Following the ideas of Descartes and Fermat, we want to show that any quadratic equation in two variables represents an ellipse, hyperbola, parabola, two lines, double line, point, or empty set. We are not going to go through all these cases but explain the procedure. First, we discuss change of variables, say, \( x, y \) to \( u, v \) obtained by rotation:

\[
\begin{align*}
\quad x &= \cos \theta \cdot u - \sin \theta \cdot v \\
\quad y &= \sin \theta \cdot u + \cos \theta \cdot v
\end{align*}
\]

\[
\begin{align*}
\quad x &= r \cos (\varphi + \theta) \\
\quad y &= r \sin (\varphi + \theta)
\end{align*}
\]

\[
\begin{align*}
\quad x &= r \cos \theta \cdot \cos \varphi - \sin \theta \cdot \sin \varphi \\
\quad y &= r \sin \theta \cdot \cos \varphi + \cos \theta \cdot \sin \varphi
\end{align*}
\]

\[
\begin{align*}
\quad u &= r \cos \varphi \\
\quad v &= r \sin \varphi
\end{align*}
\]
Now, suppose we have a quadratic equation in $x, y$:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Suppose the cross term $Bxy \neq 0$. This is the case we want to apply rotation to eliminate it.

We want $B(\cos^2 \theta - \sin^2 \theta) = (A - C) 2\sin \theta \cos \theta$.

Thus, if we rotate by $\theta$, the new equation of the curve will not have the cross term. This is the first reduction.
Example:

\[ 4x^2 + 2\sqrt{3}xy + 2y^2 + 10\sqrt{3}x + 10y = 5 \]

\[
A = 4 \quad C = 2 \quad B = 2\sqrt{3}
\]

\[
\Rightarrow \cot(2\theta) = \frac{A - C}{B} = \frac{1}{\sqrt{3}}, \quad \Rightarrow 2\theta = \frac{\pi}{3}
\]

\[
\Rightarrow \theta = \frac{\pi}{6}
\]

Hence,

\[
x = \frac{\sqrt{3}}{2}u - \frac{1}{2}v = \frac{\sqrt{3}u - v}{2}
\]

\[
y = \frac{1}{2}u + \frac{\sqrt{3}}{2}v = \frac{u + \sqrt{3}v}{2}
\]

and

\[
4\left(\frac{\sqrt{3}u - v}{2}\right)^2 + 2\sqrt{3} \left(\frac{\sqrt{3}u - v}{2} \cdot \frac{u + \sqrt{3}v}{2}\right)
\]

\[
+ 2\left(\frac{u + \sqrt{3}v}{2}\right)^2 + 10\sqrt{3} \left(\frac{\sqrt{3}u - v}{2}\right) + 10 \frac{u + \sqrt{3}v}{2} = 5
\]

\[
5u^2 + v^2 + 20u = 5
\]

\[
5(u^2 + 4u + 4) + v^2 = 25 \quad \Rightarrow 5(u + 2)^2 + v^2 = 25
\]

\[
(x)
\]

\[
\left[ \frac{(u + 2)^2}{5} + \frac{v^2}{25} \right] = 1
\]

This is not in standard form yet, because of the translation \(u + 2\). But now it is easy. Write \(z = u + 2\). Then \((x)\) becomes

\[
(x*) \left[ \frac{z^2}{25} + \frac{v^2}{25} \right] = 1
\]

\[
\text{standard form}
\]

We see that the curve is an ellipse with semiaxes \(\sqrt{5}\) and 5.
Sometimes we don't have a cross term but still need to do translations to get the curve to standard form.

**Example:** \( x^2 - 2y^2 + 2x + 4y + 10 = 0 \)

We use completion of square method to get rid of the linear terms \( 2x \) and \( 4y \):

\[
(x+1)^2 - 2(y+1)^2 = -11
\]

\[
2(y+1)^2 - (x+1)^2 = 11
\]

\[
\frac{2}{11}(y+1)^2 - \frac{1}{11}(x+1)^2 = 1
\]

\[
\frac{(y+1)^2}{(\sqrt{\frac{11}{2}})^2} - \frac{(x+1)^2}{(\sqrt{11})^2} = 1
\]

**Standard form**

\[
\frac{y^2}{(\sqrt{\frac{11}{2}})^2} - \frac{x^2}{(\sqrt{11})^2} = 1
\]

This is a hyperbola.