INSTRUCTIONS

Work all 5 problems. SHOW YOUR WORK.

1. (5 points) Use marks "T" or "F".

{T} (1) In Gregorian Calendar every year divisible by 400 is a leap year.

{T} (2) Alcuin of York was invited by Charlemagne to improve the educational system in his empire.

{T} (3) Both Fermat and Descartes are credited with creation of analytic geometry.

{T} (4) Bombelli was a mathematician bold enough to accept the existence of imaginary numbers.

{T} (5) Gerald of Cremona was a friend of Galileo.
2. (2 points) Find all complex solutions of the equation $z^9 = 1$. You may express your answers in polar form only.

$$Z_k = e^{\frac{2\pi ki}{9}} \quad k = 1, 2, \ldots, 9$$
3. (2 points) Using complex analysis, express $\sin(5x)$ in terms of $\sin x$ and $\cos x$.

Hint: You will need the formula $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

$$e^{5ix} = (e^{ix})^5$$

$$\cos 5x + i\sin 5x = (\cos x + i\sin x)^5$$

$$\cos 5x + i\sin 5x = \cos^5 x + 5i\cos^4 x \sin x + 10i\cos^3 x \sin^3 x + 10\cos^2 x \sin^4 x + 5i\cos x \sin^5 x + i^5 \sin^5 x$$

Collecting terms involving $i$ on both sides:

$$i\sin 5x = 5i\cos^4 x \sin x - 10\cos^3 x i \sin^3 x + i \sin^5 x$$

$$\Rightarrow \quad \sin 5x = 5 \cos^4 x \sin x - 10 \cos^3 x \sin^3 x + \sin^5 x$$
4. (2 points) (a) Fill out the multiplication table of \( \mathbb{Z}_7 \)

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(b) Let \( \phi \) be the Euler function. Find \( \phi(54) \).

\[
\phi(54) = \phi(27 \cdot 2) = \phi(27) \cdot \phi(2)
\]

\[
\phi(27) = \phi(3^3) = 3^3 - 3^2 = 27 - 9 = 18
\]

\[
\phi(2) = 1
\]

\[
\Rightarrow \quad \phi(54) = 18 \cdot 1 = 18
\]
5. (2 points) Express \( x^2 + xy + y^2 = 6 \) in a standard form. Identify the curve. You don’t need to graph it.

\[
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0
\]

There is a cross term \( xy \)

\( \Rightarrow \) there is rotation involved by an angle \( \theta \) such that

\[
cot(2\theta) = \frac{A-C}{B} = \frac{1-1}{1} = 0
\]

\( \Rightarrow 2\theta = \frac{\pi}{2} \), \( \Rightarrow \theta = \frac{\pi}{4} \)

\[
\begin{align*}
X &= a\sin \theta u - b\sin \theta v \\
Y &= b\sin \theta u + a\sin \theta v
\end{align*}
\]

\( \Rightarrow X = \frac{1}{\sqrt{2}} u - \frac{1}{\sqrt{2}} v = \frac{1}{\sqrt{2}} (u-v) \\
Y = \frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{2}} v = \frac{1}{\sqrt{2}} (u+v)
\]

\( \Rightarrow x^2 + xy + y^2 = \frac{1}{2} (u-v)^2 + \frac{1}{2} (u-v)(u+v) + \frac{1}{2} (u+v)^2 = 6 \\
\Rightarrow (u-v)^2 + (u-v)(u+v) + (u+v)^2 = 12 \\
\Rightarrow u^2 - 2uv + v^2 + u^2 - v^2 + u^2 + 2uv + v^2 = 12 \\
\Rightarrow 3u^2 + v^2 = 12 \\
\Rightarrow \frac{u^2}{4} + \frac{v^2}{12} = 1
\]

\[
\frac{u^2}{2^2} + \frac{v^2}{(\sqrt{12})^2} = 1
\]

\( \Rightarrow \) ellipse