SOLVING CUBIC EQUATIONS

\[ x^3 + ax^2 + bx + c = 0 \]

**Step 1**  
Want to eliminate the quadratic term.  
Set \( x = y - \frac{a}{3} \). We get

\[
(y - \frac{a}{3})^3 + a(y - \frac{a}{3})^2 + b(y - \frac{a}{3}) + c = 0 \quad \text{i.e.}
\]

\[
y^3 - 3y^2 \frac{a}{3} + 3y \left( \frac{a}{3} \right)^2 - \left( \frac{a}{3} \right)^3 + ay^2 - 2y \frac{a^2}{3} + \frac{a^3}{9} + by - b \frac{a}{3} + c = 0
\]

Note that the quadratic terms in \( y \) cancel.

**Step 2**  
Thus we can assume that our equation is of the form

\[
y^3 = py + q
\]

Now, set \( y = u + v \). The equation above becomes

\[
(u + v)^3 = p(u + v) + q
\]

\[
u^3 + v^3 + 3u^2v + 3uv^2 = p(u + v) + q
\]

\[
u^3 + v^3 + 3uv(u + v) = q + p(u + v)
\]

If we can solve

\[
\begin{align*}
  u^3 + v^3 &= q \\
  3uv &= p
\end{align*}
\]

we get at least one solution.
We rewrite it as

\[ u^3 + \left(\frac{p}{3u}\right)^3 = 9 \]

\[ u^3 + \frac{p^3}{27u^3} = 9 \cdot u^3 \]

\[ (u^3)^2 + \frac{p^3}{27} = 9u^3 \]

\[ (u^3)^2 - 9u^3 + \frac{p^3}{27} = 0 \]

We get a quadratic equation in \( u^3 \)

which we can solve. Once we get \( u \), we get \( v \), hence \( y \), and therefore \( x \).

Let's see it in practice.
Example

\[ y^3 = 6y + 6 \]  \hspace{1cm} \left( y^3 = py + q \right) 

We can skip step 1

\[ u^3 + \left( \frac{6}{3u} \right)^3 = 6 \]

\[ u^3 + \left( \frac{2}{u} \right)^3 = 6 \]

\[ u^3 + \frac{8}{u^3} = 6 \]

\[ (u^3)^2 + 8 = 6u^3 \]

\[ (u^3)^2 - 6u^3 + 8 = 0 \]

Discriminant \( b^2 - 4ac \) = \( (-6)^2 - 4 \cdot 1 \cdot 8 = 36 - 32 = 4 \)

Hence \( u^3 = \frac{6 \pm \sqrt{2}}{2} = 2 \) or 4

Take \( u = \sqrt[3]{2} \)

Then \( v = \frac{p}{3u} = \frac{2}{\sqrt[3]{2}} = \sqrt[3]{4} \)

\[ y = \sqrt[3]{2} + \sqrt[3]{4} \]

Substitute into * to see if it works
\[ y^3 = 6y + 6 \]

\[ y^3 = \left( \sqrt[3]{2} + \sqrt[3]{4} \right)^3 = 2 + 3 \sqrt[3]{2} \cdot \sqrt[3]{4} + 3 \sqrt[3]{2} \cdot (\sqrt[3]{4})^2 + 4 = \]

\[ 6 + 3 \sqrt[3]{2} \cdot (\sqrt[3]{2})^2 + 3 \sqrt[3]{2} \cdot (\sqrt[3]{4})^4 = 6 + 6 \sqrt[3]{2} + 6 \sqrt[3]{4} = \]

\[ = 6 + 6y \]