CHAPTER 7

The Renaissance of Mathematics: Cardan and Tartaglia

A mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts.

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7.1 Europe in the Fourteenth and Fifteenth Centuries

The Italian Renaissance

If the thirteenth century can be seen as the highest point of medieval Europe, then perhaps the fourteenth century was the lowest. Although the thirteenth century had given abundant promise for the future, many events conspired to make the following century a period almost as dark as what followed the collapse of Rome. The afflictions were those classic riders of the Apocalypse: famine, plague, war, and death. The fourteenth century opened with a series of heavy rainfalls so constant and so widespread that chroniclers of the time compared it with the great flood of Genesis. Not only did the climate become wetter, but it turned significantly colder also, in what has been called the Little Ice Age. The cumulative effect was a disastrous crop failure and an attendant famine in which mortality increased alarmingly in the towns, some losing ten percent of their inhabitants in six months. Those who suffered malnutrition lacked resistance to disease. Upon a people weakened by hunger fell a worse calamity, the Black Death. The Black Death was bubonic plague, carried by brown rats—specifically by a flea parasitic on brown rats—and easily spread in the crowded, dirty conditions of the medieval towns. The outbreak of the plague reached the Mediterranean in 1347, via Italian ships from the Crimea, the port center in the Black Sea. (Because the Crimea was the terminus of the greatest of the caravan routes, it is probable that the seeds of the epidemic were brought from China.) The disease then swept in a great arc through western Europe, striking France in 1348 and afflicting England a year later. Medical knowledge was hopelessly inadequate; nothing could be done to resist the attack. The Black Death raged at its worst for three years, and even when the worst was over it returned with lesser virulence at intervals of 12 to 15 years until the late seventeenth century. The Great Plague of London in 1665 was the last English eruption. In the absence of trustworthy vital statistics, it is impossible to make firm estimates of the terrible mortality. At Paris, it is said, over 800 people died of it each day, and at Avignon 10,000 people were buried in a single mass grave in the first six weeks. The few figures that we have indicate that in some towns half, in general perhaps a third,
of the population was carried away, whereas other regions were completely depopulated. Food shortages were aggravated by sickness in the agricultural districts. At Montpellier in France, so many inhabitants died that the town fathers invited repopulation from as far away as Italy. Peculiarly at peril were those whose occupations called for them to remain in the stricken towns: officials who tried to preserve order, doctors and priests who stayed to aid and console the dying, scholars who continued their studies. These also perished in great numbers; and society, deprived of its natural leaders, was shaken and unstable for decades following.

The smoke of war hung over the whole sad century. The most famous of these wars was that series of English invasions of France extending from 1338 until 1452 and known to us as the Hundred Years’ War. It dragged on for generations before either side won a permanent victory. Even the brief interludes of peace were far from tranquil. Thousands of soldiers refused to lay down their arms and instead formed wandering bands of brigands, the Free Companies of mercenaries, who pillaged the countryside and held for ransom those whom they captured. To this litany of afflictions one must add the first social revolts by the rural peasantry and the urban poor. Savage rebellions occurred in Flanders in 1323–1328, in northern France in 1358 (the famous Jacquerie, which gave its name to all other purely peasant risings), and in England, with the Peasants’ Revolt of 1381.

People of the fourteenth century saw the future as an endless succession of evils; despair and defeat everywhere overwhelmed confidence and hope. The depressed mood of the time is preserved for us in the Danse Macabre, or Dance of Death, an actual dance in pantomime performed with public sermons, in which a figure from every walk of life confronts the corpse he must become.

Yet the ultimate ruin by which Western civilization was threatened never materialized. By approximately 1450, the calamities of war, plague, and famine had tapered off, with the result that population increased, compensating for the losses from 1300 on, and the towns began growing rapidly. Prosperity was once again possible, provided that public order could be restored. The great majority of the people of Western Europe had become convinced that the ills of a strong monarchy were less to be feared than weakness of government, that rebellion was more dangerous to society than was royal tyranny. Thus, after two centuries of chaos, political security returned with the advent of the “new monarchies” of Louis XI in France (1461), Ferdinand and Isabella in Spain (1477), and Henry VII in England (1485). The rise of these strong national states marked the demise of feudalism, and provided the solid foundation on which a new European civilization could be built.

As the long-stagnant economy responded to the stimulus of the dramatic growth in population, western Europe experienced a recovery that seemed to many a remarkable rebirth. Not only did Europeans succeed in restoring order, stability, and prosperity but also embarked on a series of undertakings that vastly expanded their literary and artistic horizons. To later generations this reawakening of the human intellect is known as the Renaissance. The word is the legacy of the great nineteenth-century historian Jacob Burkhardt, who in *The Civilization of the Renaissance in Italy* (1860) popularized the idea of the Italian Renaissance as a distinct epoch in cultural history, differentiated clearly from the preceding period and from the contemporary culture north of the Alps. In recent years, the whole concept of a “renaissance” has come under suspicion by those who claim that the greater period of cultural achievement came in the twelfth century. There is no longer any general agreement about
the character of the Renaissance, its causes, or even its geographical or chronological limits. Ultimately, the Renaissance cannot be disregarded; for medieval civilization—founded as it was on a basis of land tenure and an almost purely agricultural economy—could not continue indefinitely to absorb an expanding urban population and accommodate a money economy founded on trade without changing into something recognizably different. Thus, depending on context, we shall use the term Renaissance in either of its current senses: as a great revival of literature and the arts, with its reverence for classical culture, or as that period of transition (roughly, 1350–1550) in which the decisive change from a largely feudal and ecclesiastical culture to a predominantly secular, lay, urban, and national culture took place.

The reason that a cultural rebirth was experienced and nurtured first in Italy was doubtless that Italy had not been as seriously affected by war and economic dislocation as the northern countries. (It had experienced many small wars but no great conflict.) At the beginning of the fifteenth century, feudalism had disappeared in central and northern Italy, giving place to a vigorous urban society of politically independent city-states. The intellectuals and artists of these prosperous territorial states, hoping to bolster or replace the tottering traditions of medieval culture, thought they had found a model for their secular, individualistic society in the classical past. A cultivation of the Latin and Greek classics flourished with an intensity unknown since the decline of Rome. This “revival of classical culture” was one of the distinguishing characteristics of the Renaissance and one of the chief forces in its changing civilization.

Two events helped to hasten this upsurge of interest in the literary remains of antiquity: the fall of Constantinople to the Turks (1453) and Johann Gutenberg’s invention of printing with movable, metallic type (about 1450). Long before the Arabs had subjugated Egypt, fugitive scholars from Alexandria had reached Constantinople with their books, making the fortress city the chief resting place of what was left of classical literature in the original Greek. On May 29, 1453, the Ottoman Turks seized the great city; even though Constantinople had long been a mere enclave in Turkish territory, its fall stunned Christendom. This final collapse of the Byzantine empire drove a host of Greek scholars to seek refuge on Italian soil, bringing with them a precious store of classical manuscripts. Many of the treasures of Greek learning, hitherto known indirectly through Arabic translations, could now be studied from the original sources.

Artificial Writing: The Invention of Printing

The invention of printing revolutionized the transmission and dissemination of ideas, thereby making the newly acquired knowledge accessible to a large audience. Handwritten books were scarce and dear, and they had necessarily been the monopoly of the wealthy and scholars under their patronage. Those few books that were available to the public had to be chained down, and as further insurance against their loss, many bore maledictions damning anyone who stole, mutilated, or even approached them without washing his hands. When it became possible to issue books not in single copies, but in the hundreds or even thousands, the world of letters and learning was opened up to the moderately well-to-do classes everywhere.
There is no need to labor the importance of printing with movable type. Still, it should be stressed that the first printing presses were made in the early fifteenth century in medieval Germany, not in Renaissance Italy, and that Italian scholars for a long time scorned the new process. Moreover, the stimulus that had led to the invention of printing was the typically medieval desire for quicker and cheaper ways of producing religious texts. There had been printing before Gutenberg, but Gutenberg’s Bible certainly heralded a new day.

The first form of printing in Europe, perhaps the first form of printing on paper, was the block printing (the transference of ink from carved wooden blocks) of playing cards in the latter decades of the fourteenth century. Among the many block books produced, some attained considerable popularity. The best-known was the Poor Man’s Bible, a 40-page book of religious pictures with a minimum of inscriptions, intended for the instruction of the uneducated in the principal lessons of the Bible. What came in the fifteenth century therefore was not the invention of printing but the notion of separate metal type for each letter. Of course, the production of paper made from linen helped popularize the new discovery; there would have been little use in a cheap method of duplication if the only material available had been expensive parchment.

On this last point, a digression may be permitted. By the eighth century, when the advance of Islam produced the final separation of East and West, Egyptian papyrus was no longer available. The monastic scholars therefore wrote on parchment made from the skins of animals, usually sheepskin or goatskin. Parchment was prepared for scribal use by a slow process that involved soaking the skin in a lye solution to dissolve organic materials, stretching it on a frame and rubbing it with a pumice stone for smoothness, and finally pressing the skin and cutting it to size. Parchment had many advantages over papyrus; it resisted dampness, and if a text were no longer required, it could be scraped off and the same writing surface could be used again. Even so, parchment was expensive, and without a cheaper material to print on, the invention of printing would not have been so useful and significant.

The first use of paper from hemp, tree bark, fish nets, and rags is carefully dated in Chinese dynastic records as belonging to the year 105, but this discovery like most was probably a gradual process. The secret of its manufacture was taught by Chinese prisoners to their Arab captors at Samarkand in the eighth century. For the next 500 years, paper-making was an Arab monopoly until passed on by the Moors in Spain to the Christian conquerors. At the opening of the fourteenth century, paper was still a fairly rare material in Europe, imported from Damascus and turned out in small quantities from several newly established mills in Italy. By the end of the century, it was manufactured in Italy, Spain, France, and southern Germany and had largely displaced parchment as the standard writing material of all but the wealthy. Gutenberg’s famous Bible was one of the few early books printed on parchment, and each of his Bibles is said to have required the skins of 300 sheep.

Once invented, the “divine art” of printing from cast movable type spread like wildfire through central and Western Europe, so that by the end of the century the names of 1500 printers were known. To ascertain accurately the number of books that all these presses produced before the year 1500 is impossible. According to the titles collected in various catalogs of incunabula, about 30,000 printed works appeared. Assuming that the editions
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were small, averaging about 300 copies, there would have been nearly 9 million books (including pamphlets) in Europe by 1500, as against the few score thousand manuscripts that lately had held all the irrecoverable lore of the past.

The first printed books were little concerned with mathematics. Many mathematical works written in the mid-1400s, such as Regiomontanus's treatise *De Triangulis*, did not appear in print until very much later. The principal standbys of the earlier printers were the Bible (which appears in many editions, both in Latin and in the popular languages), books of meditation, and religious tracts of various sorts. Those mathematical works that did come off the presses were unoriginal, falling far below the level of the great thirteenth- and fourteenth-century mathematicians. The first popular textbook, the *Treviso Arithmetic*, was published in 1478 at Treviso, an important mercantile town not far to the north of Venice. Essentially a list of rules for performing common calculations, it was written, claims the anonymous author, at the request of young people preparing to enter commercial careers. The *Treviso Arithmetic* was significant not so much for its content as for initiating a remarkable movement. Before the close of the fifteenth century, over 200 mathematical books had been printed in Italy alone. Euclid's *Elements*, with the Latin commentary by Campanus of Novara, was published in 1482 at Venice and again in 1491 at Vicenza. Campanus lacked linguistic competence in Arabic, so this version contained numerous errors and barbarous terminology. In 1505, Zamberti brought out a new translation, working from a recovered Greek manuscript.

One of the earliest European scholars to take advantage of the recovery of the original Greek texts was the mathematician-astronomer Johannes Müller (1436–1476), better known as Regiomontanus, from the Latin name of his native town of Königsberg. The most distinguished scientific man of his time, Regiomontanus was active in translating and publishing the classical manuscripts available, including Ptolemy's treatise on astronomy, the *Almagest*. The fruits of this study were shown in his greatest publication, *De Triangulis Omnimodis* (*On Triangles of All Kinds*). The work was finished about 1464 but remained unprinted until 1533. Trigonometry was one of the few branches of mathematics to receive substantial development at the hands of the Greeks and the Arabs. In the *De Triangulis*, Regiomontanus systematically summed up the work of these pioneers and went on to solve all sorts of problems relating to plane and spherical triangles. The only trigonometric functions introduced were the sine and the cosine, but at a later date Regiomontanus computed a table of tangents. For all practical purposes, *De Triangulis* established trigonometry as a separate branch of mathematics, independent of astronomy.

Calender revision was a growing concern at this time, particularly in regard to the calculation of the date of Easter. The Council of Nicaea (325 A.D.) stipulated that Easter must be celebrated on the first Sunday following the first full moon after the vernal equinox, and it fixed the date of the vernal equinox, the first day of spring, at March 21 for all future years. The Roman or Julian calendar, introduced by Julius Caesar, was based on a year of 365\(\frac{1}{4}\) days with a leap year every fourth year. This was not a precise enough measure, because the length of a solar year—the time it takes for the earth to complete an orbit around the sun—is apparently 365.2422 days. This small error meant that Easter receded a day from its solar norm every 128 years.

Regiomontanus had set up an observatory and a private printing press in the city of Nuremberg. He published two calendars in 1472, one in Latin and the other in German.
Although each calendar had appended to it the ecclesiastical dates of Easter for the years 1475–1531, the Latin version also contained a differing set of dates calculated from Regiomontanus’s astronomical observations. His calendars enjoyed great popularity—calendars are among the oldest examples of printing with movable type—as evidenced by their sales and numerous reprints. In 1475, Regiomontanus was invited to Rome by Pope Sextus IV to give advice on coordinating the calendar with astronomical events. He died shortly thereafter, suddenly and somewhat mysteriously. Some said he was poisoned by his enemies, but more likely he became a victim of a plague that was raging after the Tiber had overflowed its banks.

Calendar reform was forgotten after the death of Regiomontanus, and was not again viewed as imperative until the reign of Pope Gregory XIII. He brought together a large number of mathematicians, astronomers, and prelates in 1552 finally to remedy the defects in the Church’s reckoning of the dates of Easter. The Jesuit mathematician Christoph Clavius was put in charge of carrying out the necessary calculations. For this, he relied upon Erasmus Reinhold’s *Tabulae Prutenicae* (1551), named for Reinhold’s patron, the Duke of Prussia. These were far superior to any other astronomical tables available, having freely used observations that Copernicus had provided in his *De Revolutionibus*.

The new calendar that was imposed on the predominantly Catholic countries in Europe, known as the Gregorian calendar, decreed that ten days were to be omitted from the year 1582. This was accomplished by having October 15 immediately follow October 4 in that year. At the same time, Clavius amended the scheme for leap years: these would be the years divisible by four, except for those marking centuries; century years would be leap years only if they were divisible by four hundred. Because the edict came from Rome, Anglican England and her possessions resisted the changes. When England finally adopted the Gregorian calendar in 1752, the countryside erupted in riots as people demanded the return of their “lost days.”

It is difficult if not impossible to assess the influence of this new trigonometric learning on the great voyages of discovery in the late 1400s. At one time, historians thought that the Portuguese navigators in venturing south of the equator along the coast of Africa had used the tables of solar declination in Regiomontanus’s almanac, the *Ephemerides Astronomicae*; but it appears that the first editions (1474) of this work contain no such tables. What is known is that Columbus carried a copy of the *Ephemerides* with him on his four trips to the New World. On one occasion, having read that Regiomontanus predicted a total eclipse of the moon for February 29, 1504, Columbus took advantage of this knowledge to frighten the natives into reprovisioning his ships.

The period of Regiomontanus was also the time of Luca Pacioli (1445–1514), a Franciscan friar who was commonly called Fra Luca di Borga. Many scholars of this time felt the compulsive urge to bring together, within the pages of a large book, all known information in some given field. There was a systematic compendium, or “summa,” for every interest and taste. Pacioli’s *Summa de Arithmetica Geometria Proportioni et Proportionalita*, published in Venice in 1494, was the most influential mathematical book of that period. The first comprehensive work to appear after the *Liber Abaci* of Fibonacci, it contained almost nothing that could not be found in Fibonacci’s treatise, which indicate how little European mathematics had progressed in nearly 300 years. But as an encyclopedic account of the main mathematical facts inherited from the Middle Ages, the *Summa* goes far beyond what was taught in the universities. Written carelessly in Italian, it is notable historically for its wide circulation (perhaps due to the author’s explanation of the mechanics of double-entry bookkeeping).
The area of a triangle: from Regiomontanus's De Triangulis Omnimodis (1533 edition). (From A Short History of Mathematics by Vera Sanford. Reproduced by permission of the publisher, Houghton Mifflin Company.)

Founding of the Great Universities

The universities that were being established were to become prominent in the cultivation and spread of learning. The Latin universitas was originally a mere synonym for communitas, a general word indicating a collection of individuals loosely associated for communicating ideas. Initially, the only educational centers were monasteries. Their primary function was religious service, not intellectual, and they were disinclined to teach outsiders. They preserved, rather than added to, literature. As the number of laymen seeking education grew, schools attached to the churches of bishops became prominent as centers of learning. Cathedral schools were provided mainly for those who would enter the ranks of the “secular clergy” and carry on the work of the Church in the world, not apart from it. Such schools flourished as a sideline to the work of the bishop and were prone to be affected by the reputation of the local teachers, waxing and waning with the comings and goings of particular personalities.

Cathedral schools were of course hardly conducive to the free flow of ideas. Thus, long before the formal beginnings of universities, assemblies of students gathered around an individual master or two who had no connection with the Church—who, however, still needed the permission of the bishop to teach. An excellent teacher became a celebrated
figure, and students traveled from town to town in pursuit of some famous scholar whose reputation had reached their homelands. The force of personality of Peter Abelard (1079–1142) is said to have attracted students from every corner of Europe to his crowded lecture hall in Paris. Twenty of his pupils subsequently became cardinals, and more than 50 became bishops. There is an oft-told tale that when the theological writings of Abelard were condemned by the Church, the king of France suddenly forbade Abelard to teach in his lands. On hearing the news, Abelard climbed a tree and his students flocked to hear him from below. When the king then prohibited him from teaching in the air, Abelard began lecturing in a boat; at this point the king relented. Abelard is especially important because his brilliance as a teacher popularized the cathedral school at Notre Dame as a center of higher learning, thus opening the way for the foundation of the University of Paris.

The growth of the universities in the twelfth and thirteenth centuries was a natural consequence of a demand that the older cathedral and monastery schools were unable to satisfy. A developing body of secular knowledge that had a marked professional value (medicine, and especially law) and that required for its mastery protracted study under an eminent specialist began to make the university an indispensable institution. Students living in the centers made famous by the cathedral schools began to find it necessary to organize in order to regulate their own conduct, to protect themselves from extortion by local citizens, and—because many were not native to the area—to secure legal rights. Thus the students in voluntary association tended, like the merchants and craftsmen of those days, to form self-governing guilds and eventually to gain legal recognition through the charter of a king or a pope. The universities at Bologna (1158), Paris (1200), Padua (1222), Oxford (1214), and Cambridge (1231) can all trace their inception to this period. These embryonic universities bore little physical resemblance to what they later became, and there were great variations among the institutions of different towns. Not until the fifteenth century did the universities acquire permanent buildings. Before this, teachers lectured in their own quarters or in rented halls, and general meetings took place in churches or monastic halls. Competition for eminent scholars gradually led to contracted salaries, so that as early as 1180 Bologna paid several professors from municipal funds; the selection of professors, however, remained a student prerogative. Students individually paid the master who taught liberal arts, because teaching skill was equivalent to the skill of any other tradesman. Teachers of theology, on the other hand, were forbidden to stipulate charges in advance—theology being a “spiritual gift”—but were allowed to accept donations after a lecture was concluded.

Paris and Bologna were the great “mother” universities, serving as models for the later universities that sprang up in every part of Europe during the next two centuries. The universities of Italy and southern France followed the academic pattern of Bologna, whereas those in northern Europe looked to Paris as the standard. Both schools developed much the same methods of teaching and came to grant the same degrees, but they emphasized different studies and were organized differently. The rise of secular administrative governments in Italy made legal studies the door to high civil office and profitable employment. Thus, at Bologna jurisprudence always dominated and little attention was given to theology and philosophy. For these subjects, the student went by preference to Paris, where canon law was secondary and civil law was not taught at all. Education in the North was everywhere still in the hands of the Church, so it was a matter of course that ecclesiastical studies should predominate at Paris, and that the church authorities should claim a large share in university
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government. In Bologna, the university was a union of student guilds, which gained control over all academic affairs, save only the bestowing of degrees—which were licenses to teach. In Paris the system of organization was the reverse, with the governance of the university in the hands of the masters. One reason for this difference will be found in the differing ages of the students. At Bologna, with its interest in the “lucrative science” of law, many students were mature men who had already attained high civil position. The students at the faculty of arts at Paris, much the largest faculty there, were too young (possibly 12 or 14 years old) and too poor to assert themselves in any similar way.
Universities enjoyed an enormous prestige as custodians of learning in an age in which education was esteemed as at almost no other period in history. A city’s trade, population, and notability depended on the presence of a university, so that cities without schools were willing to underwrite universities that might secede from their established seats. The medieval universities had no permanent buildings and little corporate property, so it was simple for students to migrate to another city when for any reason they were dissatisfied. The masters, because they were entirely dependent for their livelihood on meager tuition fees, had no choice but to follow. (Cambridge, for example, was raised to university status by a migration from Oxford in 1209.)

Students obtained numerous privileges—including the right of trying their own members in practically all civil and criminal cases—through the potent threat of withdrawing to rival cities. A threatened secession from Bologna in 1321 was withdrawn on the terms that an offending magistrate should be publicly flogged and that the city should erect a chapel for the university. The students, having humbled the municipal authorities by a ruthless boycotting of recalcitrant teachers, went on to enforce a series of statutes governing all phases of instruction. Each master had to give a certain number of lectures covering a prescribed minimum of work, might be fined for tardiness or for evading difficult material on which he was supposed to expound, could not leave town without the permission of the student rector, and even on the occasion of his wedding was only allowed one day off. In the long run, the strength of the student body proved its undoing. As the local authorities began to pay teachers’ salaries in order to propitiate the students and keep up the reputation of the local university, the state gradually gained the responsibility for appointments and supervision of faculty.

Because the academic base consisted of the seven arts of the traditional trivium (grammar, logic, and rhetoric) and quadrivium (arithmetic, music, geometry, and astronomy), superficially mathematics seemed to be important. Little attention was paid to the quadrivium, however, ostensibly because these studies had practical applications. Paris, Oxford, and Cambridge systematically discouraged all technical instruction, holding that a university education should be general and not technical. The real reason seems to have been that distinction could be more easily attained in theology and philosophy than in the sciences. By 1336, in an effort to stimulate interest in mathematics, a statute was passed at the University of Paris that no student could graduate without attending lectures on “some mathematical books.” It also appears that after 1452, candidates for the degree master of arts at Paris had to take an oath that they had read the first six books of Euclid. Although the Renaissance was to prove to have been as much a landmark in mathematics as in other branches of learning, the university curriculum continued to provide for a literary rather than a scientific education.

A Thirst for Classical Learning

As the revival of commerce and the growth of town life in the fourteenth century gradually altered medieval culture, many efforts were made to shore it up or to replace it with something new. When neither the feudal nor the ecclesiastical tradition of the earlier period proved adequate, intellectuals of the Italian city-states looked to a more remote past to find a congenial civilization. Most of the Latin authors—and as they later discovered, Greek
authors—had written for an urban, secular, and individualistic society not unlike their own. Italian scholars devoted themselves with a passionate zeal to the study of classical writings, interpreting them in the light of the present age. Behind this “cult of the classics” lay the belief that antiquity, both Latin and Greek, offered a model of perfection by which to judge all civilizations; and in its literature could be found new solutions to all political, social, and ethical problems. There began a systematic and astonishingly successful search for mislaid or forgotten manuscripts, many of which still existed in only a few scattered copies. From one end of Europe to another, scholars rummaged through old libraries in towns and monasteries. The collecting, copying (at first by hand and later, when the printing trade had developed, by press), and diffusion of the treasures they had unearthed was just the beginning.

Manuscripts had to be edited to purify them from the many errors medieval copyists had made and to secure the correct form for each passage. Bibliophiles compiled grammars and lexicons and composed guides to ancient works, and commentaries on them. A tradition of critical judgment in dealing with authoritative texts emerged—a development quite impossible when the Church had had a monopoly on learning; this would be of great value once the interests of the educated turned toward scientific research.

The Renaissance thirst for antique culture inspired a growing fashion of collecting libraries. This enthusiasm pervaded all branches of society, as the princes of the church, state, and commerce vied with one another in assembling books. Renaissance men venerated manuscripts just as their grandfathers had adored the relics of the Holy Land. (A sort of snobbery existed among some wealthy owners, who boasted that their collections contained no printed works.) The development of the Vatican Library in Rome during this period was largely the work of Pope Nicholas V (1395–1455), who had been, before his elevation to the papal throne, librarian to Cosimo de Medici. It cannot be said that the Vatican Library had substantial reality at this time, containing as it did a mere 350 volumes in various states of repair. Nicholas dispatched agents all over Europe to collect manuscripts, with the authority to excommunicate those who refused to give them up. At the same time, some of the most distinguished scholars in Rome were set to making translations of the Greek works into Latin. By the time of his death, Nicholas had built the library to over 5000 volumes and made it one of the finest in Italy. While the primary purpose was to collect and preserve works on the history and doctrines of the Church, an increasing number of secular works found their way into the collection. Vespasiano da Bisticci, a writer of the time, said with some exaggeration, “Never since the time of Ptolemy had half so large a number of books of every kind been brought together.”

For a time it seemed that the people of the Renaissance had far less intention of creating something new than of reviving something old, less an idea of moving forward to the future than of returning to the past. Like every fad, this exaltation of ancient life was carried to absurd extremes by some of its devotees. Literary clubs, called “academies” in the ancient Greek fashion, were formed, at which discourses on classical subjects were read and followed by discussion and debate. Greek was the language of the meetings and Greek names were adopted by the members. In imitation of the ancient custom, successful poets were crowned with wreaths of laurel. Classical ways of feeling, thinking, and writing cast such a deep spell over some scholars that they slipped into the habit of pretending to be Greeks or Romans, even going through the motions of reviving pagan religious rituals. Despite these excesses, the Renaissance Italians of the fifteenth century performed an invaluable service to future generations by restoring the whole surviving heritage of
Greek literature, editing all of it, and finally bringing out printed editions of the entirety. The accomplishment becomes even more impressive when we recall that the knowledge of ancient Greek script had almost disappeared in the West during the Middle Ages. As the range of Hellenistic prose and verse was brought back into the mainstream of Western scholarship, there developed an ideal of education for general human cultivation. This new attitude toward learning differed so markedly from the strictly utilitarian or professional objective of study that had dominated the centuries just preceding that it engendered an entirely new experience.

All this activity on behalf of the classics directly influenced the universities, gradually transforming the prevailing curriculum to the humanities. The term “humanities” is simply a translation of the ancient Latin phrase *studia humanitatis* and was used in the Renaissance to mean a clearly defined set of scholarly disciplines (grammar, rhetoric, poetry, history, and moral philosophy) based on the study of the classics of Greece and Rome. The humanities of the Renaissance were not the seven liberal arts of the Middle Ages under another name; for the humanities omitted not only the mathematical disciplines of the quadrivium, but also logic, adding three subjects that are best implied in the trivium, namely poetry, history, and moral philosophy. Thus, the Renaissance thinkers created a version of the classical curriculum that in all its variations was to become one of the great staples of the university, until pushed aside in the nineteenth and twentieth centuries by science, modern languages, and social science.

Another sign of the breakup of the traditional disciplines was the conscious and deliberate creation of a new educational model, the gentleman. Gentlemanly training demanded that one be schooled in the classical writings, graceful in deportment, proper in style of dress, and of discriminating taste in music, painting, and the literary arts. In the universities an atmosphere of largely verbal scholarship arose, resting primarily on grammar—which meant reading, writing, and rigorous analysis of language and style of literary works—and on rhetoric, the art of persuasion and eloquence in speaking. Elegant Latin was regarded as essential for public documents, and Ciceronian phrases were henceforth reckoned among the tools of diplomacy. Close study and imitation of the ancients were held necessary to achieve this style.

What distinguished the Greek revival of the Renaissance from its medieval forerunners was not simply that Greek became part of the general curriculum of studies, but that the whole focus of interest was on the literary and historical masterpieces of Greek literature. By emphasizing the scholarly worth of the humanities as a molder of the gentlemanly character, the Renaissance educators subordinated, and sometimes even impugned, learning from experience and direct observation. The effect was to impede the study of the physical sciences and mathematics, which were beyond the scope of literary treatment, and if anything, offended the aesthetic senses of these men of letters. Although the Renaissance movement as a whole made relatively little progress in science, it nevertheless indirectly opened the way to the Scientific Revolution of the 1600s by recovering more of the ancient learning than the medieval scholars had possessed. Although Euclid and Ptolemy, and even much of Archimedes, were known in the Middle Ages, such advanced authors as Diophantus and Pappus were first translated during the Renaissance. By the 1600s, almost all the extant corpus of Greek mathematics was easily available to those interested in the subject. The result, apparent from the middle 1500s on, was a rapid and noticeable rise in the level of sophistication of European mathematics.
7.2 The Battle of the Scholars

Restoring the Algebraic Tradition: Robert Recorde

The Renaissance produced little brilliant mathematics commensurate with the achievements in literature, painting, and architecture. The generally low level of prevailing mathematical knowledge stood in the way of any intellectual breakthrough. Although mathematics was included in the curriculum of most universities, it was maintained only in a halfhearted manner. Indeed, during the late 1400s, Bologna was practically the only place where the teaching of the subject was properly organized, and even there it appeared chiefly as a sideline to astronomy. There were few university chairs in mathematics, and no mathematician could command respect from the learned world without also being a teacher, scholar, or patron of the Renaissance humanities.

Regiomontanus set the pattern for combining mathematics with humanistic learning. At the University of Vienna, he lectured enthusiastically on the classical Latin poets Virgil, Juvenal, and Horace, drawing a larger audience than if his subject had been astronomy or mathematics. On a visit to Rome he copied the tragedies of Seneca while learning Greek in order to undertake a more comprehensive translation of the *Almagest* and subsequently the *Conic Sections* of Apollonius. To make the Greek tradition generally available, Regiomontanus became an ardent advocate of the new craft of printing, even installing a printing press in his house for publishing his and other people’s manuscripts. Starting with Regiomontanus, mathematicians displayed an astute appreciation of the power of printing. A recurrent feature of the mathematical revival was an ambitious printing program designed to achieve a rapid dissemination of texts and translations.

Mathematics benefited immensely from the humanist passion—almost missionary zeal—for the discovery, translation, and circulation of ancient Greek texts. Though their main interest was in the literary classics, the humanists took all classical learning as their province, and mathematical works were cherished equally with literary ones in their retrieval. These manuscript collectors were responsible for assembling in Italy an almost complete corpus of Greek mathematical writings. The medieval scholar had generally been limited to Euclid, Ptolemy, and sometimes Archimedes, all in translation from the Arabic. By the fifteenth century, typical holdings encompassed not only the works of the aforementioned authors in both Latin and Greek, but also Diophantus, Apollonius, Pappus, and Proclus. The mathematician, like many of his Renaissance contemporaries, often tried little more than to comprehend what the ancients had done, certain that this was the most that could be known. Although much of this effort was wasted, the return to original sources made a first step toward an intellectual advance. It would then be only a matter of time before mathematicians were stimulated to go beyond the strict letter of the texts to develop new concepts and results strictly unforeseen by the Greeks. Besides, it was far better to read an author, say Euclid, directly than to read what some commentator thought an Arabic paraphrase of the author meant.

By 1500 the situation had changed radically. The newly translated works had been absorbed, and scholars, discontented with looking backward to antiquity, were prepared to go beyond the mathematical knowledge possessed by the Greeks. It came as an enormous and exhilarating surprise when the Italian algebraists of the early 1500s showed how to solve the cubic equation, something the ancient Greeks and the Arabs had missed. (The advance in algebra, however, that proved to be the most significant was the introduction of
better symbolism.) In arithmetic, developing commercial and banking interests stimulated improved methods of computation, such as the use of decimal fractions and logarithms. Trigonometry, in connection with its increasing use in navigation, surveying, and military engineering, began to break away from astronomy and acquire a status as a separate branch of mathematics. Refined astronomical instruments necessitated the computation of more extended tables of trigonometric functions. In a monument to German diligence and perseverance, Georg Joachim (generally called Rhaeticus, 1514–1576) worked out a table of sines for every 10 seconds to 15 decimal places. Only in geometry was the progress less pronounced. Renaissance geometers tended to accept the elementary properties found in Euclid’s Elements as an exclusive model for their conduct and to ignore developments that could not claim Greek paternity.

Mathematicians were eager to make known the newly discovered ways in which they could aid the ordinary person, from teaching the merchant how to reckon profits to showing the mapmaker the principles underlying the projection of a spherical surface onto a plane. Even the sixteenth century found the rules of simple arithmetic and geometry difficult to comprehend, and long division was truly long in the time required to accomplish it. Thus, the middle 1500s saw an increasing number of books of elementary instruction, written in plain and simple language. These practical textbooks, although producing nothing new, were important in diffusing mathematics to an ever-increasing public. The great majority were in Latin, but a good many appeared in the vernacular. Fair examples are the works of the English mathematician Robert Recorde (1510–1558): The Grounde of Artes (1542, a popular arithmetic text that ran through 29 editions), The Patheway of Knowledge (1551, a geometry containing an abridgment of the Elements), and The Whetstone of Witte (1557, on algebra). Books on algebra became so numerous in Germany that the subject was long known in Europe as the “cossic art,” after the German word coss for “unknown” (literally, “thing”). Through the trend of producing textbooks in the popular languages, mathematics assumed increasing importance in the education of all cultured people and not just of specialists training for an occupation.

Although hailed as the founder of the English school of mathematics writers, Robert Recorde was neither the author of the first mathematics text printed in England nor the first whose works appeared in the English language. The churchman Cuthbert Tonstall published the Latin De Arte Supputandi (1552), based largely on Italian sources, in the same year that he became Bishop of London; and the anonymous vernacular text An Introduction for to Lerne to Recken with the Pen and with the Counters came out in 1537. Nevertheless, Recorde’s series of works enjoyed the widest popularity, going through innumerable printings in his own and the next century.

Recorde was educated at Oxford and then received the degree doctor of medicine in 1545 from Cambridge. He gave mathematics lessons privately in both university towns, before setting up a medical practice in London. It is said that he was physician to King Edward VI and to Queen Mary. Sometime around 1551, he was appointed to the position of Surveyor of [silver] Mines and Monies in Ireland. Recorde’s good fortune must have been temporary, because he died in prison a few years later. The reason for his incarceration is not known, but it is most likely connected to his conduct in political office.

Recorde’s Castle of Knowledge (1556), a textbook on astronomy written as a dialogue between scholar and master, is equally noteworthy for containing the first discussion in England of the Copernican hypothesis of the earth’s motion. His position was guarded
and noncommittal, perhaps in fear of ridicule or, worse, religious persecution. When the young scholar describes Copernicus’s ideas as “vaine phantasies,” the master counters, “You are too younge . . . you were best to condemne no thynge that you do not well under-
stand.”

Algebraists, who had floundered under the weight of a cumbersome syncopated no-
tation, began to introduce a symbolism that would make algebraic writing more efficient and compact, one that was also better suited to the needs of typography. These improve-
ments came intermittently, and there was a lack of uniformity in symbols, even for common arithmetic operations (the present division sign ÷ was often used to indicate subtraction). Also, different symbols were proposed in different countries, tried, and often discarded. The Italian algebraists were slow in taking up new notation, preferring the initial letters p
and m for “plus” and “minus” at a time when the Germans, less fettered by tradition, were adopting the familiar mathematical signs + and −. Although the development of symbols for opera-
tions in algebra was proceeding rapidly, the quantities described in equations were still represented by actual instead of general numbers. As a result, there could not be a complete treatment of, say, the quadratic equation. Instead, methods of solution were described and direct solutions were offered for many special cases, each illustrated by equations having appropriately chosen particular numerical coefficients.

The liberation of algebra from the necessity of dealing only with concrete examples was largely the work of the great French mathematician Francois Vi` et (1540–1603), who initiated using consonants to represent known quantities and vowels for the unknowns. This one step marked a decisive change, not only in convenience of notation but also in the abstraction of mathematical thought. In moving from varied but specific examples such as $3x^2 + 5x + 10 = 0$ to the general $ax^2 + bx + c = 0$, an entire class of equations could be considered at once, so that a solution to the abstract equation would solve all the specific equations at one fell swoop.

The Italian Algebraists: Pacioli, del Ferro, and Tartaglia

Italian mathematics of the 1500s can be summarized in the names of del Ferro, Tartaglia, Cardan, Ferrari, and Bombelli. The collective achievement of the first four was the solution of the cubic and biquadratic equations and implicitly a deeper understanding of equations in general. This feat was perhaps the greatest contribution of algebra since the work of the Babylonians some 3000 years earlier. Third-degree, or cubic, equations were in no sense peculiar to the Renaissance, attempts at their solution going back to classical antiquity. We have seen that the problem of duplicating the cube, the so-called Delian problem, attained special celebrity among the Greeks. This problem is nothing more than the attempt to find two mean proportionals between a (the length of the edge of the given cube) and 2a; that is, to solve

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a},$$

which requires substantially the solution to the cubic equation $x^3 = 2a^3$. Another noteworthy cubic equation is encountered in Diophantus’s Arithmetica in connection with Problem 17 of Book VI:
Find a right triangle such that the area added to the hypotenuse gives a square, while the perimeter is a cube.

The manner in which Diophantus set up the problem leads to the cubic $x^3 + x = 4x^2 + 4$. We do not know how the solution was obtained, for he said simply that $x$ was found to be 4. Perhaps he reduced the equation to the form $x(x^2 + 1) = 4(x^2 + 1)$ and saw that it was satisfied by $x = 4$. Arab writers contributed solutions to special cubics but seem to have believed that many cases could not be solved. Part of the poet Omar Khayyam's (circa 1100) fame as a mathematician rests on his claim of being the first to handle any type of
Fra Luca Pacioli
(circa 1445–1514)

(The Bettmann Archive.)

cubic having a positive root. In the thirteenth century, John of Palermo proposed solving the equation \(x^3 + 2x^2 + 10x = 20\) as one of his challenge problems to Fibonacci in their contests. Fibonacci showed by geometry that no rational solution was possible, but he gave an approximate value for a root. Over the next several hundred years, mathematicians searched for a “cubic formula” that could be used to solve cubic equations in much the same way the quadratic formula was used for quadratic equations. The credit for finally discovering such a formula belongs to the Italian mathematical school at Bologna during the 1500s.

The most complete and detailed fifteenth century mathematical treatise was the Summa de Arithmetica, Geometria, Proportioni, et Proportionalita (1494) of Fra Luca Pacioli, a work in which the author borrowed shamelessly from earlier writers. The main contribution of the Summa (which was, after all, a summary) was to lay out the boundaries of contemporary mathematical knowledge and so to supply a program of sorts for the renaissance of mathematics. Pacioli ended his Summa by asserting that the solution of the cubic equation was as impossible as the quadrature of the circle. This put off some mathematicians from the attempt but only induced others to try. In the first or second decade of the sixteenth century, Scipione del Ferro (1465–1526) of the University of Bologna shattered Pacioli’s prediction by solving the cubic equation for the special case \(x^3 + px = q\), where \(p\) and \(q\) are positive.

Pacioli may have personally stimulated this first great achievement of Renaissance algebra, for in 1501–1502 he lectured at the University of Bologna, where one of his colleagues was del Ferro. (Pope Nicholas V had, in 1450, proclaimed a general reorganization of the university and allocated four chairs to the mathematical sciences. By 1500 there were as many as eight professors at a time teaching mathematics there.)

It was the practice in those days to treat mathematical discoveries as personal properties, disclosing neither method nor proof, to prevent their application by others to similar problems. This was because scholarly reputation was largely based on public contests. Not only could an immediate monetary prize be gained by proposing problems beyond the reach of one’s rival, but the outcomes of these challenges strongly influenced academic appointments; at that time, university positions were temporary and subject to renewal based on demonstrated achievement. (As the printing of scientific periodicals became commonplace, this attitude of secrecy gradually shifted to the view that publication of results was the scholar’s best path to recognition.) At any rate, loath to surrender an advantage over other competitors, del Ferro never published his solution and divulged the secret only to a few
close friends, among them his pupil and successor Antonio Maria Fiore. This exchange was to lead to one of the most famous of mathematical disputes, its origin being a problem-solving contest at Venice in 1535 in which Fiore challenged Nicolo Tartaglia to solve various kinds of cubics.

One of the most important restorers of the algebraic tradition, Nicolo Tartaglia (1500–1557), was also one of the least influential. Tartaglia (whose actual family name was Fontana) was born in Brescia, in northern Italy. When the French sacked Brescia in 1512, many of the inhabitants sought refuge in the local cathedral. The soldiers however violated the cathedral’s sanctuary and massacred the townspeople. The boy Nicolo’s father was among those killed in the butchery, and he himself was left for dead after receiving a severe sabre cut that cleft his jaw and palate. Although his mother found the lad and treated the wounds as best she knew, he was left with an impediment in his speech that earned him the cruel nickname Tartaglia, “the stammerer.” Later in life he used the nickname formally in his published works; he wore a long beard to cover the monstrous scars, but he could never overcome the stuttering.

Although his early years were spent in direst poverty, Tartaglia was determined to educate himself. His widowed mother had accumulated a small sum of money so that he might be tutored by a writing-master. The funds ran out after 15 days, but the boy stole a copybook from which he subsequently learned to read and write. It is said that lacking the means to buy paper, Tartaglia made use of the tombstones in the cemetery as slates on which to work out his exercises. Possessing a mind of extraordinary power, he eventually acquired such proficiency in mathematics that he earned his livelihood by teaching the subject in Verona and Venice. It is ironic that Tartaglia, a man disfigured by a sabre, contributed to the ultimate obsolescence of the sabre by his pioneering work *Nova Scientia* (1537), on the application of mathematics to artillery fire. Tartaglia’s “new science” was, of course, ballistics. Even though the theories he developed were often completely wrong, he was the first to offer a theoretical discussion as against the so-called experience of gunners.
Anticipating Galileo, Tartaglia taught that falling bodies of different weights traverse equal distances in equal times.

Tartaglia’s unfortunate early experiences may have encouraged a suspicious character. Self-taught, he was jealous of his prerogatives and constantly impelled to try to establish his intellectual credentials. Either through intent or simple ignorance of the literature, he had a habit of claiming other people’s discoveries as his own. An instance of this is the “arithmetic triangle” commonly attributed to Pascal, which Tartaglia asserted was his invention although it had previously appeared in print. Tartaglia seems to have felt that his lack of a classical education placed him at a disadvantage as a humanist; and in his General Trattato di Numeri et Misure (1556–1560), intended to replace Pacioli’s Summa, he adorned the preface with quotations from both Cicero and Ptolemy.

In 1530, Tartaglia was sent two problems by a friend, namely:

1. Find a number whose cube added to three times its square makes 5; that is, find a value of \( x \) satisfying the equation \( x^3 + 3x^2 = 5 \).

2. Find three numbers, the second of which exceeds the first by 2, and the third of which exceeds the second by 2 also, and whose product is 1000; that is, solve the equation \( x(x + 2)(x + 4) = 1000 \), or equivalently, \( x^3 + 6x^2 + 8x = 1000 \).

For some time Tartaglia was unable to solve these problems, but in 1535 he finally managed to do so, and he also announced that he could effect the solution of any equation of the type \( x^3 + px^2 = q \). Fiore, believing Tartaglia’s claim to be a bluff, challenged him to a public problem-solving contest. Each contestant was to propose 30 problems, the victor being the one who could solve the greatest number within 50 days. Tartaglia was aware that his rival had inherited the solution of some form of cubic equation from a deceased master, and he worked frantically to find the general procedure. Shortly before the appointed date, he devised a scheme for solving cubics that lacked the second-degree term. Thus, Tartaglia entered the competition prepared to handle two types of cubics, whereas his opponent was equipped for but one. Within two hours, Tartaglia had reduced all 30 problems posed to him to particular cases of the equation \( x^3 + px = q \), for which he knew the answer. Of the problems he himself put to Fiore, the latter failed to master a single one (most of which led to equations of the form \( x^3 + px^2 = q \)).

Cardan, A Scoundrel Mathematician

Girolamo Cardano (1501–1576), better known as Cardan, now appears on the scene. Cardan’s life was deplorable even by the standards of the times. He saw one son executed for wife-poisoning; he personally cropped the ears of a second son who attempted the same offense; he was imprisoned for heresy after having published the horoscope of Christ; and in general he divided his time between intensive study and extensive debauchery. Yet in his range of interests as well as vices, Cardan was a true Renaissance man: physician, philosopher, mathematician, astrologer, dabbler in the occult, and prolific writer.

After a frivolous youth devoted mainly to gambling, Cardan began his university studies at Pavia and completed them at Padua in 1525 with a doctorate in medicine. Ostensibly on the grounds of his illegitimate birth but more likely owing to his reputation as a gambler, Cardan’s repeated applications to the College of Physicians in Milan were all turned down. It is not surprising that his first published work, De Malo Recentiorum Medicorum Medendi
Usu Libellus (On the Bad Practices of Medicine in Common Use), ridiculed the practitioners in Milan. By the time he was 50 years old, Cardan stood second only to Vesalius among European physicians and traveled widely to treat the well-known. So great was his fame that the archbishop of Scotland was among his patients. The archbishop was believed to be suffering from consumption; and Cardan, on the strength of a statement—later admitted to be false—that he could cure this complaint, journeyed to Edinburgh to treat the archbishop. Fortunately for the patient, and also for Cardan’s reputation, it turned out that he was suffering from attacks of asthma. When Cardan passed through London on the return trip, he was received by the young King Edward VI, whose horoscope he obligingly cast. The comfortable predictions of a long life and prosperous future proved to be a great embarrassment when the boy died shortly thereafter. At various times, Cardan was professor of mathematics at the universities of Milan, Pavia, and Bologna, resigning each position as a result of some new scandal connected with his name. Forbidden to lecture publicly or to write or publish books, he finally settled in Rome, where for some strange reason, he obtained a handsome pension as astrologer to the papal court. According to various accounts, having predicted that he would die on a certain day, Cardan felt obliged to commit suicide to authenticate the prediction.

When the news of the mathematical joust between Tartaglia and Fiore eventually reached Cardan in Milan, Cardan begged Tartaglia for the cubic solution, offering to include the result in his forthcoming book Practica Arithmeticae (1539) under Tartaglia’s name. Tartaglia refused on the grounds that in due time he intended to publish his own discourse on algebra. Being credited for a formula is not the same thing as having a treatise, an original work, under your own name; it is the book, not the footnote reference, that history will cite. Cardan, in the hope of learning the secret, invited Tartaglia to visit him. After many entreaties and much flattery, Tartaglia revealed his method of solution on the promise, probably given under oath, that Cardan would keep it confidential. Rumors began to circulate, however, that Tartaglia was not the first discoverer of the cubic formula, and in 1543 Cardan journeyed to Bologna to try to verify these reports. After examining the posthumous papers of del Ferro, he concluded that del Ferro was the one who had made the breakthrough. Cardan no longer felt bound by his promise to Tartaglia, and when Cardan’s work Ars Magna appeared in 1545, the formula and method of proof were fully disclosed. Cardan candidly admitted (at three places in the text) that he had gotten the solution to the special cubic equation $x^3 + px = q$ from his “friend” Tartaglia, but claimed to have carried out for himself the proof that the formula he had received was correct. Angered at this apparent breach of a solemn oath and feeling cheated out of the rewards of his monumental work, Tartaglia accused Cardan of lying. Thus began one of the bitterest feuds in the history of science, carried on with name-calling and mudslinging of the lowest order.

7.3 Cardan’s Ars Magna

Cardan’s Solution of the Cubic Equation

Cardan wrote on a wide variety of subjects, including mathematics, astrology, music, philosophy, and medicine. When he died, 131 of his works had been published and 111 existed in manuscript form, and he had claimed to have burned 170 others that were unsatisfactory. These ran the gamut from Practica Arithmeticae (1539), a book on numerical calculation based largely on Pacioli’s work of 1494, to Liber de Vita Propria (1575), an autobiography in which he did not spare the most
shameful revelations. His passion for the games of chess, dice, and cards inspired Cardan to write *Liber de Ludo Aleae (Book on Games of Chance)*. Found among his papers after his death and published in 1663, this work broke the ground for a theory of probability more than 50 years before Fermat and Pascal, to whom the first steps are usually attributed. In it he even gives advice on how to cheat, no doubt gained from personal experience. One of the ironic twists of fate in Cardan’s life is that his excessive gambling, which had cost him time, money, and reputation, should have helped him earn a place in the history of mathematics.

In permanent significance, the *Ars Magna (The Great Art)* undoubtedly stands at the head of the entire body of Cardan’s writings, mathematical or otherwise. This work, which was first printed in 1545, today would be classified as a text on algebraic equations. It makes very clear that Cardan was no mere plagiarist but one who combined a measure of honest toil with his piracy. Although negative numbers had become known in Europe through Arabic texts, most Western algebraists did not accept them as bona fide numbers and preferred to write their equations so that only positive terms appeared. Thus, there was no one cubic equation at the time, but rather thirteen of them, according to whether the terms of the various degrees appeared on the same side of the equality sign or on opposite sides. In giving Cardan the formula for \( x^3 + px = q \), Tartaglia did not automatically provide solutions for all the other forms that the cubic might take. Cardan was forced to expand Tartaglia’s discovery to cover these other cases, devising and providing the rule separately in each instance.

Hitherto, Western mathematicians had confined their attention to those roots of equations that were positive numbers. Cardan was the first to take notice of negative roots, although he called them “fictitious,” and the first to recognize that a cubic might have three roots. Another notable aspect of Cardan’s discussion was the clear realization of the existence of what we now call complex or imaginary numbers (the ghosts of real numbers, as Napier was later to call them). Cardan kept these numbers out of the *Ars Magna* except in one case, when he considered the problem of dividing 10 into two parts whose product was 40. He obtained the roots \( 5 + \sqrt{-15} \) and \( 5 - \sqrt{-15} \) as solutions of the quadratic equation \( x(10 - x) = 40 \), and then stated, “Putting aside the mental tortures involved, multiply \( 5 + \sqrt{-15} \) by \( 5 - \sqrt{-15} \), making 25 – (-15), whence the product is 40.” Cardan somehow felt obliged to accept these solutions yet hastened to add that there was no interpretation for them, remarking, “So progresses arithmetical subtlety the end of which, as is said, is as refined as it is useless.” But merely writing down the meaningless gave it a symbolic meaning, and Cardan deserves credit for having paid attention to the situation.

Among the innovations that Cardan introduced in the *Ars Magna* was the trick of changing a cubic equation to one in which the second-degree term was absent. If one starts with the equation

\[
x^3 + ax^2 + bx + c = 0,
\]

all that is needed is to make the substitution \( x = y - a/3 \). With this new variable, the given equation becomes

\[
0 = \left(y - \frac{a}{3}\right)^3 + a \left(y - \frac{a}{3}\right)^2 + b \left(y - \frac{a}{3}\right) + c
\]
Girolamo Cardano
(1501–1576)

(Source: Princeton University Press.)

\[ y^3 - 3y^2\left(\frac{a}{3}\right) + 3y\left(\frac{a}{3}\right)^2 - \left(\frac{a}{3}\right)^3 \]
\[ + a\left[ y^2 - 2y\left(\frac{a}{3}\right) + \left(\frac{a}{3}\right)^2 \right] + b\left( y - \frac{a}{3} \right) + c \]
\[ = y^3 + \left( b - \frac{a^2}{3} \right) y + \left( \frac{2a^3}{27} - \frac{ab}{3} + c \right). \]

If one sets
\[ p = b - \frac{a^2}{3} \quad \text{and} \quad q = -\left( \frac{2a^3}{27} - \frac{ab}{3} + c \right), \]
then the last equation can be written
\[ y^3 + py = q, \]
which is the so-called reduced form of the cubic. It lacks a term in \( y^2 \), but otherwise the coefficients are arbitrary.

Cardan solved the cubic equation \( x^3 + 20x = 6x^2 + 33 \) by this reduction technique. Through the substitution \( x = y - (-6)/3 = y + 2 \), it is transformed to the equation
\[ (y^3 + 6y^2 + 12y + 8) + 20(y + 2) = 6(y^2 + 4y + 4) + 33, \]
or simplified,
\[ y^3 + 8y = 9. \]
This last equation has one obvious solution, namely, \( y = 1 \); hence, \( x = y + 2 = 3 \) will satisfy the original cubic.
Let us examine how Cardan managed to arrive at the general solution of the reduced cubic. Because the Renaissance was a period of the highest veneration of Greek mathematics, it is not unexpected that his proofs should be based on geometric arguments, emulating (as Cardan himself emphasized) the reasoning of Euclid. The technique for dealing with the cubic

\[ x^3 + px = q, \quad p > 0, \quad q > 0, \]

although geometric, is equivalent to using the algebraic identity

\[ (a - b)^3 + 3ab(a - b) = a^3 - b^3. \]

If \( a \) and \( b \) are chosen so that \( 3ab = p \) and \( a^3 - b^3 = q \), then identity (2) becomes

\[ (a - b)^3 + p(a - b) = q, \]

which shows that \( x = a - b \) will furnish a solution to the cubic (1). The problem therefore involves solving the pair of simultaneous equations

\[
\begin{align*}
    a^3 - b^3 &= q, \\
    ab &= \frac{p}{3},
\end{align*}
\]

for \( a \) and \( b \). To do so, one squares the first equation and cubes the second, to get

\[
\begin{align*}
    a^6 - 2a^3b^3 + b^6 &= q^2, \\
    4a^3b^3 &= \frac{4p^3}{27}.
\end{align*}
\]

When the equations are added, it follows that

\[
(a^3 + b^3)^2 = a^6 + 2a^3b^3 + b^6 = q^2 + \frac{4p^3}{27},
\]

and so

\[
a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}.
\]

If the equations

\[
a^3 - b^3 = q \quad \text{and} \quad a^3 + b^3 = \sqrt{q^2 + \frac{4p^3}{27}}
\]

are now solved simultaneously, then \( a^3 \) and \( b^3 \) can be determined; the result is

\[
\begin{align*}
a^3 &= \frac{1}{2} \left( q + \sqrt{q^2 + \frac{4p^3}{27}} \right) = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \\
b^3 &= \frac{1}{2} \left( -q + \sqrt{q^2 + \frac{4p^3}{27}} \right) = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}},
\end{align*}
\]
But then

\[ a = \sqrt[3]{\frac{q}{2}} + \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}}, \]

\[ b = \sqrt[3]{-\frac{q}{2}} + \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}}, \]

and consequently,

\[ x = a - b = \sqrt[3]{\frac{q}{2}} + \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}} - \sqrt[3]{-\frac{q}{2}} + \sqrt[3]{\frac{q^2}{4} + \frac{p^3}{27}}. \]

As Tartaglia feared, this last formula has forever since been known as Cardan’s formula for the solution of the cubic equation. The mathematician to whom we owe the chief contribution made to algebra in the sixteenth century is largely forgotten, and the discovery goes by the name of a scoundrel.

Cardan illustrated his method by solving the equation

\[ x^3 + 6x = 20. \]

In this case, \( p = 6 \) and \( q = 20 \), so that \( p^3/27 = 8 \) and \( q^2/4 = 100 \); whence the formula yields

\[ x = \sqrt[3]{108} + 10 - \sqrt[3]{108} - 10. \]

As remarked earlier, Cardan was forced to treat an elaborate list of equation types, produced largely by his failure to allow negative coefficients. In solving the equation

\[ x^3 = px + q, \quad p > 0, \quad q > 0, \]

he used a geometric argument corresponding to the identity

\[ (a + b)^3 = a^3 + b^3 + 3ab(a + b), \]

to arrive at the solution

\[ x = \sqrt[3]{\frac{q}{2}} + \sqrt[3]{\frac{q^2}{4} - \frac{p^3}{27}} + \sqrt[3]{\frac{q}{2}} - \sqrt[3]{\frac{q^2}{4} - \frac{p^3}{27}}. \]

There is one difficulty connected with this last formula, which Cardan observed but could not resolve. When \( (q/2)^2 < (p/3)^3 \), the formula leads inevitably to square roots of negative numbers. That is, \( \sqrt{q^2/4 - p^3/27} \) involves “imaginary numbers.”

Consider, for example, the historic equation

\[ x^3 = 15x + 4, \]

treated by Rafael Bombelli, the last great sixteenth century Bolognese mathematician, in his _Algebra_ (1572). A direct application of the Cardan-Tartaglia formula would lead to

\[ x = \sqrt[3]{2} + \sqrt{-121} + \sqrt[3]{2} - \sqrt{-121}. \]
Bombelli knew, nevertheless, that the equation had three real solutions, namely $4, -2 + \sqrt{3}$, and $-2 - \sqrt{3}$. One is left in the paradoxical situation in which the formula produces a result useless for most purposes, yet in other ways three perfectly good solutions can be found. This impasse, which arises when all three roots are real and different from zero, is known as the “irreducible case” of the cubic equation.

### Bombelli and Imaginary Roots of the Cubic

Bombelli was the first mathematician bold enough to accept the existence of imaginary numbers, and hence to throw some light on the puzzle of irreducible cubic equations. A native of Bologna, he himself had not received any formal instruction in mathematics and did not teach at the university. He was the son of a wool merchant and by profession an engineer-architect. Bombelli felt that only Cardan among his predecessors had explored algebra in depth, and that Cardan had not been clear in his exposition. He therefore decided to write a systematic treatment of algebra to be a successor to Cardan’s *Ars Magna*. Bombelli composed the first draft of his treatise about 1560, but it remained in manuscript form until 1572, shortly before his death. The preparation of his *Algebra* took considerably longer than Bombelli had foreseen, for as he wrote in the work:

> A Greek manuscript in this science was found in the Vatican Library, composed by Diophantus. . . . We set to translating it and have already done five of the seven (sic) extant books. The rest we have not been able to finish because of other commitments.

Tremendously enthusiastic over the rediscovery of the *Arithmetica*, Bombelli took 143 problems and their solutions from its first four books and embodied them in his *Algebra*, interspersing them with his own contributions. Although Bombelli did not distinguish among the problems, he nonetheless acknowledged that he had borrowed freely from Diophantus. (A manuscript of the *Algebra* was found in 1923; the absence of the 143 problems borrowed from the *Arithmetica* suggests that Bombelli had not seen the Vatican copy when he first wrote the work.) Whereas the works of Pacioli and Cardan contained many problems of applied arithmetic, Bombelli’s problems were all abstract. He claimed that while others wrote for a practical, rather than a scientific purpose, he had “restored the effectiveness of arithmetic, imitating the ancient writers.” The publication of Bombelli’s *Algebra* completed a movement that began in Italy about 1200, when Fibonacci introduced the rules of algebra in the *Liber Abaci*.

Bombelli’s skill in operating with imaginary numbers enabled him to demonstrate the applicability of Cardan’s formula, even in the irreducible case (all roots real) of the cubic equation. Assuming that the complex numbers behaved like other numbers in calculations, he made a circuitous passage into, and out of, the complex domain and ended by showing that the apparently imaginary expression for the root of the equation $x^3 = 15x + 4$ gave a real value. Bombelli had the ingenious idea that the complex values of the radicals

$$\sqrt[3]{2 + \sqrt{-121}} \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}}$$

might be related much as the radicals themselves; that is, they might differ only in a sign.
This prompted him to set

\[ \sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1} \quad \text{and} \quad \sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1}, \]

where \( a > 0 \) and \( b > 0 \) are to be determined. As Bombelli said:

It was a wild thought in the judgment of many; and I too for a long time was of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.

Now the relation \[ \sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1} \] implies that

\[ 2 + \sqrt{-121} = (a + b\sqrt{-1})^3 \]

\[ = a^3 + 3a^2b\sqrt{-1} + 3ab^2(-1) + b^3(-1)^3 \]

\[ = a(a^2 - 3b^2) + b(3a^2 - b^2)\sqrt{-1}. \]

This equality would hold provided that

\[ a(a^2 - 3b^2) = 2 \quad \text{and} \quad b(3a^2 - b^2) = 11. \]

If solutions are sought in the integers, then the first of these conditions tells us that \( a \) must be equal to 1 or 2, and the second condition asserts that \( b \) has the value 1 or 11; only the choices \( a = 2 \) and \( b = 1 \) satisfy both conditions. Therefore,

\[ 2 + \sqrt{-121} = (2 + \sqrt{-1})^3 \quad \text{and} \quad 2 - \sqrt{-121} = (2 - \sqrt{-1})^3. \]

Bombelli concluded that one solution to the cubic equation \( x^3 = 15x + 4 \) was

\[ x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \]

\[ = \sqrt[3]{(2 + \sqrt{-1})^3} + \sqrt[3]{(2 - \sqrt{-1})^3} \]

\[ = (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4. \]

In proving the reality of the roots of the cubic \( x^3 = 15x + 4 \), he demonstrated the extraordinary fact that real numbers could be engendered by imaginary numbers. From this time on, imaginary numbers lost some of their mystical character, although their full acceptance as bona fide numbers came only in the 1800s.

### 7.3 Problems

1. Find all three roots of each of the following cubic equations by first reducing them to cubics that lack a term in \( x^2 \).
   
   (a) \( x^3 + 11x = 6x^2 + 6 \).
   
   (b) \( x^3 + 6x^2 + 3x = 2 \).
   
   (c) \( x^3 + 6x^2 = 20x + 56 \).
   
   (d) \( x^3 + 64 = 6x^2 + 24x \).

2. Derive Cardan’s formula

\[ x = \sqrt[3]{q/2 + \sqrt{q^2/4 - p^3/27}} + \sqrt[3]{q/2 - \sqrt{q^2/4 - p^3/27}} \]

for solving the cubic equation \( x^3 = px + q \), where \( p > 0 \) and \( q > 0 \).

3. Using Cardan’s formula, obtain one root of each of the
The following method of Viète (1540–1603) is useful
in solving the reduced cubic $x^3 + ax = b$. By substitution of $x = a/3y - y$, the given equation becomes $y^6 + by^3 - a^2/27 = 0$, a quadratic in $y^3$. By

the quadratic formula,

$$y^3 = \frac{1}{2} \left( -b \pm \sqrt{b^2 + \frac{4a^2}{27}} \right),$$

from which $y$ and then $x$ can be determined. Use this method to find a root of the cubics $x^3 + 81x = 702$ and

$$x^3 + 6x + 18 + 13 = 0. \text{ [Hint: } \sqrt[3]{142.884} = 5.3\text{].}$$

13. By making the substitution $x = y + 5/y$, find a root of the cubic equation $x^3 = 15x + 126$.

14. Use Cardan’s formula to find, in these examples of the irreducible case in cubics, a root of the given equations.

(a) $x^3 = 63x + 162$. 
   \[\text{[Hint: } 81 \pm 30\sqrt{-3} = (-3 \pm 2\sqrt{-3})^3\text{]}\]

(b) $x^3 = 7x + 6$.
   \[\text{[Hint: } 3 \pm \frac{10}{9} \sqrt{-3} = \left(\frac{3}{2} \pm \frac{1}{6} \sqrt{-3}\right)^3\text{]}\]

(c) $x^3 + 6 = 2x^2 + 5x$.
   \[\text{[Hint: } -\frac{28}{27} - \frac{5}{3} \sqrt{-3} = \left(\frac{1}{6} + \frac{5}{6} \sqrt{-3}\right)^3\text{]}\]

15. The great Persian poet, Omar Khayyam (circa 1050–1130), found a geometric solution of the cubic equation $x^3 + ax^2 = b$ by using a pair of intersecting conic sections. In modern notation, he first constructed the parabola $x^2 = ay$. Then he drew a semicircle with diameter $AC = b/a^2$ on the $x$-axis, and let $P$ be the point of intersection of the semicircle with the parabola (see the figure). A perpendicular is dropped from $P$ to the $x$-axis to produce a point $Q$.

\[x^2 = ay\]

Complete the details in the following proof that the $x$-coordinate of $P$, that is, the length of segment $AQ$, is the root of the given cubic.

(a) $AQ^2 = a(PQ)$.

(b) Triangles $AQP$ and $PQC$ are similar, so that

$$\frac{AQ}{PQ} = \frac{PQ}{QC}, \text{ or } PQ^2 = AQ \left(\frac{b}{a^2} - AQ\right).$$

(c) Substitution gives $AQ^3 + a^2AQ = b$. 

\[\text{[Diagram]}\]
16. It is also possible to use the parabola $y = x^2$ for duplicating a cube of edge $a$. Draw a circle with center $(a/2, 1/2)$ that passes through the origin $(0, 0)$. Then the $x$-coordinate of the point of intersection of the circle and the parabola $y = x^2$ will serve as the edge of a cube double in volume to the given cube. Prove this conclusion.

7.4 Ferrari’s Solution of the Quartic Equation

The Resolvant Cubic

After the cubic had been solved, it was only natural that mathematicians should attack the quartic (fourth-degree) equation. The solution was discovered during work on a problem proposed to Cardan in 1540. Divide the number 10 into three proportional parts so that the product of the first and second parts is 6. If the numbers are called $\frac{6}{x}$, $x$, and $\frac{x^3}{6}$, the conditions laid down are clearly fulfilled. In particular, the requirement that

$$\frac{6}{x} + x + \frac{x^3}{6} = 10$$

is equivalent to the quartic

$$x^4 + 6x^2 + 36 = 60x.$$

After an unsuccessful attempt at solving this equation, Cardan turned it over to his disciple Ludovico Ferrari (1522–1565). Ferrari, using the rules for solving the cubic, eventually succeeded where his master had failed. At least, Cardan had the pleasure of incorporating the result in the *Ars Magna*, with due credit given Ferrari.

Ferrari, the son of poor parents, was taken into Cardan’s household as a servant boy at the age of 14. Although he had not received any formal education, Ferrari was exceptionally gifted, and Cardan undertook to instruct him in Latin, Greek, and mathematics. Cardan soon made him his personal secretary, and after four years of service, Ferrari left to become public lecturer in mathematics at the University of Milan. He became professor of mathematics at Bologna in 1565 and died in the same year, having been poisoned with white arsenic—by his own sister, as rumor had it.

Ferrari joined the fray surrounding the solution of the cubic by swearing that he had been present at the fateful meeting between Cardan and Tartaglia and that there had been no oath of secrecy involved. Always eager to defend his old master, Ferrari then challenged Tartaglia to a public disputation on mathematics and related disciplines, writing in a widely distributed manifesto: “You have written things that falsely and unworthily slander Signor Cardan, compared with whom you are hardly worth mentioning.” Tartaglia’s counterstatement asked Ferrari either to let Cardan fight his own battles or to admit that he was acting on Cardan’s behalf; the challenge would be accepted if Cardan were willing to countersign Ferrari’s letter and if (because Tartaglia feared some sort of trickery) topics from the *Ars Magna*...
were excluded. Another acrimonious dispute ensued in which 12 letters were exchanged, full of charges and insults, with each party trying to justify his own position. In one of these retorts, Ferrari made the mistake of calling himself Cardan’s creation, allowing Tartaglia the satisfaction of thereafter referring to him as “Cardan’s creature.”

The contest finally took place in Ferrari’s hometown of Milan in 1548 before a large and distinguished gathering. Perhaps aware of his own limitations, Cardan had the foresight to leave Milan for several days. There is no record of the proceedings except for a few
statements to the effect that the meeting soon deteriorated into a shouting match over a problem of Ferrari’s that Tartaglia had been unable to resolve. The altercation ran into the dinner hour, at which time everyone felt compelled to leave. Tartaglia departed the next morning claiming to have come off the better in the dispute, but it seems more likely that Ferrari was declared the winner. The best evidence of this is that Tartaglia lost his teaching post in Brescia, and Ferrari received a host of flattering offers, among them an invitation to lecture in Venice, Tartaglia’s stronghold.

Ferrari’s method for solving the general quartic could, in modern notation, be summarized as follows. First, reduce the equation
\[ x^4 + ax^3 + bx^2 + cx + d = 0 \]
to the special form
\[ y^4 + py^2 + qy + r = 0, \]
in which the term in \( y^3 \) is missing, by substituting \( x = y - a/4 \). Now the left-hand side of
\[ y^4 + py^2 = -qy - r \]
contains two of the terms of the square of \( y^2 + p \). Let us complete the square by adding \( py^2 + p^2 \) to each side to get
\[ (y^2 + p)^2 = y^4 + 2py^2 + p^2 = py^2 + p^2 - qy - r. \]
We now introduce another unknown for the purpose of converting the left member of this equation into \((y^2 + p + z)^2\). This is done by adding \(2(y^2 + p)z + z^2\) to each side, and leads to
\[ (y^2 + p + z)^2 = py^2 + p^2 - qy - r + 2(y^2 + p)z + z^2 \]
\[ = (p + 2z)y^2 - qy + (p^2 - r + 2pz + z^2). \]
The problem now reduces to finding a value of \( z \) that makes the right-hand side, a quadratic in \( y \), a perfect square. This will be the case when the discriminant of the quadratic is zero; that is, when
\[ 4(p + 2z)(p^2 - r + 2pz + z^2) = q^2, \]
which requires solving a cubic in \( z \); namely,
\[ 8z^3 + 20pz^2 + (16p^2 - 8r)z + (4p^3 - 4pr - q^2) = 0. \]
The last equation is known as the resolvent cubic of the given quartic equation, and it can be solved in the usual way. There are in general three solutions of the resolvent cubic, and \( y \) can be determined from any one of them by extracting square roots. Once a value of \( y \) is known, the solution of the original quartic is readily reached.

If the procedure sounds complicated, an example from the *Ars Magna* might help to clarify the sequence of steps. Cardan considered (Chapter 39, Problem 9) the quartic equation \( x^4 + 4x + 8 = 10x^2 \), or equivalently,
\[ x^4 - 10x^2 = -4x - 8. \]
Ferrari’s Solution of the Quartic Equation

Completing the square on the left-hand side, one gets

$$(x^2 - 10)^2 = -10x^2 - 4x + 92.$$  

By adding the quantity $2z(x^2 - 10) + z^2$ to each side, this equation is changed to

$$(x^2 - 10 + z)^2 = (2z - 10)x^2 - 4x + (92 - 20z + z^2),$$  

where $z$ is a new unknown. Now the right-hand expression is a perfect square if $z$ is chosen to satisfy the condition

$$4(2z - 10)(92 - 20z + z^2) = 16,$$

or after simplification,

$$z^3 - 25z^2 + 192z = 462.$$  

This is a cubic equation from which $z$ can be found. We start by letting $z = u + \frac{25}{3}$; this substitution reduces the equation to the form

$$u^3 = \frac{49}{3}u + \frac{524}{27}.$$  

A solution is $u = -\frac{4}{3}$, so that $z = 7$. It is this value of $z$ that should give squares on both sides of equation (1). The result of substituting $z = 7$ is

$$(x^2 - 3)^2 = 4x^2 - 4x + 1 = (2x - 1)^2,$$

whence $x^2 - 3 = \pm(2x - 1)$. The positive sign gives

$$x^2 - 2x - 2 = 0,$$

and the negative sign yields

$$x^2 + 2x - 4 = 0.$$  

In solving these equations by the quadratic formula, it is found that the four solutions of the original quartic equation are

$$1 + \sqrt{3}, 1 - \sqrt{3}, -1 + \sqrt{5}, \text{ and } -1 - \sqrt{5}.$$  

The Story of the Quintic Equation: Ruffini, Abel, and Galois

Our story has a postscript. We have seen that in the case of quadratic, cubic, and quartic equations, explicit formulas for the roots were found that were formed from the coefficients of the equation by using the four operations of arithmetic (addition, multiplication, subtraction, and division) and by taking radicals of various sorts. The next natural step was to seek similar solutions of equations of higher degrees, the presumption being that an equation of degree $n$ should be capable of formal solution by means of radicals and probably by radicals of an exponent not larger than $n$. For close to 300 years, algebraists wrestled with the general equation of fifth degree (the quintic equation) and made almost no progress. But these repeated failures at least had the effect of suggesting the possibility, startling at the time, that the quintic equation might not be solvable in this way.
Paolo Ruffini (1765–1822), an Italian physician who taught mathematics as well as medicine at the University of Modena, confirmed the suspicion of the impossibility of finding an algebraic solution for the general fifth-degree equation. Ruffini’s proof, which appeared in his two-volume *Teorie generale delle equazioni* of 1799, was sound in general outline although faulty in some details. The Norwegian genius Niels Henrik Abel (1802–1829), when he was about 19 years old, made a study of the same problem. At first he thought he had found a solution of the general quintic by radicals, but later he established the unsolvability of the equation, using a more rigorous argument than Ruffini’s. Abel fully realized the importance of his discovery and had it published in 1824, at his own expense, in a pamphlet that bore the title *Memoire sur les equations algebriques ou on demontre l’impossibilite de la resolution de l’equation generale du cinquieme degre*. So that expenses could be kept down, the whole pamphlet had to be condensed to six pages of actual print, making it difficult to follow the reasoning. Thus, the significance of Abel’s masterpiece went unnoticed by contemporary scholars. When Europe’s leading mathematician, Carl Friedrich Gauss, duly received his copy, he tossed it aside unread with the disgusted exclamation, “Here is another of those monstrosities!”

Abel’s opportunity came when he had the great good fortune to make the acquaintance of August Leopold Crelle, a German civil engineer and enthusiastic mathematical amateur. At this time, Crelle was making plans to launch a new journal, which would be the first periodical devoted exclusively to mathematical research. Abel eagerly accepted the invitation to submit articles, and the first three volumes of the *Journal fur die reine und angewandte mathematik* (Journal for Pure and Applied Mathematics), or *Crelle’s Journal* as it is commonly called, contained 22 papers by Abel. In the founding volume (1826), he expanded his earlier research into what now is known as the Abel-Ruffini theorem: It is impossible to find a general formula for the roots of a polynomial equation of degree five or higher if the formula for the solution is allowed to use only arithmetic operations and extraction of roots. When Abel composed his paper, he was not aware that he had a precursor. He was later to write, however, in a manuscript *Sur la resolution algebreique des equations* (dated 1828 but only published after his death): “The only one before me, if I am not mistaken, who has tried to prove the impossibility of the algebraic solution of the general equation is the mathematician Ruffini, but his paper is so complicated that it is difficult to judge the correctness of his arguments.”

Abel’s theorem on the unsolvability of higher equations applied to general equations only. Many special equations existed that were solvable by radicals, and the characterization of these remained an open question. It was reserved for another young mathematician, Evariste Galois (1811–1832) to definitively answer what specific equations of a given degree admit an algebraic solution. The posthumous publication of Galois’s manuscripts in Liouville’s *Journal de Mathematiques* in 1846 represented both the completion of Abel’s research and the foundation of group theory, one of the most important branches of modern mathematics. Considering the significance of his discovery, one naturally asks why it required 14 years after Galois’s death for the essential elements of his work to become available in print. The reason is a combination of sheer bad luck and negligence. The original memoir was mislaid by the editor appointed to examine it, and after resubmission, it was returned by a second editor, who judged the contents incomprehensible.

The sequence of events seems to be this. Galois first submitted his results on the algebraic solution of equations to the Academy of Sciences in May 1829, while he was...
still only 17 years old. Augustin-Louis Cauchy (1789–1857), a member of the Academy and a professor at the École Polytechnique, was appointed referee. Cauchy either forgot or lost the communication, as well as another presented a week later. Galois then (February 1830) submitted a new version of his investigations to the Academy, hoping to enter it in the competition for the Grand Prize in Mathematics, the pinnacle of mathematical honor. This time it was entrusted to the permanent secretary, Joseph Fourier (1768–1830), who died shortly thereafter, before examining the manuscript. It was never retrieved from among his papers. A further disappointment awaited Galois. In January 1831, he submitted his paper for the third time under the title “Une mémoire sur les conditions de résolubilité des équations par radicaux.” After a delay of some six months, during which Galois wrote to the president of the Academy asking what had happened, it was rejected by the referee Simeon-Denis Poisson (1781–1840). At the conclusion of his report, Poisson remarked:

His arguments are not sufficiently clear, nor developed enough for us to judge their correctness. . . . It is hoped that the author would publish his work in its entirety so that we can form a definite opinion.

In May 1832, Galois was provoked into a duel in unclear circumstances. (The theory has been advanced that the challenger was hired by the police, who arranged the confrontation to eliminate what they considered to be a dangerous radical.) On the eve of the duel, apparently certain of death, Galois wrote a letter to a friend describing the contents of the memoir Poisson had rejected. Its seven pages, hastily written, contain a summary of the discoveries he had been unable to develop. The letter ends with the plea:

Eventually there will be, I hope, some people who will find it profitable to decipher this mess.

Galois spent the rest of the night annotating and making corrections to some of his papers; next to a theorem, he scrawled:

There are a few things left to be completed in this proof. I do not have time.

The duel took place on May 30, 1832, early in the morning. Galois was grievously wounded by a shot in the abdomen, and lay where he had fallen until found by a passing peasant, who took him to the hospital. He died the next morning of peritonitis, attended by his younger brother. Galois tried to console him, saying, “Do not cry. I need all my courage to die at twenty.” He was buried in a common ditch at the cemetery of Montparnasse; the exact location is unknown.

By 1843, Galois’s manuscripts had found their way to Joseph Liouville (1809–1882), who after spending several months in the attempt to understand them, became convinced of their importance. He addressed the Academy of Sciences on July 4, 1843, opening with the words:

I hope to interest the Academy in announcing that among the papers of Evariste Galois I have found a solution, as precise as it is profound, of this beautiful problem: whether or not it [the general equation of fifth degree] is solvable by radicals.

Liouville announced that he would publish Galois’s papers in the December 1843 issue in his recently founded periodical *Journal de Mathématiques Pures et Appliquées*. But for some reason, publication of the heavily edited version of the celebrated 1831 memoir did not occur until the October–November 1846 issue.
Although no trace of Galois’s grave remains, his enduring monument lies in his ideas. During the late 1800s, Galois’s theory—as well as a new topic it brought to life, group theory—became an integral and accepted part of mathematics. Galois’s theory appears to have been taught in the German universities for the first time by Richard Dedekind, who lectured on the topic at Göttingen in the winter of 1856–1857; it is said that only two students came to hear him. The first full and clear presentation of the Galois theory was given by Camille Jordan in his book Traité des substitutions et des équations algébriques (1870).

7.4 Problems

1. Solve the following quartic equations by Ferrari’s method.
   (a) \( x^4 + 3 = 12x \).
   (b) \( x^4 + 6x^2 + 8x + 21 = 0 \).
   \[ \text{Hint: The reduced form of the resolvent cubic is} \quad u^3 + 24u + 32 = 0, \quad \text{with} \quad u = 4 \text{ as a solution.} \]
   (c) \( x^4 + 9x + 4 = 4x^2 \).
   \[ \text{Hint: The resolvent cubic} \quad z^3 - 10z^2 + 28z - \frac{273}{8} = 0 \quad \text{has} \quad z = \frac{13}{2} \quad \text{as a solution.} \]

2. Solve the quartic \( x^4 + 4x^3 + 8x^2 + 7x + 4 = 0 \).
   \[ \text{Hint: First replace the given quartic by} \quad y^4 + 2y^2 - y + 2 = 0. \quad \text{The resolvent cubic of this last equation is} \quad z^3 + 5z^2 + 6z + \frac{12}{5} = 0, \quad \text{with} \quad z = -\frac{1}{2} \quad \text{as a solution.} \]

3. Solve the quartic \( x^4 + 8x^3 + 15x^2 = 8x + 16 \).
   \[ \text{Hint: First replace the given quartic by} \quad y^4 - 9y^2 - 4y + 12 = 0. \quad \text{The reduced form of the resolvent cubic of this last equation is} \quad u^3 + \frac{75}{4}u + \frac{125}{4}, \quad \text{with} \quad u = 5 \quad \text{as a solution.} \]

4. Use Ferrari’s method to show that the quartic equation
   \[ x^4 + 9 = 4x^3 + 6x^2 + 12x \]
   has the four roots \( 3 + \sqrt{5}, \quad 3 - \sqrt{5}, \quad -1 + \sqrt{-2}, \quad \text{and} \quad -1 - \sqrt{-2} \).

5. Find a solution to the following problem from the Ars Magna.
   Chapter 26, Problem 1. Four men form an organization. The first deposits a given quantity of aurei; the second deposits the fourth power of one-tenth of the first; the third, five times the square of one-tenth the first; and the fourth, 5. Let the sum of the first and second equal the sum of the third and fourth. How much did each deposit? \[ \text{Hint: If it is assumed that the first deposited} \ 10x, \quad \text{then the conditions imply that} \ x^4 + 10x = 5x^2 + 5. \]

6. The following method of Viète was a notable improvement in Ferrari’s technique for solving the quartic \( y^4 + py^3 + qy + r = 0 \). To both sides of the equation, add \( y^2z^2 + \frac{4}{3}z^3 \), where \( z \) is a new unknown, so that
   \[ (y^2 + \frac{1}{2}z^2)^2 = y^4 + \frac{4}{3}z^3 - r - qy - py^2 \]
   \[ = y^4(z^3 - p) - qy + (\frac{1}{4}z^3 - r). \]
   The right-hand side is a perfect square if \( z \) is chosen to satisfy
   \[ q^2 = 4(z^3 - p)(\frac{1}{4}z^3 - r) \]
   \[ = z^6 - p^4 - 4rz^3 + 4rp, \]
   which is a cubic in \( z^2 \) and therefore solvable. Use Viète’s procedure to find one root of the quartic equation
   \[ y^4 - y^3 + y^2 - y = 10. \]

Bibliography


Bibliography


CHAPTER 8

The Mechanical World: Descartes and Newton

The discoveries of Newton have done more for England and for the race, than has been done by whole dynasties of British monarchs.

THOMAS HILL

8.1 The Dawn of Modern Mathematics

The Seventeenth Century

Spread of Knowledge

The Renaissance, which by the sixteenth century was well under way in Italy, soon spread north and west, first to Germany, then to France and the Low Countries, and finally to England. By the late 1600s, scientific, technological, and economic leadership centered on the English Channel—in those countries that had been galvanized by the commerce arising from the great voyages of discovery. At the start, the revival was mainly literary, but gradually scholars began to pay less attention to what was written in ancient books and to place more reliance on their own observations. The age was characterized by an eagerness to experiment, and above all to determine how things happened. Seventeenth-century science may be said to have begun with the appearance of William Gilbert’s De Magnete in 1600, the first treatise on physical science whose content was based entirely on experimentation; and the culmination would have been Isaac Newton’s Opticks in 1704.

In between the De Magnete and the Opticks came the contributions of Johannes Kepler, who was convinced that planetary bodies moved not in Aristotle’s “ideal circles” but in elliptical orbits, and he thereby formulated the laws of terrestrial motion (1619). Also there were the demonstrations by William Harvey (1628) of the circulatory route of the blood from the heart through arteries and veins by way of the lungs; the laying down of the principles of modern chemistry by Robert Boyle in his Sceptical Chymist (1661); and the publication of Robert Hooke’s Micrographia (1665), the earliest large-scale work on the microscopic observation of cellular structure. However, no brief summing can do justice to the achievements of a period that saw so many new discoveries and so many advances in scientific methods.

Whereas the Renaissance marked a return to classical concepts, the seventeenth century set mathematics on entirely new foundations. So extensive and radical were the changes that historians have come to regard the half-century from 1637 to 1687 as the fountainhead of
modern mathematics—the first date alluding to the publication of Descartes’s *La Géométrie* and the second to the date of publication of Newton’s *Principia Mathematica*.

Renaissance mathematics had added little to the geometry of the ancient Greeks, but 1600 ushered in an unexpected revival in the subject. In 1637 the French mathematical community witnessed one of those strange coincidences, once thought rare but which the history of science has shown to be frequent. Two men, Pierre de Fermat and René Descartes, simultaneously wedded algebra to geometry, to produce a remarkable innovation, analytic geometry. About the time when Fermat and Descartes were laying the foundations of a coordinate geometry, two other equally original mathematicians, Pascal and Desargues, were rendering a similar service in the area of synthetic projective geometry. But it was not only on account of the far-reaching developments in geometry that the seventeenth century has become illustrious in the history of mathematics, for the activities of the mathematicians of the period stretched into many fields, new and old. Number mysticism gave way to number theory, in Fermat’s reflections on diophantine analysis. The mathematical theory of probability, a subject to which Cardan contributed in his book *Liber de Ludo Aleae*, took its first full steps in an exchange of letters between Pascal and Fermat concerning the calculation of probabilities. Leibniz’s attempt to reduce logical discussion to systematic form was the forerunner of modern symbolic logic; but it was so far in advance of its time that not until 200 years later was the idea realized through the work of the English mathematician George Boole. Hardly less important were the studies of Galileo, Descartes, Torricelli, and Newton, which were to turn mechanics into an exact science during the next two centuries.

During the middle years of the Renaissance, trigonometry had become a systematic branch of mathematics in its own right in place of serving as handmaiden to astronomy. The aim of facilitating work with complicated trigonometric tables was responsible for one of the greatest computational improvements in arithmetic, the invention of logarithms, by John Napier (1550–1617). Napier worked at least twenty years on the theory, which he explained in his book *Mirifici Logarithmorum Canonis Descriptio* (*A Description of an Admirable Table of Logarithms*, 1614). Seldom has a new discovery won such universal acclaim and acceptance. With logarithms, the operations of multiplication and division can be reduced to addition and subtraction, thereby saving an immense amount of calculation, especially when large numbers are involved. Astronomy was notorious for the time-consuming computations it imposed; the French mathematician Pierre de Laplace was later to assert that the invention of logarithms “by shortening the labors, doubled the life of the astronomer.”

Above all, for mathematics the seventeenth century was the century of the rise of calculus. Although we normally ascribe the invention of calculus to two brilliant contemporaries, Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), great advances in mathematics are seldom the work of single individuals. Cavalieri, Torricelli, Barrow, Descartes, Fermat, and Wallis had all paved the way to the threshold but had hesitated when it came to crossing it. By the second half of the seventeenth century, the raw materials lay at hand out of which the calculus would emerge. All that remained was for a Leibniz or a Newton to fuse these ideas in a tremendous synthesis. Newton’s well-known statement to Hooke, “If I have seen farther than others, it is because I have stood on the shoulders of giants,” shows his appreciation of this cumulative and progressive growth of mathematics.
Galileo’s Telescopic Observations

Probably no single figure from the 1600s is as well known as the mathematician-physicist-astronomer Galileo Galilei (1564–1642). His name is associated with events of profound significance: with the birth of modern science, with the Copernican revolution, with the dethronement of Aristotle as the supreme authority in the schools, and with the struggle against external restrictions on scientific inquiry. Galileo’s original intention was to enter the lucrative profession of medicine, and in 1581, he enrolled at the University of Pisa as a medical student. While a student at Pisa, Galileo is supposed to have made his first independent discovery, the isochronism (equality of time) of the pendulum. Tradition has it that this came about through his observation that a chandelier in the cathedral, set in motion while being lit, performed all its swings at equal intervals of time although its successive swings gradually grew narrower in amplitude.

Galileo’s formal introduction to mathematics came late. There is a story that Galileo, after listening at the door of a classroom in which Ostilio Ricci (a pupil of the famous Italian mathematician Tartaglia) was lecturing on the geometry of Euclid, became so fascinated with mathematics that he abandoned his medical plans. Indeed, he appears to have had little fondness for medicine and left the university in 1585 without a degree. Under the tutelage of Ricci, Galileo spent the next year pursuing the study of Euclid; he then went on to the other Greek geometers, winding up with the mechanical works of Archimedes. From 1585 to 1589, he earned money by giving private lessons in mathematics. Then, at the age of 25, Galileo succeeded in obtaining a lectureship at the University of Pisa. The appointment was only for three years, and the salary a mere pittance, but he gained academic standing.

Galileo is alleged to have performed, during his stay at Pisa, a public demonstration at the Leaning Tower to show that bodies of the same material but different weights fall with equal speed. This was an open challenge to the prevailing Aristotelian physics, according to which “the downward movement of a mass of gold or lead, or any body endowed with weight, is quicker in proportion to its size;” that is, the heavier the body, the faster the fall. Aristotelians claimed that the simultaneous arrival of the two weights—if the demonstration was actually carried out—was the effect of sorcery and not a refutation of Aristotle. They managed to pack the young professor’s lectures and hiss at his every word. Because he had aroused antagonism in the faculty, Galileo had little hope of reappointment at Pisa at the end of his three-year contract. He left, in 1592, to become professor of mathematics at the famed University of Padua, a post he held for the next 18 years.

In 1609, Galileo heard rumors that Dutch spectacle-makers had invented a remarkable contrivance for making distant objects appear quite close. Surmising how such a device, called a “telescope,” might be constructed, he set to work to fashion one for himself. (It is noteworthy that the Dutch instrument was of a totally different type from what Galileo designed.) Although Galileo by no means invented the telescope, he seems to have been the first to look at the sky systematically with one and to publish findings.

In a series of observations made in 1610, Galileo was able to distinguish the four satellites revolving about Jupiter—perhaps the most dramatic disproof of the Aristotelian view that the earth is at the center of all astronomical motions. Within a month he published this truly earth-shattering news in a 29-page booklet entitled the Sidereus Nuncius (The Starry Messenger), “unfolding great and marvelous sights” such as the existence of unknown stars,
the nature of the Milky Way, and the rugged surface of the moon. Such ideas were so disturbing that there were professors at Padua who refused to credit Galileo’s discoveries, refused even to look into his telescope for fear of seeing in it things that would discredit the infallibility of Aristotle and Ptolemy, and even the Church. His open publication of Copernican views made Galileo’s position as a teacher at Padua, a stronghold of Aristotelianism, untenable. Later in the year, he accepted an appointment as “First Mathematician” of the University of Pisa, and also the post of court mathematician to the Grand Duke of Tuscany.

At the beginning of the sixteenth century, most people still believed in the ancient description of the universe. As conceived by Aristotle and elaborated by Ptolemy, this system placed the earth at the center; and then at increasing distances from it came nine crystalline and concentric spheres. The first seven spheres carried the sun, the moon, and the five known planets, and the fixed stars were attached to the eighth one, often called the “firmament.” An elaborate theory of epicycles, deferents, equants, and eccentrics accounted for each planet’s motion within its own sphere. On the outside lay the ninth sphere, known as the “primum mobile” and representing the Prime Mover, or God; this was held to provide in some inexplicable fashion the motive power for all the others. Beyond this last sphere, there was nothing, no matter, no space, nothing at all. The Aristotelian universe was a finite one contained within the primum mobile.

From the standpoint of Aristotle, the earth was the main body in the universe, and everything else existed for its sake and the sake of its inhabitants. In the new cosmology produced by Nicolaus Copernicus (1473–1543), the sun changed places with the earth; the sun became the unique central body and the earth merely one of several planets revolving about the stationary sun. The ancient theory, because it made the earth the center, is known as the “geocentric theory,” and the Copernican, because it treated the sun as central, is called “heliocentric.” (In Greek, the words for “earth” and “sun” are ge and helios.) From the theory’s inception, theologians—both Protestant and Catholic—viewed with extreme dislike a theory in which the earth became a comparatively insignificant part of Creation. Had not God created the universe for man’s enjoyment and put the earth at the center to
prove this? Indeed the psalmist declared, in the ninety-third Psalm, “He hath made the round world so sure, that it cannot be moved.” Moving the earth was like displacing God’s throne.

Underlying the issue whether the Copernican pattern of celestial motion was physically correct was the matter of authority. Copernicanism was so incompatible with the traditional interpretation of various passages in the Bible that if it should prevail, it seemed that the Bible would lose authority, and Christianity would suffer. Besides, if freedom of judgment could be exercised to the extent of deciding between rival astronomical theories, it was but a short step to questioning authority itself.

Over the next several years, Galileo found himself involved in disputes about the relation of his astronomical views to the Bible. A question frequently raised to confound the adherents of the heliocentric theory was how to explain the “Miracle of Joshua.” The tenth chapter of the Book of Joshua relates that God, at Joshua’s prayer, made the sun stand still and lengthened the day so that the Israelites could pursue their enemies; had the sun gone down, the victory would not have been total. In several widely circulated letters (1613), Galileo maintained that “this passage shows manifestly the impossibility of the Aristotelian and Ptolemaic world systems and on the other hand accords very well with the Copernican.” With a brilliant dialectic turn, Galileo pointed out that if one accepted the traditional cosmology of Aristotle, the account of Joshua stopping the sun could not be understood literally. For it was admitted that the originative source of the sun’s motion, as well as that of the planets and stars, was the primum mobile. Therefore, if the whole of the heavenly movement was not to be disarranged, Joshua must have stopped not the sun but the outermost celestial sphere. On the other hand, by accepting Copernican theory, one could take the story literally. For if it was assumed that the revolution of the planets was impressed on them by the sun, which is in the center of the universe, then by stopping the sun Joshua was able to stop the whole solar system without disordering the other parts. Quite simply, if one were going to interpret scriptural language in its strict meaning, it would be better in this case to be a Copernican.

Galileo went on to state that the Holy Scriptures did not have as their aim the teaching of science, and that the words of the Bible were not to be taken literally. Where the sun was described as moving around the earth, this was a reflection of the incomplete knowledge of those times; certainly it was not meant as an endorsement of a given astronomical theory. He quoted “an ecclesiastic of the most eminent degree” who once said, “The intention of the Holy Ghost is to teach us not how the heavens go, but how to go to Heaven.” Further on, he added that “before a physical proposition is condemned, it must be shown to be not rigorously demonstrated.” In implying that it was the Church that ought to give scientific proof if Galileo were to be faulted, he provided exactly the opportunity his enemies wanted. They proclaimed everywhere that Galileo had assailed the authority of the Scriptures as a privileged source of knowledge and had tried, as an outsider, to meddle in religious matters. Mathematics was denounced from the pulpit as a devilish art and all mathematicians as enemies of the true religion.

Toward the beginning of 1616, the pope submitted the following two propositions to the Holy Office for examination: (1) “The sun is the center of the world and entirely motionless as regards spatial motion” and (2) “The earth is not the center of the world and is not motionless, but moves with regard to itself and in daily motion.” After a day’s deliberation, a special commission of theologians ruled that the first of these was “foolish and absurd,
philosophically and formally heretical, inasmuch as it expressly contradicts the doctrine of the Holy Scripture in many passages.” As for the second proposition, it could be “equally censured philosophically and was at least erroneous in faith.”

As an avowed protagonist of a moving earth, Galileo also had to be disciplined. He was summoned to the palace of Cardinal Bellarmine and in the presence of witnesses, admonished to “abstain altogether from teaching or defending this opinion and doctrine, and even from discussing it.” If he did not acquiesce, he was to be imprisoned. Galileo declared that he submitted. Immediately afterwards, the Holy Office proceeded against the writings of Copernicus. The work that had caused the upheaval in astronomical thought, the De Revolutionibus Orbium Coelestium (On the Revolution of the Heavenly Spheres, 1543), and all other texts that affirmed the earth’s motion were put on the Index of Prohibited Books “pending correction” of various passages.

Galileo was more or less silenced until 1623, when a new pope, a longtime admirer of his, was elected and took the name Urban VIII. After several friendly audiences, Urban granted Galileo permission to write about Copernicus’s theory provided that he represent it not as reality but as a convenient scientific hypothesis, and provided that arguments for the Ptolemaic view were given equal and impartial discussion. Galileo began work on a book that he believed would comply with these instructions. The writing went slowly, as Galileo was in ill health, but by 1630 he had finally completed the manuscript of the astronomical treatise that was to lead to his celebrated trial: Dialogo Sopra Due Massimi Sistemi del Mondo (Dialogue Concerning the Two Chief World Systems). After many arguments and delays in getting the necessary license for printing it, Galileo had the book approved by the Church authorities and it was published in Florence in 1632.

To reach the widest possible audience, Galileo wrote the Dialogue not in Latin, the academic language of the universities, but in Italian. It is addressed “To the Discerning Reader,” whom he wished to win over to his cause. The immediate response from the public was enormous, with the work sold out as it came off the presses. One reason is that the Dialogue was the most readable of the three great masterpieces of contemporary astronomical literature, of which the other two were the De Revolutionibus of Copernicus and the Principia of Newton. It was not a severely technical treatise like the others—mathematics would have been out of place in it—but a piece of brilliant polemic, directed at the clerical establishment. As far as the text itself was concerned, the Dialogue consisted of a lively conversation extending over four successive days in which three people discussed the arguments for and against Copernicanism, though coming to no definite conclusion. Of the three speakers, Salviati, the Copernican scholar, represented Galileo; Simplicio, the archetype of the bumbling Aristotelian philosopher, stood for authority; and Sagredo, the intelligent and cultivated layman, acted as moderator. The form of a dialogue was chosen partly for literary reasons, but still more because it would enable Galileo to claim that certain views expressed were not really his own but those of an imaginary character. Needless to say, the Aristotelian cause came out a miserable second best, as Simplicio was made to look foolish and forced to withdraw from the conversation.

With the publication of the Dialogue, Galileo’s enemies in science as well as in the Church redoubled their denunciation of him. An ecclesiastical commission that examined the work reported that he had transgressed orders by treating Copernicanism not as hypothesis but as fact. Moreover, their search of the Inquisition’s records for 1616 disclosed a notary’s unsigned statement to the effect that Galileo had been personally ordered not to