e^{-t}$ and only the first one blows up as $t \to \infty$, since
\[
G(\tau) e^{-t} \to 0 \quad \text{as} \quad t \to \infty.
\]

Hence, the non-resonance condition is $F(\tau) = 0$, or
\[
\partial_\tau A = A - A^2. \quad (6.12)
\]

A phase-plane analysis shows that the system (6.12) has an unstable fixed point at $A = 0$ and a stable fixed point at $A = 1$. Thus, we conclude that $A(\tau) \to 1$ as $\tau \to \infty$, provided $A(0) > 0$. \[\blacksquare\]

**Problem 4.** Consider the differential equation
\[\ddot{x} + x = -\epsilon f(x, \dot{x}), \quad \text{with} \quad |\epsilon| \ll 1.\]

Let $y = \dot{x}$.

1. Show that if $E(t) = E(x(t), y(t)) = (x(t)^2 + y(t)^2)/2$, then
\[
\dot{E} = -\epsilon f(x, y) y.
\]

Hence, show that $E(t)$ is approximately $2\pi$-periodic with $x = A_0 \cos(t) + O(\epsilon)$ provided
\[
\int_0^{2\pi} f(A_0 \cos \tau, -A_0 \sin \tau) \sin \tau \, d\tau = 0.
\]

**Proof.** With $y = \dot{x}$, we have
\[
\dot{E} = xx + yy = xy + y[-x - \epsilon f(x, y)],
\]
and it follows that
\[
\dot{E} = -\epsilon f(x, y) y. \quad (6.13)
\]
To investigate the periodicity of $E(t)$, we perform integration over $[2n\pi, 2(n+1)\pi]$

$$E(2n\pi + 2\pi) - E(2n\pi) = -\epsilon \int_{2n\pi}^{2n\pi+\pi} f(x(\tau), y(\tau))y(\tau)d\tau.$$  \hspace{1cm} (6.14)

Denote the energy at time $t = 2n\pi$ by $E_n = E(2n\pi)$. Imposing the assumption of $x$ and expanding $f$ by Taylor series yields

$$E_{n+1} - E_n = \epsilon A_0 \int_{2n\pi}^{2n\pi+\pi} f(A_0 \cos \tau, -A_0 \sin \tau) \sin \tau d\tau + O(\epsilon^2).$$

Applying given condition

$$E_{n+1} - E_n = 0 + O(\epsilon^2),$$

and it means that the energy is approximately $2\pi$-periodic. \hfill ■

2. Suppose that the periodicity condition on part (a) does not hold. Let $E_n = E(x(2\pi n), y(2\pi n))$. Show that to lowest order $E_n$ satisfies a difference equation of the form

$$E_{n+1} = E_n + \epsilon F(E_n),$$

with

$$F(E_n) = \int_{0}^{2\pi} \sqrt{2E_n} f \left( \sqrt{2E_n} \cos \tau, -\sqrt{2E_n} \sin \tau \right) \sin \tau d\tau.$$  \hspace{1cm} (Hint: Take $x \sim A \cos t$ with $A = \sqrt{2E}$ slowly varying over a single period of length $2\pi$.)

Proof. Using the method of multiple scales with two time variables $t$ and $\tau = \epsilon t$, then the time derivative becomes

$$\frac{d}{dt} \rightarrow \frac{d}{d\tau} + \epsilon \frac{d}{d\tau}.$$  \hspace{1cm} (6.15)

Expand $x \sim x_0 + \epsilon x_1 + \cdots$ and substitute it to the differential equation gives

$$\left[ \left( \frac{d}{d\tau} + \epsilon \frac{d}{d\tau} \right)^2 + 1 \right] x = -\epsilon f \left( x, \left( \frac{d}{d\tau} + \epsilon \frac{d}{d\tau} \right) x \right).$$

Collecting $O(1)$ terms from (6.15) gives

$$x_0 + x_0 = 0,$$

and its general solution is

$$x_0(t, \tau) = A(\tau) \cos t + B(\tau) \sin t.$$  \hspace{1cm} (6.16)

Without loss of generality, one can assume that

$$x_0(t, \tau) = A(\tau) \cos t.$$  \hspace{1cm} (6.16)

In the same fashion, expanding $E \sim E_0 + \epsilon E_1 + \cdots$

$$E_0 + \epsilon E_1 + \cdots = \frac{1}{2} \left( x^2 + x^2 \right),$$
and the $O(1)$ equation gives
\[ E_0 = \frac{1}{2} A^2(\tau) \implies A(\tau) = \sqrt{2E_0(\tau)}. \] (6.17)

It implies that $E_0$ is slowly varying numbers and guaranteed by followings: Applying the multiple scales on the energy differential equation yields
\[
\left( \frac{d}{dt} + \epsilon \frac{d}{d\tau} \right) E = -\epsilon f \left( x, \left( \frac{d}{dt} + \epsilon \frac{d}{d\tau} \right) x \right) \cdot \left( \frac{d}{dt} + \epsilon \frac{d}{d\tau} \right) x,
\]
and collecting $O(1)$ terms gives
\[
\frac{d}{dt} E_0 = 0.
\]

Finally, since the lowest order of $E_n$ is slowly varying, then one can approximate $x$ as
\[
x(2\pi n + t) \sim \sqrt{2E_n} \cos t.
\]

Substituting it into (6.14) gives the following equation
\[ E_{n+1} - E_n = \epsilon F(E_n), \] (6.18)
with
\[
F(E_n) = \int_0^{2\pi} \sqrt{2E_n} f(\sqrt{2E_n} \cos \xi, -\sqrt{2E_n} \sin \xi) \sin \xi d\xi = 0.
\] (6.19)

3. Hence, deduce that a periodic orbit with approximate amplitude $A^* = \sqrt{2E^*}$ exists if $F(E^*) = 0$ and this orbit is stable if
\[
\epsilon \frac{dF}{dE}(E^*) < 0.
\]

Hint: Spiralling orbits close to the periodic orbit $x = A^* \cos t + O(\epsilon)$ can be approximated by a solution of the form $x = A \cos t + O(\epsilon)$.

Proof. From part 2., we have a one-dimensional map
\[ E_{n+1} = E_n + \epsilon F(E_n). \]

If it has a fixed point, then it satisfies
\[ E^* = E^* + \epsilon F(E^*), \]
that is $F(E^*) = 0$. This implies that the periodic amplitude exists. Consider a perturbed point from the fixed point $E^*$ as $E^* + \Delta E$. Then the energy is evolved after time $2\pi$ as
\[
E' = E^* + \Delta E + \epsilon F(E^* + \Delta E) \sim E^* + \Delta E + \Delta E \cdot \epsilon \frac{dF}{dE}(E^*),
\]
with local approximation. If $\Delta E > 0$, then $E' < E^* + \Delta E$ and it means that the energy decreases back to $E^*$. On the other hand, if $\Delta E < 0$, then
\[ E' > E^* + \Delta E \text{ and the energy increases up to } E^*. \text{ Since it is converse when } c dF/dE(E^*) > 0. \text{ Therefore, it is stable when } \epsilon \frac{dF}{dE}(E^*) < 0. \]

4. Using the above result, find the approximate amplitude of the periodic orbit of the Van der Pol equation \[ \ddot{x} + x + \epsilon (x^2 - 1) \dot{x} = 0 \]
and verify that it is stable.

\textbf{Proof.} To utilize the result in part 3., find \( F \) in this case. Since \( f(x, y) = (x^2 - 1)y \), then we have
\[
F(E) = \int_0^{2\pi} \sqrt{2E(2E \cos^2 \xi - 1)}(-\sqrt{2E \sin^2 \xi})d\xi.
\]
Expanding it
\[
F(E) = \int_0^{2\pi} E \cdot 2 \sin^2 \xi - E^2 \cdot 4 \sin^2 \xi \cos^2 \xi d\xi,
\]
and imposing trigonometric formula gives
\[
F(E) = \int_0^{2\pi} E[1 - \cos(2\xi)] - \frac{E^2}{2}[1 - \cos(4\xi)]d\xi.
\]
Performing integration gives
\[
F(E) = \pi E(2 - E). \tag{6.20}
\]
Therefore, the zeros of \( F \) are \( E^* = 0, 2 \) and the approximate amplitude of the periodic orbit of the Van der Pol equation is \( A^* = \sqrt{2E^2} = 2 \). This orbit is stable with positive \( \epsilon \) since
\[
\epsilon \frac{dF}{dE}(E^*) = 2\pi(1 - E^*) = -2\pi < 0.
\]

\textbf{Problem 5.} Consider the Van der Pol equation
\[ \ddot{x} + x + \epsilon (x^2 - 1) \dot{x} = \Gamma \cos(\omega t), \ 0 < \epsilon \ll 1, \]
with \( \Gamma = O(1) \) and \( \omega \neq 1/3, 1, 3 \). Use the method of multiple scales to show that the solution is attracted to
\[
x(t) = \left( \frac{\Gamma}{1 - \omega^2} \right) \cos(\omega t) + O(\epsilon)
\]
when \( \Gamma^2 \geq 2(1 - \omega^2)^2 \) and
\[
x(t) = 2 \left[ 1 - \frac{\Gamma^2}{2(1 - \omega^2)^2} \right]^{1/2} \cos t + \left( \frac{\Gamma}{1 - \omega^2} \right) \cos(\omega t) + O(\epsilon)
\]
when \( \Gamma^2 < 2(1 - \omega^2)^2 \). Explain why this result breaks down when \( \omega = 1/3, 1, 3 \).
Proof. (Sketch) Use the method of multiple scales with \( t \) and \( \tau = \epsilon t \). Solve the \( O(1) \) by setting
\[
x_0(t, \tau) = a(\tau) \cos t + \theta \cos \omega t,
\]
where \( \theta = \Gamma / 2(1 - \omega^2) \). Then find required equations by removing secular terms. Investigate its long term behavior and get the two different equations depending on parameters. ■

Problem 6. Multiple scales with nonlinear wave equations. The Korteweg-de Vries (KdV) equation is
\[
u_t + \nu_x + auu_x + \beta \nu_{xxxx} = 0, \quad x \in \mathbb{R}, \quad t > 0,
\]
where \( a, \beta \) are positive real constants and \( u(x, 0) = \epsilon f(x) \) for \( 0 < \epsilon \ll 1 \).

1. Let \( \theta = kx - \omega t \) and seek traveling wave solutions using an expansion of the form
\[
u(x, t) \sim \epsilon [u_0(\theta) + \epsilon u_1(\theta) + \ldots],
\]
where \( \omega = k - \beta k^3 \) and \( k > 0 \) is a constant. Show that this can lead to secular terms.

Proof. (Sketch) For given expansion, one can see the following
\[
u_t = \epsilon [u'_0(\theta) + \ldots], \quad \nu_x = \epsilon [u'_0(\theta) + \ldots].
\]
Then get \( O(\epsilon) \) equation and find secular term in \( O(\epsilon^2) \) equation. ■

2. Use multiple scales (variables \( \theta, \epsilon x, \epsilon t \)) to eliminate the secular terms in part (a) and find a first-term expansion. In the process, show that \( f(x) \) must have the form
\[
f(x) = A \cos(kx + \phi)
\]
for constants \( A, B, \phi \) in order to generate a traveling wave? Hint: Use the fact that \( f(x) \) is independent of \( \epsilon \).

Proof. (Sketch) Apply the method of multiple scales with \( \theta, x, t \) and rescaling gives
\[
\partial_t \rightarrow -w \partial_\theta + \epsilon \partial_t, \quad \partial_x \rightarrow k \partial_\theta + \epsilon \partial_x.
\]
Then \( O(\epsilon) \) equation gives
\[
u_0(\theta, x, t) = F(x, t) e^{i\theta} + \text{complex conjugate} + G(x, t),
\]
and find conditions for \( F, G \) from removing secular terms in \( O(\epsilon^2) \) equation. Therefore, we have the following
\[
F_t + A_x (1 - \beta k^2) + iakFG = 0, \quad G_t + G_x = 0.
\]
From this condition, determine the form of \( f(x) \). ■
REFERENCES

