INTRODUCTION TO ASYMPTOTIC APPROXIMATION

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ABSTRACT

These notes are largely based on Math 6730: Asymptotic and Perturbation Methods course, taught by Paul Bressloff in Fall 2017, at the University of Utah. Additional examples or remarks or results from other sources are added as we see fit, mainly to facilitate our understanding. These notes are by no means accurate or applicable, and any mistakes here are of course our own. Please report any typographical errors or mathematical fallacy to us by email hkim@math.utah.edu or tan@math.utah.edu.

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Our main goal is to construct approximate solutions of differential equations to gain insight of the problem, since they are nearly impossible to solve analytically in general due to the nonlinear nature of the problem. Among the most important machinery in approximating functions in some small neighbourhood is the Taylor’s theorem. It says that given \( f \in C^{N+1}(B_\delta(x_0)) \), we can write \( f(x) \) as

\[
f(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + R_{N+1}(x),
\]

where \( R_{N+1}(x) \) is the remainder term

\[
R_{N+1}(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} (x - x_0)^{N+1},
\]

for some \( \xi \) between \( x \) and \( x_0 \), provided \( x \in B_\delta(x_0) \). Taylor’s theorem can be used to solve the following problem:

**Given a fixed \( \epsilon = |x - x_0| > 0 \), how many terms should we include in the Taylor polynomial to achieve a certain accuracy?**

Asymptotic approximation concerns about a slightly different problem:

**Given a fixed number of terms \( N \), how accurate is the asymptotic approximation as \( \epsilon \to 0 \)?**

We want to avoid from including as many terms as possible as \( \epsilon \to 0 \) and in contrast to Taylor’s theorem, we do not care about convergence of the asymptotic approximation. In fact, most asymptotic approximations diverge as \( N \to \infty \) for a fixed \( \epsilon \).

**Remark 1.** If the given function is sufficiently differentiable, then Taylor’s theorem offers a reasonable approximation and we can easily analyse the error as well.

## 1 Asymptotic Expansion

We begin the section with a motivating example. Suppose that we want to evaluate the integral

\[
f(\epsilon) = \int_0^\infty \frac{e^{-t}}{1 + \epsilon t} dt, \quad \epsilon > 0.
\]

We can develop an approximation of \( f(\epsilon) \) for small \( \epsilon \) by repeatedly integration by parts

\[
f(\epsilon) = 1 - \epsilon \int_0^\infty \frac{e^{-t}}{(1 + \epsilon t)^2} dt = \cdots = \sum_{k=0}^{N} (-1)^k k! \epsilon^k + R_N(\epsilon),
\]

where

\[
R_N(\epsilon) = (-1)^{N+1}(N + 1)! \epsilon^{N+1} \int_0^\infty \frac{e^{-t}}{(1 + \epsilon t)^{N+2}} dt.
\]
In practice, it is hard to estimate a bound for the remaining term. In this case, since
\[
\int_0^\infty \frac{e^{-t}}{(1+et)^{N+2}} dt \leq \int_0^\infty e^{-t} dt = 1,
\]
then
\[
|R_N(e)| \leq |(-1)^{N+1}(N + 1)!e^{N+1}| \ll |(-1)^{N}N!e^{N}|.
\]
For fixed \(N\),
\[
\lim_{\epsilon \to 0} \left| \frac{f(\epsilon) - \sum_{k=0}^{N} a_k \epsilon^k}{e^{N}} \right| = 0.
\]
That is,
\[
f(\epsilon) = \sum_{k=0}^{N} a_k \epsilon^k + O(e^{N+1}).
\]
The formal series \(\sum_{k=0}^{N} a_k \epsilon^k\) is said to be an asymptotic expansion of \(f(t)\) such that for fixed \(N\) in practice? A thumb rule is beginning at \(N = 2\) and increasing to see physical phenomena) \(N\) it provides a good approximation to \(f(\epsilon)\) as \(\epsilon \to 0\). Note that asymptotic expansion in general is not convergent for fixed \(\epsilon\) since
\[
(-1)^{N}N!e^{N} \to \infty, \quad N \to \infty,
\]
that is the correction term actually blows up!

Remark 2. Observe that for sufficiently small \(\epsilon > 0\),
\[
R_N(\epsilon) \ll |(-1)^{N}N!e^{N}|,
\]
which means that the remainder \(R_N(\epsilon)\) is dominated by the \(N + 1\)th term of the approximation. In other words the error is of higher-order of the approximating function. This property is something that we would want to impose on the asymptotic expansion, and this idea can be made precise using the Landau symbols.

Definition 1. (a) \(f(\epsilon) = O(g(\epsilon))\) as \(\epsilon \to 0\) means that there exists a finite \(M\) for which \(|f(\epsilon)| \leq M|g(\epsilon)|\).
(b) \(f(\epsilon) = o(g(\epsilon))\) as \(\epsilon \to 0\) means \(\lim_{\epsilon \to 0} |f(\epsilon)/g(\epsilon)| = 0\).
(c) The ordered sequence of function \(\{\phi_k(\epsilon)\}, k = 0, 1, \ldots\) is called an asymptotic sequence as \(\epsilon \to 0\) if
\[
\phi_{k+1}(\epsilon) = o(\phi_k(\epsilon)) \quad \text{as} \quad \epsilon \to 0.
\]
(d) Let \(f(\epsilon)\) be a continuous function of \(\epsilon\) and let \(\{\phi_k(\epsilon)\}\) be an asymptotic sequence. The formal series expansion
\[
\sum_{k=0}^{N} a_k \phi_k(\epsilon)
\]
(1.1)
is called an asymptotic expansion (a.e.) valid to order \(\phi_N(\epsilon)\) if for all \(N \geq 0\)
\[
\lim_{\epsilon \to 0} \left| \frac{f(\epsilon) - \sum_{k=0}^{N} a_k \phi_k(\epsilon)}{\phi_N(\epsilon)} \right| = 0.
\]
(1.2)
One typically write \(f(\epsilon) \sim \sum_{k=0}^{N} a_k \phi_k(\epsilon)\) as \(\epsilon \to 0\).
In solution to ODEs we will need to consider time-dependent asymptotic expansions. Suppose \( x'(t) = f(x, \epsilon) \) for \( x \in \mathbb{R}^n \) where \( \epsilon \) is a parameter of vector field. Then

\[
x(t, \epsilon) \sim \sum_{k=0}^{N} a_k(t) \phi_k(\epsilon)
\]

(In case of PDE, \( u(x, t, \epsilon) \sim \sum_{k=0}^{N} a_k(x, t) \phi_k(\epsilon) \)) which will tend to be valid over some range of times \( t \). It is often useful to characterize the time interval over which a.e. exists, i.e. say that estimate is valid on a time scale \( 1/\delta(\epsilon) \) if

\[
\lim_{\epsilon \to 0} \left| \frac{x(t, \epsilon) - \sum_{k=0}^{N} a_k(t) \phi_k(\epsilon)}{\phi_N(\epsilon)} \right| = 0 \text{ for } 0 \leq t \leq C/\delta(\epsilon).
\]

where \( C \) is independent of \( \epsilon \).

### 1.1 Accuracy and convergence

In the case of a Taylor series expansion, one can increase accuracy (for fixed \( \epsilon \)) by including more terms (assuming convergence). This is not usually the case for an a.e. – a.e. concerns limit \( \epsilon \to 0 \) whereas the number of terms concerns \( N \to \infty \) for fixed \( \epsilon \).

### 1.2 Manipulating asymptotic expansion

Two asymptotic expansions can be added together term by term, assuming both involve the same basis functions \( \{\phi_k(\epsilon)\} \). Multiplication can also be carried out provided the asymptotic sequences can be ordered in a particular way. What about differentiation? Suppose

\[
f(x, \epsilon) \sim \phi_1(x, \epsilon) + \phi_2(x, \epsilon), \quad \epsilon \to 0.
\]

It is not necessarily the case

\[
\frac{d}{dx} f(x, \epsilon) \sim \frac{d}{dx} \phi_1(x, \epsilon) + \frac{d}{dx} \phi_2(x, \epsilon), \quad \epsilon \to 0.
\]

There are two possible scenarios:

**Example 1.** Consider \( f(x, \epsilon) = e^{-x/\epsilon} \sin(e^{x/\epsilon}) \). Note that \( f \) is fast oscillation with different scale. Observe that for \( x > 0 \)

\[
\lim_{\epsilon \to 0} \left| \frac{f(x, \epsilon)}{e^n} \right| = 0 \text{ for all finite } n,
\]

which means that

\[
f(x, \epsilon) \sim 0 + 0 \cdot \epsilon + 0 \cdot \epsilon^2 + \cdots,
\]

as \( \epsilon \to 0 \). However,

\[
\frac{d}{dx} f(x, \epsilon) = -\frac{1}{\epsilon} e^{-x/\epsilon} \sin(e^{x/\epsilon}) + \frac{1}{\epsilon} \cos(e^{x/\epsilon}) \to \infty, \quad \epsilon \to 0,
\]

that is derivative cannot be expanded using \( \epsilon, \epsilon^2, \cdots \to 0 \).

**Example 2.** Even if \( \{\phi_k(x, \epsilon)\} \) is an ordered asymptotic sequence, its derivative \( \{\phi'_k(x, \epsilon)\} \) need not be. Consider \( \phi_1(x, \epsilon) = 1 + x \) and \( \phi_2(x, \epsilon) = \epsilon \sin(x/\epsilon) \) for \( x \in [0, 1] \). Then

\[
\phi'_1(x, \epsilon) = 1, \quad \phi'_2(x, \epsilon) = \cos(x/\epsilon)
\]

which are not ordered.
In practice, if
\[ f(x, \epsilon) \sim a_1(x)\phi_1(\epsilon) + a_2(x)\phi_2(\epsilon), \quad \epsilon \to 0, \quad (1.3) \]
and
\[ d_x f(x, \epsilon) \sim b_1(x)\phi_1(\epsilon) + b_2(x)\phi_2(\epsilon), \quad \epsilon \to 0, \quad (1.4) \]
then \( b_k(x) = d_x a_k(x) \). Throughout this course, we assume that it holds whenever we are given (1.3) which almost holds in practice. Integration, on the other hand, is less problematic. If (1.3) holds on \( x \in [a, b] \) and all functions are integrable, then
\[ \int_a^b f(x, \epsilon)dx \sim \left( \int_a^b a_1(x)dx \right)\phi_1(\epsilon) + \left( \int_a^b a_2(x)dx \right)\phi_2(\epsilon), \quad \epsilon \to 0. \quad (1.5) \]

2 \ ASYMPTOTIC SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

We study three examples where approximate solutions are found using asymptotic expansions, but each uses different method. They serve to illustrate the important point that instead of performing the routine procedure with standard asymptotic sequence, we should tailor our asymptotic expansion to extract the physical property or behavior of our problem.

2.1 Example: algebraic equation

Consider the quadratic equation
\[ \epsilon x^2 + 2x - 1 = 0. \quad (2.1) \]
This is singular problem since order of polynomial changes when \( \epsilon = 0 \), see Fig. 1.
In this case, the unique solution is \( x = 1/2 \). Therefore, try
\[ x \sim \frac{1}{2} + x_1 e^a + \cdots . \]
Substitute it into the equation and we have
\[ \epsilon \left( \frac{1}{4} + x_1^2 e^{2a} + x_1 e^a + \cdots \right) + 2 \left( \frac{1}{2} + x_1 e^a + \cdots \right) - 1 = 0. \]
The $O(1)$ terms cancel as we anticipate. To cancel $O(\epsilon)$ terms, set $\alpha = 1$ and obtain
\[
\frac{1}{4} + 2x_1 = 0 \implies x_1 = -\frac{1}{8} \implies x \sim \frac{1}{2} - \frac{1}{8} \epsilon.
\]

One method to generate other root is to consider $(x - x_1)(x - x_2) = 0$. More systematic method, applicable to ODEs, is to avoid the $O(1)$ solution. Take $x \sim \epsilon^\gamma(x_0 + x_1e^\delta + \cdots)$. Now we have
\[
\epsilon^{1+2\gamma}(x_0^2 + 2x_0x_1e^\delta + \cdots) + 2\epsilon^\gamma(x_0 + x_1e^\delta + \cdots) + (-1) = 0.
\]

Terms on LHS must balance and there are three possibilities:

1. Set $\gamma = 0$ and we recover the previous solution balancing (a) and (c).

2. Balance (a) $\sim$ (c) and make (b) to be higher-order. Require $1 + 2\gamma = 0 \implies \gamma = -1/2$ such that (a) $\sim$ (c) for $O(1)$ with (b) = $O(\epsilon^{1/2})$, not higher-order. Thus, this is not a good approximation.

3. Balance (a) $\sim$ (b) and make (c) to be higher order. Require $1 + 2\gamma = \gamma$, i.e. $\gamma = -1$. Then (a), (b) = $O(\epsilon^{-1})$ and (c) = $O(1)$. Setting $\gamma = -1$ gives
\[
(x_0^2 + 2x_0x_1e^\delta + \cdots) + 2(x_0 + x_1e^\delta + \cdots) - \epsilon = 0.
\]

Balancing $O(1)$ gives $x_0^2 + 2x_0 = 0 \implies x_0 = 0, -2$. If $x_0 = 0$, then it recovers the original solution. Now root given by taking $x_0 = -2$. To balance $\epsilon$ in the last term take $\alpha = 1$ and balance $O(\epsilon)$. Then we have $2x_0x_1 + 2x_1 - 1 = 0$ implies $x_1 = -1/2$. Therefore,
\[
x \sim \frac{1}{\epsilon}\left(-2 - \frac{\epsilon}{2}\right).
\]

### 2.2 Example: transcendental equation

For transcendental equations, one needs to determine how many solution exist graphically. Consider
\[
x^2 + e^{x} = 5,
\]
and one can figure out that it has two solution for $\epsilon > 0$, see Fig. 2 (and this is not singular because $\epsilon = 0$ provides $x = \pm 2$). Applying same argument in example of algebraic equation, assume asymptotic expansion
\[
x x_0 + x_1e^\delta + \cdots
\]
and Taylor expand the exponent and have
\[
(x_0^2 + 2x_0x_1e^\delta + \cdots) + (1 + \epsilon x_0 + \cdots) = 5.
\]
Balance \( \mathcal{O}(1) \) provides \( x_0^2 + 1 = 4 \Rightarrow x_0 = \pm 2 \). Take \( \alpha = 1/2 \) and balance \( \mathcal{O}(\epsilon) \) follows that \( x_1 = -1/2 \). Therefore,

\[
x \sim \pm 2 - \frac{\epsilon}{2}.
\]

Consider one more transcendental equation

\[
x + 1 + \epsilon \cdot \text{sech} \left( \frac{x}{\epsilon} \right) = 0.
\] (2.3)

One can determine it has one solution graphically. If you try with polynomials \( x \sim x_0 + x_1 \epsilon + \cdots \), then one cannot balance terms. In fact, it suffices to find an asymptotic sequence \( \{\phi_1(\epsilon), \phi_2(\epsilon), \cdots\} \) such that \( \phi_1(\epsilon) \ll 1, \phi_2(\epsilon) \ll \phi_1(\epsilon) \) and so on. Therefore, try \( x \sim x_0 + \mu(\epsilon) \) with \( \mu(\epsilon) \ll 1 \) where \( \epsilon \ll 1 \). Substituting into original equation gives

\[
x_0 + \mu(\epsilon) + 1 + \epsilon \cdot \text{sech} \left( \frac{x_0}{\epsilon} + \frac{\mu(\epsilon)}{\epsilon} \right) = 0.
\]

Balance \( \mathcal{O}(1) \) yields \( x_0 = -1 \). Then we have

\[
\mu(\epsilon) + \epsilon \cdot \text{sech} \left( -\frac{1}{\epsilon} + \frac{\mu(\epsilon)}{\epsilon} \right) = 0.
\]

Since \( \mu(\epsilon) \ll 1 \), then one can approximate

\[
\text{sech} \left( -\frac{1}{\epsilon} + \frac{\mu(\epsilon)}{\epsilon} \right) \sim \text{sech} \left( -\frac{1}{\epsilon} \right) = \frac{2}{e^{1/\epsilon} + e^{-1/\epsilon}} \sim 2e^{-1/\epsilon}.
\]

Therefore, it requires \( \mu(\epsilon) = -2\epsilon e^{-1/\epsilon} \). To proceed further, take

\[
x \sim -1 - 2\epsilon e^{-1/\epsilon} + \nu(\epsilon)
\]

with \( \nu(\epsilon) \ll \epsilon e^{-1/\epsilon} \).

### 3 Differential Equations – Regular Perturbations

Roughly speaking, the regular perturbation theory is a variant of Taylor’s theorem, in the sense that we seek power series solution in \( \epsilon \). More precisely, we assume that the solution takes the form

\[
x \sim x_0 + x_1 \epsilon + x_2 \epsilon^2 + \cdots,
\]

as \( \epsilon \to 0 \). In this chapter, we apply this idea to some differential equations.

#### 3.1 Projectile problem

Consider the motion of a gerbil projected radially upward from the surface of the earth. Let \( x(t) \) be the height of the gerbil from the surface of the Earth. Newton’s law of motion asserts that

\[
\frac{d^2}{dt^2} x = -g \frac{R^2}{(x + R)^2}
\] (3.1)

where \( R \) is radius of the earth and \( g \) is the gravitational constant. If \( x \ll R \), then to a first approximation

\[
\frac{d^2}{dt^2} x = -g \Rightarrow x_0(t) = -\frac{1}{2} gt^2 + v_0 t,
\]
where \( v_0 \) is given initial velocity, see Fig. 3. It has characteristic values that \( t_c = v_0 / g \) is time reaching highest height and \( x_c = v_0^2 / 2g \) is highest reached height. Before perturbing the ODE, non-dimensionalize by \( \tau = t / t_c \) and \( y = x / x_c \). Then it provides that

\[
\frac{d^2 y}{d\tau^2} = -\frac{1}{(1 + \epsilon y)^2} \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1. \tag{3.2}
\]

Consider an asymptotic expansion with power basis \( y(\tau) \sim y_0(\tau) + \epsilon y_1(\tau) + \cdots \). Assume we can differentiate term by term. Then by Taylor expanding to RHS

\[
y''_0 + \epsilon y''_1 + \cdots = -\frac{1}{(1 + \epsilon(y_0 + \cdots))^2} = -1 + 2\epsilon y_0 + \cdots
\]

with initial condition

\[
y_0(0) + \epsilon y_1(0) + \cdots = 0 \quad \text{and} \quad y'_0(0) + \epsilon y'_1(0) + \cdots = 1.
\]

Balance \( \mathcal{O}(1) \) with \( y_0(0) = 0 \) and \( y'_0(0) = 1 \) and it yields \( y''_0 = -1 \) and

\[
y_0(\tau) = -\frac{1}{2} \tau^2 + \tau.
\]

To balance the next term \( a = 1 \), determined by RHS. Balance \( \mathcal{O}(\epsilon) \), then one can obtain \( y''_1 = 2y_0 \) with \( y_1(0) = y'_1(0) = 0 \) and

\[
y_1(\tau) = \frac{1}{3} \tau^3 - \frac{1}{12} \tau^2.
\]

Therefore, the solution of asymptotic expansion would be

\[
y(\tau) \sim \tau \left(1 - \frac{1}{2} \tau^2\right) + \frac{1}{3} \epsilon \tau^3 \left(1 - \frac{\tau}{4}\right). \tag{3.3}
\]

### 3.2 Non-linear potential problem

Let us consider diffusion of ions through a solution containing charged molecule. Electrostatic potential \( \phi(x) \) satisfies **Poisson-Boltzmann equation**

\[
\nabla^2 \phi = -\sum_{i=1}^{k} \alpha_i z_i e^{-z_i \phi}, \quad x \in \Omega, \tag{3.4}
\]

where \( \alpha_i \) is constant and \( z_i \) is valence of \( i \)th ionic species, with electro-neutrality and boundary conditions

\[
\sum_{i=1}^{k} \alpha_i z_i = 0 \quad \text{and} \quad \partial_n \phi = n \cdot \nabla \phi = e, \quad \partial \Omega. \tag{3.5}
\]
Until now, there is no known exact solution. Try

$$\phi \sim \varepsilon \phi_0(x) + \varepsilon \phi_1(x) + \cdots$$

because one can get trivial solution as $$\varepsilon \to 0$$. Then we have

$$\varepsilon (\nabla^2 \phi_0 + \varepsilon \nabla^2 \phi_1 + \cdots) = - \sum_i \alpha_i z_i e^{-\varepsilon z_i (\phi_0 + \varepsilon \phi_1 + \cdots)}.$$ 

Expanding the right-hand side by Taylor series

$$\varepsilon (\nabla^2 \phi_0 + \varepsilon \nabla^2 \phi_1 + \cdots) \sim - \sum_i \alpha_i z_i \left( 1 - \varepsilon z_i (\phi_0 + \varepsilon \phi_1 + \cdots) + \frac{1}{2} \varepsilon^2 z_i^2 (\phi_0 + \cdots)^2 + \cdots \right),$$

and rearranging it gives

$$\varepsilon (\nabla^2 \phi_0 + \varepsilon \nabla^2 \phi_1 + \cdots) \sim \varepsilon \sum_i \alpha_i z_i^2 \phi_0 + \varepsilon^2 \sum_i \alpha_i z_i^2 \left( \phi_1 - \frac{1}{2} \varepsilon z_i \phi_0^2 \right) + \cdots.$$ 

Set $$\kappa^2 = \sum_i \alpha_i z_i^2$$ and it follows an equation of balancing $$O(\varepsilon)$$

$$\begin{cases} (\nabla^2 - \kappa^2) \phi_0 = 0, & \chi \in \Omega \\ \partial_n \phi_0 = 1, & \chi \in \partial \Omega \end{cases}$$

Balance $$O(\varepsilon^2)$$ and we get

$$\begin{cases} (\nabla^2 - \kappa^2) \phi_1 = -\lambda \phi_0, & \chi \in \Omega \\ \partial_n \phi_1 = 0, & \chi \in \partial \Omega \end{cases}$$

where $$\lambda = \frac{1}{2} \sum \alpha_i z_i^3$$. The solutions are called basis of classical Debye-Huckel theory in electrochemistry.

Specifically, take $$\Omega$$ to be the domain outside of the unit sphere. It follows that solutions have radial symmetry. By the radial Laplace operator, we obtain

$$\begin{cases} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi_0}{dr} \right) - \kappa^2 \phi_0 = 0, & 1 < r < \infty \\ \phi_0(1) = -1 \end{cases}$$

It yields $$\phi_0(r) = \frac{1}{(1+x)^2} e^{x(1-r)}$$. In the same fashion, one can get

$$\begin{cases} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d \phi_1}{dr} \right) - \kappa^2 \phi_0 = -\frac{\lambda}{(1+x)^2} e^{2x(1-r)}, & 1 < r < \infty \\ \phi_1(1) = 0 \end{cases}$$

and it yields the solution

$$\phi_1(r) = \frac{\lambda}{r} e^{-x} + \frac{\gamma}{\kappa r} \left[ e^{x} E_1(3kr) - e^{-x} E_1(kr) \right]$$

where $$\gamma = \lambda e^{2x}/2\kappa$$ and $$E_1$$ is the exponential integral $$E_1(z) = \int_x^\infty e^{-t^{-1}} dt$$.

3.3 Fredholm alternative

Let $$L_0$$ and $$L_1$$ be differential operators on $$L^2(\mathbb{R})$$ with inner product $$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) dx$$. Consider eigenvalue problem

$$(L_0 + \varepsilon L_1) \phi = \lambda \phi.$$ 

Suppose that $$\varepsilon = 0$$ and equation has a unique solution

$$L_0 \phi = \lambda_0 \phi.$$
with $\lambda_0$ is non-degenerate. Take $L_0$ to be self-adjoint operator. Introduce asymptotic expansion

$$\phi \sim \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 \quad \text{and} \quad \lambda \sim \lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2,$$

and it yields

$$(L_0 + \epsilon L_1)(\phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots) = (\lambda_0 + \epsilon \lambda_1 + \epsilon^2 \lambda_2)(\phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \cdots).$$

Balance $O(1)$ and we have

$$L_0 \phi_0 = \lambda_0 \phi_0, \quad \text{i.e.} \quad (L_0 - \lambda_0 I) \phi_0 = 0.$$

Balance $O(\epsilon)$ and get equation for $\phi_1$

$$(L_0 - \lambda_0 I) \phi_1 = \lambda_1 \phi_0 - L_1 \phi_0.$$

Use the fact that $L_0 - \lambda_0 I$ is self-adjoint and $\phi_0 \in \ker(L_0 - \lambda_0 I)$. Then we get

$$0 = \lambda_1 \langle \phi_0, \phi_0 \rangle + \langle \phi_0, L_1 \phi_0 \rangle \implies \lambda_1 = \frac{\langle \phi_0, L_1 \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle}.$$

Therefore, $\lambda \sim \lambda_0 + \epsilon \lambda_1$ and solve for $\phi_1$. With the same analogy, one can find $\lambda_n$ and $\phi_n$.

## 4 Exercises

**Problem 1.** Consider the transcendental equation

$$1 + \sqrt{x^2 + \epsilon} = e^x. \quad (4.1)$$

Explain why there is only one small root for small $\epsilon$. Find the three term expansion of the root:

$$x \sim x_0 + x_1 \epsilon^\alpha + x_2 \epsilon^\beta, \quad \beta > \alpha > 0.$$

**Proof.** Consider two graphs $f(x) = \sqrt{x^2 + \epsilon}$ and $g(x) = e^x - 1$. If $x < 0$, then $f(x) > 0 > g(x)$. It means that there is no solution in negative region. If $x > 0$, then $f(x) \to x$ as $x \to \infty$ starting its curve from $f(0) = \epsilon$. One can draw graph of $f(x)$ and $g(x)$ on $x > 0$, then it yields there is only one solution.

To obtain first expansion, set $\epsilon = 0$. Then we get

$$1 + x = e^x \implies x = 0.$$

Since there is only on solution for all $\epsilon > 0$, then one can set $x_0 = 0$ and expand $x$ further at this point. To do so, rewrite the equation $(4.1)$ as

$$x^2 + \epsilon = (e^x - 1)^2$$

and $x \ll 1$ as $\epsilon \ll 1$, it is reasonable to expand RHS as Taylor series. Then one can obtain

$$x^2 + \epsilon = \left( x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots \right)^2 = x^2 + x^3 + \frac{7}{12} x^4 + \cdots. \quad (4.2)$$

Before we balance both sides, consider the leading order of both sides. Without doubt, the leading order is $\epsilon^{2\alpha}$ with coefficient $x_0^2$. For LHS, we have three cases.
1. If $2\alpha > 1$, then the leading order is $\epsilon$ with coefficient $1$. It leads to a contradiction when balancing both sides because $2\alpha = 1$. (×)

2. If $2\alpha = 1$, the balancing equation yields $x_1^2 + 1 = x_1^2$ and it does not make sense. (×)

3. Thus, the only case is $2\alpha < 1$.

Then one can rewrite equation (4.2) as

$$\epsilon = x^3 + \frac{7}{12} x^4 + \cdots = (x_1^3 e^{3\alpha} + 3 x_1^2 x_2 e^{2\alpha + \beta} + \cdots) + \frac{7}{12} (x_1^4 e^{4\alpha} + \cdots).$$

Since the leading order of RHS is $e^{3\alpha}$, it provides that $1 = 3\alpha$ and $1 = x_1^3$. Thus, $\alpha = 1/3$ and $x_1 = 1$. The next leading term is $e^{4\alpha}$. Since there is no remaining term on LHS, then balance RHS as

$$2\alpha + \beta = 4\alpha \quad \text{and} \quad 0 = 3x_1^2 x_2 + \frac{7}{12} x_1^4,$$

yields $\beta = 2\alpha = 2/3$ and $x_2 = -7/36$. Therefore, the three term expansion of root is

$$x \sim 0 + \epsilon^{1/3} - \frac{7}{36} \epsilon^{2/3}. \quad (4.3)$$

**Problem 2.** A classical eigenvalue problem is the transcendental equation

$$\lambda = \tan(\lambda).$$

1. After sketching the two functions in the equation, establish that there is an infinite number of solutions, and for sufficiently large $\lambda$ take the form

$$\lambda = \pi n + \frac{\pi}{2} - x_n$$

with $x_n$ small.

**Proof.** Tangent is $\pi$-periodic function with asymptotic line $\lambda_n = \pi n + \pi/2$. $\tan(\lambda) \to \infty$ as $\lambda \to \lambda_n^+$. Since $f(\lambda) = \lambda$ passes through all the asymptotic line and tangent function is close to the asymptotic line, then for sufficiently large $n$, $\lambda$ takes the form

$$\lambda = \lambda_n - x_n$$

where $x_n$ is a small number and tends to zero as $n \to \infty$. (×)

2. Find an asymptotic expansion of the large solutions of the form

$$\lambda \sim \epsilon^{-\alpha} (\lambda_0 + \epsilon^\beta \lambda_1)$$

and determine $\epsilon, \alpha, \beta, \lambda_0, \lambda_1$.

**Proof.** Set $\lambda = 1/\epsilon$ and see the asymptotic behavior of $x_n$. Then one can figure out an asymptotic expansion of $\lambda$. For convenience, set $x_n = x$. Then one can get

$$\frac{1}{\epsilon} - x = \tan \left( \frac{1}{\epsilon} - x \right) = \cot(x)$$
because \( \tan(1/\epsilon) = 0 \). By multiplying \( \epsilon \tan(x) \) on both sides and we get
\[
\tan(x) - \epsilon x \tan(x) = \epsilon.
\]
Since we know that \( x \to 0 \) as \( \epsilon \to 0 \), take \( x \sim x_0 \epsilon^\theta \), \( \theta > 0 \). It follows that
\[
(x_0 \epsilon^\theta + \cdots) - \epsilon(x_0 \epsilon^\theta + \cdots)(x_0 \epsilon^\theta + \cdots) = \epsilon
\]
Since \( \theta > 0 \), the leading order of LHS is \( \epsilon^\theta \). To balance both sides with \( O(\epsilon^2) \), set \( \theta = 1 \) and get \( x_0 = 1 \). Therefore, an asymptotic expansion of \( \lambda \) is
\[
\lambda = \frac{1}{\epsilon} - x = \frac{1}{\epsilon} - \epsilon^{-1}(1 + \epsilon^2(-1))
\]
It follows that \( \alpha = -1, \beta = 2, \lambda_0 = 1 \) and \( \lambda_1 = -1 \).

**Problem 3.** In the study of porous media one is interested in determining the permeability \( k(s) = F'(c(s)) \), where
\[
\int_0^1 F^{-1}(c - \epsilon r) dr = s \quad \text{and} \quad F^{-1}(c) - F^{-1}(c - \epsilon) = \beta,
\]
and \( \beta \) is a given positive constant. The functions \( F(c) \) and constant \( c \) both depends on \( \epsilon \), whereas \( s \) and \( \beta \) are independent of \( \epsilon \). Find the first term in the expansion of the permeability for small \( \epsilon \). [Hint: consider an asymptotic expansion of \( c \) and use the fact that \( s \) is independent of \( \epsilon \).]

**Proof.** Take \( c \sim c_0 + c_1 \epsilon + \cdots \). Substituting it into given integral equation and expanding \( F^{-1} \) as Taylor series centered at \( c = c_0 \) yields
\[
\int_0^1 F^{-1}(c_0) + \epsilon(c_1 - r) \frac{dF^{-1}}{dc}(c_0) + O(\epsilon^2) dr
= F^{-1}(c_0) + \epsilon \left(c_1 - \frac{1}{2}\right) \frac{dF^{-1}}{dc}(c_0) + O(\epsilon^2) = s.
\]
Since \( s \) is independent of \( \epsilon \), then it gives us that
\[
F^{-1}(c_0) = s \quad \text{and} \quad c_0 - \frac{1}{2} = 0 \implies c_0 = F(s) \quad \text{and} \quad c_1 = \frac{1}{2}.
\]
From the second condition, expanding \( F^{-1} \) as Taylor series follows provides that
\[
F^{-1}(c_0) + \epsilon c_1 \frac{dF^{-1}}{dc}(c_0) = F^{-1}(c_0) - \epsilon(c_1 - 1) \frac{dF^{-1}}{dc}(c_0) + O(\epsilon^2) = \beta.
\]
It follows that
\[
\epsilon \frac{1}{F'(s)} + O(\epsilon^2) = \beta \implies k(s) = F'(s) \sim \frac{\epsilon}{\beta}.
\]

**Problem 4.** Let \( A \) and \( D \) be real \( n \times n \) matrices.

1. Suppose \( A \) is symmetric and has \( n \) distinct eigenvalues. Find a two-term expansion of the eigenvalues of the perturbed matrix \( A + \epsilon D \), where \( D \) is positive definite.
Proof. Take $\lambda \sim \lambda_0 + \lambda_1 \epsilon$ and its corresponding eigenvector $x \sim x_0 + x_1 \epsilon$. It follows that

$$(A + \epsilon D)(x_0 + x_1 \epsilon + \cdots) = (\lambda_0 + \lambda_1 \epsilon + \cdots)(x_0 + x_1 \epsilon + \cdots).$$

Balance for $O(1)$ and we have

$$Ax_0 = \lambda_0 x_0.$$ 

Since $A$ has $n$ distinct eigenvalues, then $\lambda_0$ can be one of them and $x_0$ is corresponding eigenvector. Similarly, balance for $O(\epsilon)$ and it yields

$$Ax_1 + Dx_0 = \lambda_0 x_1 + \lambda_1 x_0 \Rightarrow (A - \lambda_0 I)x_1 = -Dx_0 + \lambda_1 x_0.$$ 

Take inner product with $x_0$ and by symmetry of $A - \lambda_0 I$, one can obtain

$$-\langle x_0, Dx_0 \rangle + \lambda_1 \langle x_0, x_0 \rangle = \langle x_0, (A - \lambda_0 I)x_1 \rangle = \langle (A - \lambda_0 I)x_0, x_1 \rangle = 0.$$ 

Thus $\lambda_1 = \langle x_0, Dx_0 \rangle / \langle x_0, x_0 \rangle$. It is well-defined because $x_0 \neq 0 \Rightarrow \langle x_0, x_0 \rangle \neq 0$. One can find $x_1$ by solving

$$(A - \lambda_0 I)x_1 = -Dx_0 + \lambda_1 x_0.$$ 

$\blacksquare$

2. Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$ 

Use this example to show that the $O(\epsilon)$ perturbation of a matrix need not result in a $O(\epsilon)$ perturbation of the eigenvalues, nor that the perturbation is smooth (at $\epsilon = 0$).

Proof. Its characteristic equation is

$$p(\lambda) = \det(A + \epsilon D - \lambda I) = \lambda^2 - \epsilon$$

and it provides two eigenvalues $\lambda = \pm \sqrt{\epsilon}$. At $\epsilon \to 0$, $\frac{d}{d\epsilon}\lambda \to \infty$ and it means that the perturbation is not smooth at $\epsilon = 0$. 

$\blacksquare$

Problem 5. The eigenvalue problem for the vertical displacement $y(x)$ of an elastic string with variable density is

$$y'' + \lambda^2 \rho(x, \epsilon)y = 0, \quad 0 < x < 1,$$

where $y(0) = y(1) = 0$. For small $\epsilon$, assume $\rho \sim 1 + \epsilon \mu(x)$, where $\mu(x)$ is positive and continuous. Consider the asymptotic expansions

$$y(x) \sim y_0(x) + \epsilon y_1(x) \quad \text{and} \quad \lambda \sim \lambda_0 + \epsilon \lambda_1.$$ 

1. Find $y_0$, $\lambda_0$ and $\lambda_1$. (The latter will involve an integral expression.)

Proof. Substituting all asymptotic expansions into given differential equation leads that

$$(y_0'' + y_0'' \epsilon + \cdots) +$$

$$(\lambda_0 + \lambda_1 \epsilon + \cdots)^2(1 + \mu \epsilon + \cdots)(y_0 + y_1 \epsilon + \cdots) = 0.$$
Balancing $O(1)$ terms yields
\[ y''_0(x) + \lambda^2_0 y_0(x) = 0. \]

The solution of equation is known as
\[ y_0(x) = \sin(n \pi x) \quad \text{with} \quad \lambda_0 = n \pi, \quad \text{for} \quad n = 1, 2, \ldots . \]

Similarly, balance $O(\epsilon)$ terms and get
\[ y''_1(x) + 2 \lambda_0 \lambda_1 y_0(x) + \lambda_0^2 \mu(x) y_0(x) + \lambda_0^2 y_1(x) = 0, \]

that is
\[ y''_1(x) + \lambda_0^2 y_1(x) = -\lambda_0^2 \mu(x) y_0(x) - 2 \lambda_0 \lambda_1 y_0(x). \]

Taking integral with $y_0(x)$ and integration by parts yields that
\[ 0 = -\lambda_0^2 \int_0^1 \mu(s) y_0^2(s) ds - 2 \lambda_0 \lambda_1 \int_0^1 y_0^2(s) ds. \]

Therefore,
\[ \lambda_1 = -\lambda_0 \int_0^1 \mu(s) y_0^2(s) ds = -n \pi \int_0^1 \mu(s) \sin^2(n \pi s) ds. \]

2. Using the equation for $y_1$, explain why the asymptotic expansion can break down when $\lambda_0$ is large.

Proof. From the previous results, one can find the equation for $y_1$ as
\[ y''_1(x) + \lambda_0^2 y_1(x) = \lambda_0^2 \left( -\mu(x) y_0(x) + 2 \int_0^1 \mu(s) y_0^2(s) ds \right). \]

Notice that the RHS proportional to $\lambda_0^2$ and it follows that the particular solution of $y_1$ is proportional to $\lambda_0^2$, then it implies that $y_1 \to \infty$. This can break down the expansion mixed with $\epsilon$. ■

Problem 6. Consider the following eigenvalue problem:
\[ \int_0^a K(x, s) y(s) ds = \lambda y(x), \quad 0 < x < a. \]

This is a Fredholm integral equation, where the Kernel $K(x, d)$ is known and is assumed to be smooth and positive. The eigenfunction $y(x)$ is taken to be positive and normalized so that
\[ \int_0^a y^2 ds = a. \]

Both $y(x)$ and $\lambda$ depends on the parameter $a$, which is assumed to be small.

1. Find the first two terms in the expansion of $\lambda$ and $y(x)$ for small $a$.  

Proof. Since the LHS of Fredholm integral equation is proportional to $a$, because integral contains $a$, the leading order of eigenvalue $\lambda$ is $O(a)$. So, take

$$\lambda \sim \lambda_0 a + \lambda_1 a^2$$

and $y(x) \sim y_0(x) + y_1(x)a$.

Expand $K(x,s)$ and $y(s)$ as Taylor series in terms of $s$ centered at $s = 0$ because $0 < s < a$ is also small. Then we get

$$\int_0^a (K(x,0) + K_d(x,0)s + \cdots)(y(0) + y'(0)s + \cdots)ds = \lambda y(x),$$

and it follows that

$$aK(x,0)y(0) + \frac{a^2}{2}(K(x,0)y'(0) + K_d(x,0)y(0)) + \cdots = \lambda y(x).$$

Balance $O(a)$ terms and one can obtain

$$K(x,0)y_0(0) = \lambda_0 y_0(x). \quad (4.4)$$

Balance $O(a^2)$ terms and one can find

$$K(x,0)y_1(0) + \frac{1}{2}(K(x,0)y'_0(0) + K_d(x,0)y_0(0)) = \lambda_1 y_0(x) + \lambda_0 y_1(x). \quad (4.5)$$

In the same fashion, find one more asymptotic equation from given normalization equation

$$\int_0^a (y(0) + y'(0)s + \cdots)^2ds = a$$

and it follows that

$$a \cdot (y(0))^2 + \frac{a^2}{2} \cdot 2y(0)y'(0) + \cdots = a.$$ 

Balance $O(a)$ terms and one can obtain

$$(y_0(0))^2 = 1. \quad (4.6)$$

Balance $O(a^2)$ terms and one can find

$$2y_0(0)y_1(0) + \frac{1}{2} \cdot 2y_0(0)y'_0(0) = 0. \quad (4.7)$$

From equation (4.4, 4.6), one can find

$$y_0(0) = 1 \quad \text{and} \quad \lambda_0 = K(0,0).$$

This implies that

$$y_0(x) = \frac{K(x,0)}{K(0,0)}.$$

Similarly, one can find

$$\lambda_1 = \frac{1}{2}(K(0,0)y'_0(0) + K_d(0,0)) = \frac{1}{2}(K(x,0)K_d(x,0))$$

and

$$y_1(x) = \frac{1}{\lambda_0} \left[ K(x,0)y_1(0) + \frac{1}{2} \left( \frac{K(x,0)K_d(x,0)}{K(0,0)} + K_d(x,0) \right) - \lambda_1 y_0(x) \right].$$
and it follow that
\[
y_1(x) = \frac{1}{2\lambda_0} \left[ -\frac{K(x, 0)}{K(0, 0)} [K_x(x, 0) - K_x(0, 0)] + K_d(x, 0) - 2\lambda_1 y_0(x) \right].
\]

2. By changing variables, transform the integral equation into
\[
\int_0^1 K(a\xi, ar)\phi(r)dr = \frac{\lambda}{a}\phi(\xi), \quad 0 < \xi < 1.
\]
Write down the normalization condition for \(\phi\).

**Proof.** Substituting \(x = a\xi\) and \(s = ar\) into the Fredholm integral equation yields
\[
\int_0^1 K(a\xi, ar)y(ar)adr = \lambda y(ar),
\]
and set \(\phi(r) = y(ar)\). It follows that
\[
\int_0^1 K(a\xi, ar)\phi(r)dr = \frac{\lambda}{a}\phi(r).
\]

In the same fashion, consider the normalization equation
\[
\int_0^a y^2(s)ds = a \Rightarrow \int_0^1 \phi^2(r)adr = a \Rightarrow \int_0^1 \phi^2(r)dr = 1.
\]

3. From part (b) find the two-term expansion for \(\lambda\) and \(\phi(\xi)\) for small \(a\).

**Proof.** Take \(\lambda \sim a\lambda_0 + a^2\lambda_1\) and \(\phi \sim \phi_0 + a\phi_1\). In the same fashion we did in part (a), expand \(K\) inside of integral centered at zero
\[
\int_0^1 (K(0, 0) + K_x(0, 0)a\xi + K_d(0, 0)ar + \cdots)\phi(r)dr =
\]
\[
K(0, 0)\int_0^1 \phi(r)dr + a\xi K_x(0, 0)\int_0^1 \phi(r)dr +
\]
\[
ak K_d(0, 0)\int_0^1 r\phi(r)dr + \cdots.
\]

Then balance \(O(1)\) terms in both sides of the equations and we get
\[
\begin{cases}
K(0, 0)\int_0^1 \phi_0(r)dr = \lambda_0\phi_0(\xi) \\
\int_0^1 (\phi_0(r))^2dr = 1
\end{cases}
\]

It implies that \(\phi_0\) is constant, and it yields that
\[
\phi_0(\xi) = 1 \quad \text{and} \quad \lambda_0 = K(0, 0).
\]

Similarly, balance \(O(a)\) terms of the equations and one can obtain
\[
\begin{cases}
K(0, 0)\int_0^1 \phi_1(r)dr + \xi K_x(0, 0)\int_0^1 \phi_0(r)dr + K_d(0, 0)\int_0^1 r\phi_0(r)dr = \lambda_0\phi_1(\xi) + \lambda_1\phi_0(\xi) \\
\int_0^1 \phi_0(r)\phi_1(r)dr = 0
\end{cases}
\]
It follows that \( \int_0^1 \phi_1(r)dr = 0 \) and one can have
\[
\xi K_x(0,0) + \frac{1}{2} K_d(0,0) = \lambda_0 \phi_1(\xi) + \lambda_1.
\]
Integrate both side with respect to \( \xi \) on \([0,1]\) and get
\[
\lambda_1 = \frac{1}{2} K_x(0,0) + \frac{1}{2} K_d(0,0).
\]
This eigenvalue yields that
\[
\phi_1(\xi) = \frac{K_x(0,0)}{\lambda_0} \left( \xi - \frac{1}{2} \right).
\]
(4.9)

4. Explain why expansion in part (a) and (c) are the same for \( \lambda \) but not the eigenfunction.

**Proof.** The eigenvalue is coordinate invariant, so it is not affected by change of variables. However, the eigenfunctions are.

**Problem 7.** In quantum mechanics, the perturbation theory for bound states involves the time-independent Schrödinger equation
\[
\psi'' - [V_0(x) + \epsilon V_1(x)] \psi = -E \psi, \quad -\infty < x < \infty,
\]
where \( \psi(-\infty) = \psi(\infty) = 0 \). In this problem, the eigenvalue \( E \) represents energy and \( V_1 \) is a perturbing potential. Assume that the unperturbed \( (\epsilon = 0) \) is nonzero and nondegenerate.

1. Assuming \( \psi(x) \sim \psi_0(x) + \epsilon \psi_1(x) + \epsilon^2 \psi_2(x) \) and \( E \sim E_0 + \epsilon E_1 + \epsilon^2 E_2 \), write down the equation for \( \psi_0(x) \) and \( E_0 \). We will assume in the following that
\[
\int_{-\infty}^{\infty} \psi_0^2 dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} |V_1(x)| dx < \infty.
\]

**Proof.** Substituting expansion of \( \psi \) and \( E \) into the Schrödinger equation, one can balance \( O(1) \) terms and get
\[
\psi_0''(x) - V_0(x) \psi_0(x) = -E \psi_0(x).
\]

2. Substituting \( \psi(x) = \exp(\phi(x)) \) into the Schrödinger equation, derive the equation for \( \phi(x) \).

**Proof.** Take derivative twice to \( \psi \) and it yields that
\[
\psi'(x) = \phi'(x) e^{\phi(x)} \quad \text{and} \quad \psi''(x) = (\phi''(x) + (\phi'(x))^2) e^{\phi(x)}.
\]
Plug them into the Schrödinger equation and drop common term \( e^{\phi(x)} \). Then it follows that
\[
\phi''(x) + (\phi'(x))^2 - (V_0(x) + \epsilon V_1(x)) = -E.
\]
3. By expanding $\phi(x)$ for small $\epsilon$, determine $E_1$ and $E_2$ in terms of $\psi_0$ and $V_1$.

Proof. Assume that $\phi(x) \sim \phi_0(x) + \epsilon \phi_1(x) + \epsilon^2 \phi_2(x)$. Substituting it into the new Schrodinger equation and balance $O(\epsilon)$ terms. Then one can obtain

$$\phi_1'' + 2\phi_0' \phi_1' = V_1 - E_1.$$ 

Define an differential operator $L = d^2/dx^2 + 2\phi_0' \cdot d/dx$. Notice that for sufficiently smooth $f$,

$$\langle \psi_0^2, Lf \rangle = \int_{-\infty}^{\infty} e^{2\phi_0(x)} (f''(x) + 2\phi_0'(x)f'(x))dx.$$ 

Performing integration by parts

$$\langle \psi_0^2, Lf \rangle = \int_{-\infty}^{\infty} 2e^{2\phi_0(x)} f''(x)dx + e^{2\phi_0(x)} f'(x) |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{2\phi_0(x)} f'''(x)dx$$

and it follows that $\langle \psi_0^2, Lf \rangle = 0$. From the observation, take inner product with $\psi_2$ to the first order balance equation and get

$$\langle \psi_0^2, L\phi_1 \rangle = 0 = \langle \psi_0^2, V_1 \rangle - E_1 \langle \psi_0^2, 1 \rangle = \langle \psi_0^2, V_1 \rangle - E_1.$$

Therefore,

$$E_1 = \langle \psi_0^2, V_1 \rangle = \int_{-\infty}^{\infty} V_1(x) |\psi_0(x)|^2dx. \quad (4.11)$$

To find $\phi_1$, solve the first order inhomogeneous ODE of $\phi_1'$ by integrating factor, or observe that

$$\int_{-\infty}^{x} \psi_0^2 L\phi_1 dy = \int_{-\infty}^{x} d (\psi_0^2 \frac{d \phi_1}{dy}) dy = \psi_0^2(x) \phi_1(x)$$

$$= \int_{-\infty}^{x} \psi_0^2(V_1 - E_1)dy$$

and it yields that

$$\phi_1'(x) = \frac{1}{\psi_0^2(x)} \int_{-\infty}^{x} \psi_0^2(V_1 - E_1)dy.$$ 

Similarly, find the second order balance equation

$$\phi_2'' + 2\phi_0' \phi_2' + (\phi_1')^2 = -E_2 \implies L\phi_2 = -(\phi_1')^2 - E_2.$$ 

Therefore, $E_2 = -\langle \psi_0^2, (\phi_1')^2 \rangle$. ■

REFERENCES

