Suppose $f$ is analytic in an open set $U$ containing the annulus $r_1 \leq |z-a| \leq r_2$.

Then \[
\int_{|z-a|=r_1} f(z) \, dz = \int_{|z-a|=r_2} f(z) \, dz
\]

Why?

We can create closed paths $A_1, A_2, \ldots$ such that $A_1 + A_2 + \ldots = C_2 - C_1$, each the support of each $A_i$ lies in a convex open subset of $U$. Hence $\int_{C_2} f(z) \, dz = 0$.

This means that changing the circles does not affect the value of the integral of $f$ over the circle, as long as $f$ is "analytic in between".

This idea will be used later in discussion of independence of integrals on homotopy.
Let $\gamma = C_n - C_r$. This is a cycle.

We can extend the definition of an index to cycles, by the same formula:

$$\text{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z'} dz$$

We have shown that

$$\int_{\gamma} g(z) dz = 0 \quad \text{for every } g \text{ analytic}$$

on an open set containing the closed annulus.

Applying it to $G(z) = \begin{cases} \frac{f(z) - f(\pi)}{\pi - z} & \pi \neq z \\ f'(\pi) & \pi = z \end{cases}$

we get a general Cauchy integral formula.
Laurent's Expansion

Let \( f(z) \) be an analytic function in the annulus
\[
R_1 < |z| < R_2, \quad R_1, R_2 > 0
\]

We allow \( R_2 \) to be equal \( \infty \)

\[
f_1(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
\[
f_2(z) = -\frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]

By the previous "independence" result, value of \( f_1, f_2 \)
does not depend on the choice of \( r \), as long as
(1) and (2) are satisfied.

Hence \( f_1(z), f_2(z) \) both define analytic functions
on \( |z| < R_1 \), \( |z| > R_2 \), respectively.

What is \( f_1(z) + f_2(z) \)?

By Cauchy Integral Formula \( f(z) = f_1(z) + f_2(z) \)
In the case $(\ast)$

\[ \frac{1}{\sqrt{z-2}} = \frac{1}{(\sqrt{3}-\sqrt{2})-(\sqrt{3}-\sqrt{2})} = \frac{1}{(\sqrt{3}-\sqrt{2})(1-\frac{z-a}{\sqrt{3}})} = \sum_{k=0}^{\infty} \frac{(z-a)^{k+1}}{(\sqrt{3}-a)^{k+1}} \]

In the case $(\ast \ast)$

\[ -\frac{1}{\sqrt{z-2}} = \frac{1}{(\sqrt{3}-\sqrt{2})-(\sqrt{3}-\sqrt{2})} = \frac{1}{(\sqrt{3}-\sqrt{2})(1-\frac{z-a}{\sqrt{3}})} = \]

\[ = \sum_{k=0}^{\infty} \frac{(\sqrt{3}-a)^{k}}{(z-a)^{k+1}} = \frac{1}{\sum_{k=0}^{\infty} \frac{(\sqrt{3}-a)^{k}}{(z-a)^{k+1}}} \]

\[ \\
\]

Thus

\[ f_1(z) = \sum_{k=0}^{\infty} \left( \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi-a)^{k+1}} d\xi \right) (z-a)^{k} \]

\[ f_2(z) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi-a)^{k+1}} d\xi \right) (z-a)^{k} \]

\[ f(z) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{2\pi i} \int \frac{f(\xi)}{(\xi-a)^{k+1}} d\xi \right) (z-a)^{k} \]

This is the Laurent's series of \( f(z) \).
Laurent's Series:

\[ f(z) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(\xi)}{(|z-a|^{k+1}} d\xi \right)(z-a)^k \]

Suppose \( f(z) = f_1(z) + f_2(z) \)

\( f_G(z) = g_1(z) + g_2(z) \)

\( 0 = h_1(z) + h_2(z) \)

\( h_2(z) = -h_1(z) \) on \( 0 < |z-a| < R_2 \)

Define \( H_2(z) = \begin{cases} h_1(z) & \text{if } |z-a| > R_1 \\ -h_1(z) & \text{if } |z-a| < R_2 \end{cases} \)

Then \( H_2(z) \) is bounded entire function on \( C \)

\[ \Rightarrow H_2(z) \equiv \text{const} \Rightarrow H_1(z) \equiv 0 \]

\( \Rightarrow h_2 = 0 \Rightarrow h_1 = 0 \]