11.1 Cartesian Coordinates in Three-Space

We have reached an important transition point in our study of calculus. Until now, we have been traveling across that broad expanse known as the Euclidean plane, or two-space. The concepts of calculus have been applied to functions of a single variable, functions whose graphs can be drawn in the plane. We are now going to study calculus in three dimensions. All the familiar ideas (such as limit, derivative, integral) are to be explored again from a loftier perspective.

To begin, consider three mutually perpendicular coordinate lines (the \( x \)-, \( y \)-, and \( z \)-axes) with their zero points at a common point \( O \), called the \textit{origin}. Although these lines can be oriented in any way one pleases, we follow a custom in thinking of the \( y \)- and \( z \)-axes as lying in the plane of the paper with their positive directions to the right and upward, respectively. The \( x \)-axis is then perpendicular to the paper, and we suppose its positive end to point toward us, thus forming a \textit{right-handed system}. We call it right-handed because, if the fingers of the right hand are curled so that they curve from the positive \( x \)-axis toward the positive \( y \)-axis, the thumb points in the direction of the positive \( z \)-axis (Figure 1).

The three axes determine three planes, the \( yz \), \( xz \), and \( xy \)-planes, which divide space into eight octants (Figure 2). To each point \( P \) in space corresponds an ordered triple of numbers \((x, y, z)\), its \textit{Cartesian coordinates}, which measure its directed distances from the three planes (Figure 3).

Plotting points in the first octant (the octant where all three coordinates are positive) is relatively easy. In Figures 4 and 5, we illustrate something more difficult by plotting two points from other octants, the points \( P(2, -3, 4) \) and \( Q(-3, 2, -5) \).
The Distance Formula. Consider two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in three-space $(x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2)$. They determine a parallelepiped (i.e., a rectangular box) with $P_1$ and $P_2$ as opposite vertices and with edges parallel to the coordinate axes (Figure 6). The triangles $P_1QP_2$ and $P_1RQ$ are right triangles and, by the Pythagorean Theorem,

$$|P_1P_2|^2 = |P_1Q|^2 + |QP_2|^2$$

and

$$|P_1Q|^2 = |P_1R|^2 + |RQ|^2$$

Thus,

$$|P_1P_2|^2 = |P_1R|^2 + |RQ|^2 + |QP_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

This gives us the Distance Formula in three-space, which applies even if some coordinates are identical.

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Example 1.** Find the distance between the points $P(2, -3, 4)$ and $Q(-3, 2, -5)$, which were plotted in Figures 4 and 5.

**Solution**

$$|PQ| = \sqrt{(-3 - 2)^2 + (2 + 3)^2 + (-5 - 4)^2} = \sqrt{131} \approx 11.45$$

Spheres and Their Equations. It is a small step from the Distance Formula to the equation of a sphere. By a sphere, we mean the set of all points in three-dimensional space that are a constant distance (the radius) from a fixed point (the center). (Recall that a circle is defined as the set of points in a plane that are a constant distance from a fixed point.) In fact, if $(x, y, z)$ is a point on the sphere of radius $r$ centered at $(h, k, l)$, then (see Figure 7)

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

**What Is a Sphere?**

We have defined a sphere to be the set of points a given distance away from some point, that is, those points $(x, y, z)$ satisfying $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$. Just as by “circle” we sometimes mean the points on and inside the circle’s boundary (e.g., when we talk about the “area” of a circle being $\pi r^2$), there are times when by “sphere” we mean the boundary together with the interior. (This is sometimes called a ball or a solid sphere.) In other words, we sometimes mean the set of points satisfying $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$. When we say that the volume of a sphere is $\frac{4}{3} \pi r^3$, we of course mean this latter interpretation. The context of a problem will usually dictate which “sphere” we are talking about.

In expanded form, the boxed equation may be written as

$$x^2 + y^2 + z^2 + Gx + Hy + Ix + J = 0$$

Conversely, the graph of any equation of this form is either a sphere, a point (a degenerate sphere), or the empty set. To see why, consider the following example.
EXAMPLE 2  Find the center and radius of the sphere with equation
\[ x^2 + y^2 + z^2 - 10x - 8y - 12z + 68 = 0 \]
and sketch its graph.

SOLUTION  We use the process of completing the square.
\[
\begin{align*}
(x^2 - 10x) + (y^2 - 8y) + (z^2 - 12z) &= -68 \\
(x^2 - 10x + 25) + (y^2 - 8y + 16) + (z^2 - 12z + 36) &= -68 + 25 + 16 + 36 \\
(x - 5)^2 + (y - 4)^2 + (z - 6)^2 &= 9
\end{align*}
\]
Thus, the equation represents a sphere with center at (5, 4, 6) and radius 3. Its graph is shown in Figure 8.

If, after completing the square in Example 2, the equation had been
\[(x - 5)^2 + (y - 4)^2 + (z - 6)^2 = 0\]
then the graph would be the single point (5, 4, 6); if the right side were negative, the graph would be the empty set.

Another simple result that follows from the Distance Formula is the Midpoint Formula. If \(P_1(x_1, y_1, z_1)\) and \(P_2(x_2, y_2, z_2)\) are end points of a line segment, then the midpoint \(M(m_1, m_2, m_3)\) has coordinates
\[
m_1 = \frac{x_1 + x_2}{2}, \quad m_2 = \frac{y_1 + y_2}{2}, \quad m_3 = \frac{z_1 + z_2}{2}
\]
In other words, to find the coordinates of the midpoint of a segment, simply take the average of corresponding coordinates of the end points.

EXAMPLE 3  Find the equation of the sphere that has the line segment joining \((-1, 2, 3)\) and \((5, -2, 7)\) as a diameter (Figure 9).

SOLUTION  The center of this sphere is at the midpoint of the segment, that is, at \((2, 0, 5)\); the radius \(r\) satisfies
\[
r^2 = (5 - 2)^2 + (-2 - 0)^2 + (7 - 5)^2 = 17
\]
We conclude that the equation of the sphere is
\[(x - 2)^2 + y^2 + (z - 5)^2 = 17\]

Graphs in Three-Space  It was natural to consider a quadratic equation first because of its relation to the Distance Formula. But, presumably, a linear equation in \(x, y,\) and \(z\), that is, an equation of the form
\[Ax + By + Cz = D, \quad A^2 + B^2 + C^2 \neq 0\]
should be even easier to analyze. (Note that \(A^2 + B^2 + C^2 \neq 0\) is a compact way of saying that \(A, B,\) and \(C\) are not all zero.) As a matter of fact, we will show in Section 11.3 that the graph of a linear equation is a plane. Taking this for granted for now, let’s consider how we might graph such an equation.

If, as will often be the case, the plane intersects the three axes, we begin by finding these intersection points: that is, we find the \(x, y,\) and \(z\)-intercepts. These three points determine the plane and allow us to draw the (coordinate-plane) traces, which are the lines of intersection of that plane with the coordinate planes. Then, with just a bit of artistry, we can shade in the plane.
**Example 4** Sketch the graph of $3x + 4y + 2z = 12$.

**Solution** To find the $x$-intercept, set $y$ and $z$ equal to zero and solve for $x$, obtaining $x = 4$. The corresponding point is $(4, 0, 0)$. Similarly, the $y$- and $z$-intercepts are $(0, 3, 0)$ and $(0, 0, 6)$. Next, connect these points by line segments to get the traces. Then shade in (the first octant part of) the plane, thereby obtaining the result shown in Figure 10.

What if the plane does not intersect all three axes? This will happen, for example, if one of the variables in the equation of the plane is missing (i.e., has a zero coefficient).

**Example 5** Sketch the graph of the linear equation

$$2x + 3y = 6$$

in three-space.

**Solution** The $x$- and $y$-intercepts are $(3, 0, 0)$ and $(0, 2, 0)$, respectively, and these points determine the trace in the $xy$-plane. The plane never crosses the $z$-axis ($x$ and $y$ cannot both be 0), and so the plane is parallel to the $z$-axis. We have sketched the graph in Figure 11.

Notice that in each of our examples the graph of an equation in three-space was a surface. This contrasts with the two-space case, where the graph of an equation was usually a curve. We will have a good deal more to say about graphing equations and the corresponding surfaces in Section 11.8.

**Curves in Three-Space** We saw parametrized curves in the plane in Section 5.4. This concept generalizes easily to three dimensions. A curve in three-space is determined by the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t); \quad a \leq t \leq b$$

We say that a curve is **smooth** if $f'(t)$, $g'(t)$, and $h'(t)$ exist and are not simultaneously zero.

The concept of arc length also generalizes easily to curves in three-space. For the parametric curve defined above, the arc length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt$$

**Example 6** An object’s position at time $t$ is given by the parametrically defined curve $x = \cos t, y = \sin t, z = t/\pi$ for $0 \leq t \leq 2\pi$. Sketch this curve and find its arc length.

**Solution** We begin by making a table of values of $t, x, y,$ and $z$; then we connect the dots in three-space; the curve is shown in Figure 12. The arc length is

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1/\pi)^2} \, dt$$

$$\quad = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1/\pi^2} \, dt$$

$$\quad = \int_0^{2\pi} \sqrt{1 + 1/\pi^2} \, dt$$

$$\quad = 2\pi \sqrt{1 + 1/\pi^2}$$
The curve in Example 6 is called a helix. Notice that if we ignore (for a moment) the motion in the z-dimension, the object is in uniform circular motion. Introducing back the motion in the z-dimension, which is up with constant speed, we see that the object is going around and around as it moves upward, much like a spiral staircase.

Here is another way to obtain the length of this curve. The helix lies entirely on the surface of a right circular cylinder as shown in Figure 13. Now imagine that the cylinder is cut as indicated and that the cylinder is "peeled" back to make a rectangle. The helix will become a diagonal of the rectangle, so it will have length \( \sqrt{4 + 4\pi^2} = \sqrt{4\pi^2(1 + 1/\pi^2)} = 2\pi \sqrt{1 + 1/\pi^2} \).

Concepts Review

1. The numbers \( x, y, \) and \( z \) in \((x, y, z)\) are called the ______ of a point in three-space.
2. The distance between the points \((-1, 3, 5)\) and \((x, y, z)\) is ______.
3. The equation \((x + 1)^2 + (y - 3)^2 + (z - 5)^2 = 16\) determines a sphere with center ______ and radius ______.
4. The graph of \(3x - 2y + 4z = 12\) is a ______ with x-intercept ______, y-intercept ______, and z-intercept ______.

Problem Set 11.1

1. Plot the points whose coordinates are \((1, 2, 3), (2, 0, 1), (-2, 4, 5), (0, 3, 0), \) and \((-1, -2, -3)\). If appropriate, show the “box” as in Figures 4 and 5.
2. Follow the directions of Problem 1 for \((\sqrt{3}, -3, 3), (0, \pi, -3), (-2, 1/2), \) and \((0, 0, e)\).
3. What is peculiar to the coordinates of all points in the yz-plane? On the z-axis?
4. What is peculiar to the coordinates of all points in the xz-plane? On the y-axis?
5. Find the distance between the following pairs of points.
   (a) \((-6, -1, 0)\) and \((1, 2, 3)\)
   (b) \((-2, -2, 0)\) and \((2, -2, -3)\)
   (c) \((e, \pi, 0)\) and \((-\pi, -4, \sqrt{3})\)
6. Show that \((4, 5, 3), (1, 7, 4), \) and \((2, 4, 6)\) are vertices of an equilateral triangle.
7. Show that \((2, 1, 6), (4, 7, 9), \) and \((8, 5, -6)\) are vertices of a right triangle. *Hint: Only right triangles satisfy the Pythagorean Theorem.*
8. Find the distance from \((2, 3, -1)\) to
   (a) the xy-plane, (b) the y-axis, and (c) the origin.
9. A rectangular box has its faces parallel to the coordinate planes and has $(2,3,4)$ and $(6,-1,0)$ as the end points of a main diagonal. Sketch the box and find the coordinates of all eight vertices.

10. $P(x,5,z)$ is on a line through $Q(2,-4,3)$ that is parallel to one of the coordinate axes. Which axis must it be and what are $x$ and $z$?

11. Write the equation of the sphere with the given center and radius.
   (a) $(1,2,3); 5$ (b) $(-2,3,-6); \sqrt{5}$
   (c) $(\pi, e, \sqrt{2}); \sqrt{\pi}$

12. Find the equation of the sphere whose center is $(2,4,5)$ and that is tangent to the $xy$-plane.

In Problems 13–16, complete the squares to find the center and radius of the sphere whose equation is given (see Example 2).

13. $x^2 + y^2 + z^2 - 12x + 14y - 8z + 1 = 0$
14. $x^2 + y^2 + z^2 + 2x - 6y - 10z + 34 = 0$
15. $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 16z - 13 = 0$
16. $x^2 + y^2 + z^2 + 8x - 4y - 22z + 77 = 0$

In Problems 17–24, sketch the graphs of the given equations. Begin by sketching the traces in the coordinate planes (see Examples 4 and 5).
17. $2x + 6y + 3z = 12$
18. $3x - 4y + 2z = 24$
19. $x + 3y - 2z = 6$
20. $-3x + 2y + z = 6$
21. $x + 3y = 8$
22. $3x + 4z = 12$
23. $x^2 + y^2 + z^2 = 9$
24. $(x - 2)^2 + y^2 + z^2 = 4$

In Problems 25–32, find the arc length of the given curve.

25. $x = t, y = t, z = 2t; 0 \leq t \leq 2$
26. $x = t/4, y = t/3, z = t/2; 1 \leq t \leq 3$
27. $x = t^{1/2}, y = 3t, z = 4t; 1 \leq t \leq 4$
28. $x = t^{1/2}, y = t^{1/2}, z = t; 2 \leq t \leq 4$
29. $x = t^2, y = (4/3)t^{3/2}, z = t; 0 \leq t \leq 8$
30. $x = t^2, y = 4\sqrt{3}t^{3/2}, z = 3t; 1 \leq t \leq 4$
31. $x = 2\cos t, y = 2\sin t, z = 3t; -\pi \leq t \leq \pi$
32. $x = 2\cos t, y = 2\sin t, z = t/20; 0 \leq t \leq 8\pi$

CAS In Problems 33–36, set up a definite integral for the arc length of the given curve. Use the Parabolic Rule with $n = 10$ or a CAS to approximate the integral.

33. $x = \sqrt{t}, y = t, z = e^t; 1 \leq t \leq 6$
34. $x = t, y = t^2, z = t^3; 1 \leq t \leq 2$
35. $x = 2\cos t, y = \sin t, z = c; 0 \leq t \leq 6\pi$
36. $x = \sin t, y = \cos t, z = \sin \pi; 0 \leq t \leq 2\pi$

37. Find the equation of the sphere that has the line segment joining $(-2,3,6)$ and $(4,-1,5)$ as a diameter (see Example 3).

38. Find the equations of the tangent spheres of equal radii whose centers are $(-3,1,2)$ and $(5,-3,6)$.

39. Find the equation of the sphere that is tangent to the three coordinate planes if its radius is 6 and its center is in the first octant.

40. Find the equation of the sphere with center $(1,1,4)$ that is tangent to the plane $x + y = 12$.

41. Describe the graph in three-space of each equation.
   (a) $z = 2$ (b) $x = y$
   (c) $xy = 0$ (d) $xyz = 0$
   (e) $x^2 + y^2 = 4$ (f) $z = \sqrt{9 - x^2 - y^2}$

42. The sphere $(x - 1)^2 + (y + 2)^2 + (z + 1)^2 = 10$ intersects the plane $z = 2$ in a circle. Find the circle’s center and radius.

43. An object’s position $P$ changes so that its distance from $(1,2,3)$ is always twice its distance from $(1,2,3)$. Show that $P$ is on a sphere and find its center and radius.

44. An object’s position $P$ changes so that its distance from $(1,2,3)$ always equals its distance from $(2,3,2)$. Find the equation of the plane on which $P$ lies.

45. The solid spheres $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 \leq 4$ and $(x - 2)^2 + (y - 1)^2 + (z - 3)^2 \leq 4$ intersect in a solid. Find its volume.

46. Do Problem 45 assuming that the second solid sphere is $(x - 2)^2 + (y - 4)^2 + (z - 3)^2 \leq 9$.

CAS 47. The curve defined by $x = a\cos t, y = a\sin t, z = c$ is a helix. Hold $a$ fixed and use a CAS to obtain a parametric plot of the helix for various values of $c$. What effect does $c$ have on the curve?

CAS 48. For the helix described in Problem 47, hold $c$ fixed and use a CAS to obtain a parametric plot for various values of $a$. What effect does $a$ have on the curve?

**Answers to Concepts Review:**
1. coordinates
2. $\sqrt{(x+1)^2 + (y-3)^2 + (z-5)^2}$
3. $(-1,3,5); 4$
4. plane: $-6; 3$

### 11.2 Vectors

Many quantities that occur in science (e.g., length, mass, volume, and electric charge) can be specified by giving a single number. These quantities (and the numbers that measure them) are called **scalars**. Other quantities, such as velocity, force, torque, and displacement, require both a magnitude and a direction for complete specification. We call such quantities **vectors** and represent them by arrows (directed line segments). The length of the arrow represents the **magnitude**, or length, of the vector; its direction is the **direction** of the vector. The vector in Figure 1 has length 2.3 units and direction 30° north of east (or 30° from the positive $x$-axis).

Arrows that we draw, like those shot from a bow, have two ends. There is the feather end (the initial point), called the **tail**, and the pointed end (the terminal...
point), called the **head**, or tip (Figure 2). Two vectors are considered to be **equivalent** if they have the same magnitude and direction (Figure 3). We shall symbolize vectors by boldface letters, such as \( \mathbf{u} \) and \( \mathbf{v} \). Since this is hard to accomplish in normal writing, you might use \( \hat{u} \) and \( \hat{v} \). The magnitude, or length, of a vector \( \mathbf{u} \) is symbolized by \( |\mathbf{u}| \).

In general, we think of vectors as being three-dimensional; that is, their initial and terminal points are points in three-space. There are many applications, however, where the vectors lie entirely in the xy-plane. The context of a problem should indicate whether the vectors are two- or three-dimensional.

**Operations on Vectors** To find the sum, or **resultant**, of \( \mathbf{u} \) and \( \mathbf{v} \), move \( \mathbf{v} \) without changing its magnitude or direction until its tail coincides with the head of \( \mathbf{u} \). Then \( \mathbf{u} + \mathbf{v} \) is the vector connecting the tail of \( \mathbf{u} \) to the head of \( \mathbf{v} \). This method (called the **Triangle Law**) is illustrated in the left half of Figure 4.

As an alternative way to find \( \mathbf{u} + \mathbf{v} \), move \( \mathbf{v} \) so that its tail coincides with that of \( \mathbf{u} \). Then \( \mathbf{u} + \mathbf{v} \) is the vector with this common tail and coinciding with the diagonal of the parallelogram that has \( \mathbf{u} \) and \( \mathbf{v} \) as sides. This method (called the **Parallelogram Law**) is illustrated on the right in Figure 4.

These two methods are equivalent ways to define what we mean by the sum of two vectors. You should convince yourself that vector addition is commutative and associative; that is,

\[
\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \\
(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})
\]

If \( \mathbf{u} \) is a vector, then \( 3\mathbf{u} \) is the vector with the same direction as \( \mathbf{u} \) but three times as long; \( -2\mathbf{u} \) is twice as long but oppositely directed (Figure 5). In general, \( c\mathbf{u} \), called a **scalar multiple** of \( \mathbf{u} \), has magnitude \( |c| \) times that of \( \mathbf{u} \) and is similarly or oppositely directed, depending on whether \( c \) is positive or negative. In particular, \( (-1)\mathbf{u} \) (usually written \( -\mathbf{u} \)) has the same length as \( \mathbf{u} \), but opposite direction. It is called the **negative** of \( \mathbf{u} \) because, when we add it to \( \mathbf{u} \), the result is a vector that is nothing more than a point. This latter vector (the only vector without a well-defined direction) is called the **zero vector** and is denoted by \( \mathbf{0} \). It is the identity element for addition; that is, \( \mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \). Finally, subtraction is defined by

\[
\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})
\]

**EXAMPLE 1** In Figure 6, express \( \mathbf{w} \) in terms of \( \mathbf{u} \) and \( \mathbf{v} \).

**SOLUTION** Since \( \mathbf{u} + \mathbf{v} = \mathbf{w} \), it follows that

\[
\mathbf{w} = \mathbf{v} - \mathbf{u}
\]

If \( P \) and \( Q \) are points in the plane, then \( \overrightarrow{PQ} \) denotes the vector with tail at \( P \) and head at \( Q \).

**EXAMPLE 2** In Figure 7, \( \overrightarrow{AB} = \frac{1}{3} \overrightarrow{AC} \). Express \( \mathbf{m} \) in terms of \( \mathbf{u} \) and \( \mathbf{v} \).
SOLUTION

\[
m = \mathbf{u} + \frac{1}{2} \mathbf{A} = \mathbf{u} + \frac{1}{2} \mathbf{A} = \mathbf{u} + \frac{1}{2} (\mathbf{v} - \mathbf{u}) = \frac{1}{2} \mathbf{u} + \frac{1}{2} \mathbf{v}
\]

More generally, if \( \mathbf{A} = t \mathbf{A} \), where \( 0 < t < 1 \), then

\[
m = (1 - t) \mathbf{u} + t \mathbf{v}
\]

The expression just obtained for \( m \) can also be written as

\[
\mathbf{u} + t(\mathbf{v} - \mathbf{u})
\]

If we allow \( t \) to range over all scalars, we obtain the set of all vectors with tails at the same point as the tail of \( \mathbf{u} \) and heads on the line \( \ell \) (see Figure 8). This fact will be important to us later in describing lines using vector language.

An Application
A force has both a magnitude and a direction. If two forces \( \mathbf{u} \) and \( \mathbf{v} \) act at a point, the resultant force at the point is the vector sum of the two forces.

EXAMPLE 3
A weight of 200 newtons is supported by two wires, as shown in Figure 9. Find the magnitude of the tension in each wire.

SOLUTION
All forces are in one plane, so the vectors in this problem are two-dimensional. The weight \( \mathbf{w} \) and the two tensions \( \mathbf{u} \) and \( \mathbf{v} \) are forces that behave as vectors (Figure 10). Each of these vectors can be expressed as a sum of a horizontal and a vertical component. The weight is in equilibrium, so (1) the magnitude of the leftward force must equal the magnitude of the rightward force, and (2) the magnitude of the upward force must equal the magnitude of the downward force. In other words, the net force is zero. Thus,

(1)

\[
||\mathbf{u}|| \cos 33^\circ = ||\mathbf{v}|| \cos 50^\circ
\]

(2)

\[
||\mathbf{u}|| \sin 33^\circ + ||\mathbf{v}|| \sin 50^\circ = ||\mathbf{w}|| = 200
\]

When we solve (1) for \( ||\mathbf{v}|| \) and substitute in (2), we get

\[
||\mathbf{u}|| \sin 33^\circ + \frac{||\mathbf{u}|| \cos 33^\circ}{\cos 50^\circ} \sin 50^\circ = 200
\]

or

\[
||\mathbf{u}|| = \frac{200}{\sin 33^\circ + \cos 33^\circ \tan 50^\circ} \approx 129.52 \text{ newtons}
\]

Then

\[
||\mathbf{v}|| = \frac{||\mathbf{u}|| \cos 33^\circ}{\cos 50^\circ} \approx \frac{129.52 \cos 33^\circ}{\cos 50^\circ} \approx 168.99 \text{ newtons}
\]

Algebraic Approach to Vectors
For a given vector \( \mathbf{u} \) in the plane we choose as its representative the arrow that has its tail at the origin (Figure 11). This arrow is uniquely determined by the coordinates \( u_1 \) and \( u_2 \) of its head; that is, the vector \( \mathbf{u} \) is completely described by the ordered pair \((u_1, u_2)\). The numbers \( u_1 \) and \( u_2 \) are called the components of the vector \( \mathbf{u} \). We write \((u_1, u_2)\) rather than \((\mathbf{u}, \mathbf{u})\) to distinguish the vector originating at the origin and terminating at the point with coordinates \( u_1 \) and \( u_2 \) from the point having coordinates \( u_1 \) and \( u_2 \).

For vectors in three-space, the generalization is straightforward. We represent the vector by an arrow starting at the origin and terminating at the point with coordinates \( u_1, u_2, u_3 \) and we denote this vector by \((u_1, u_2, u_3)\) (Figure 12). In the remainder of this section, we develop the properties of vectors in three dimensions; the results for vectors in two dimensions should be obvious.

The vectors \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) are equal if and only if the corresponding components are equal; that is, \( u_1 = v_1, u_2 = v_2, \) and \( u_3 = v_3 \). To multiply a vector \( \mathbf{u} \) by a scalar \( c \), we multiply each component by \( c \); that is,

\[
c \mathbf{u} = c \mathbf{u} = (cu_1, cu_2, cu_3)
\]
The notation $-\mathbf{u}$ indicates the vector $(-1)\mathbf{u} = (-u_1, -u_2, -u_3)$. The vector with all components equal to zero is called the zero vector; that is $\mathbf{0} = (0, 0, 0)$. The sum of the two vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

The vector $\mathbf{u} - \mathbf{v}$ is defined to be

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3)$$

Figure 13 indicates that these definitions are equivalent to the geometric ones given earlier in this section.

Three special vectors in three-space are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. These are called the standard unit vectors, or basis vectors. Every vector $\mathbf{u} = (u_1, u_2, u_3)$ can be written in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ as follows:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

The magnitude of a vector is just the length of the arrow that represents it. If the arrow begins at the origin and ends at $(u_1, u_2, u_3)$, then its length can be readily determined from the distance formula:

$$|\mathbf{u}| = \sqrt{(u_1 - 0)^2 + (u_2 - 0)^2 + (u_3 - 0)^2} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Just as $|c|$ gives the distance from the origin to a point $c$ on the number line, $|\mathbf{u}|$ gives the distance from the origin to the point in space whose ordered triple is $\mathbf{u} = (u_1, u_2, u_3)$ (Figure 14). Using this algebraic interpretation of vectors, the following rules for operating with vectors can be easily established.

**Theorem A**

For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and any scalars $a$ and $b$, the following relationships hold.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $a(b\mathbf{u}) = (ab)\mathbf{u}$
6. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
7. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$
8. $\mathbf{u} - \mathbf{u} = \mathbf{0}$
9. $|a\mathbf{u}| = |a||\mathbf{u}|$
Proof. We illustrate the proof by demonstrating Rules 6 and 9 for the case of three-dimensional vectors.

\[ a(u + v) = a((u_1, u_2, u_3) + (v_1, v_2, v_3)) \]
\[ = a(u_1 + v_1, u_2 + v_2, u_3 + v_3) \]
\[ = (a(u_1 + v_1), a(u_2 + v_2), a(u_3 + v_3)) \]
\[ = (au_1 + av_1, au_2 + av_2, au_3 + av_3) \]
\[ = (au_1, au_2, au_3) + (av_1, av_2, av_3) \]
\[ = a(u_1, u_2, u_3) + a(v_1, v_2, v_3) \]
\[ = au + av \]

This proves Rule 6. Now, for Rule 9,
\[ |au| = ||(au_1, au_2, au_3)|| \]
\[ = \sqrt{(au_1)^2 + (au_2)^2 + (au_3)^2} \]
\[ = \sqrt{a^2(u_1^2 + u_2^2 + u_3^2)} \]
\[ = a\sqrt{u_1^2 + u_2^2 + u_3^2} = |a| |u| \]

**Example 4** Let \( u = (1, 1, 2) \) and \( v = (0, -1, 2) \). Find (a) \( u + v \), and (b) \( u - 2v \), and express them in terms of \( i, j, \) and \( k \). Find (c) \( |u| \), and (d) \( ||u|| \).

**Solution**
(a) \( u + v = (1, 1, 2) + (0, -1, 2) = (1 + 0, 1 - 1, 2 + 2) \)
\[ = (1, 0, 4) = i + 0j + 4k \]
(b) \( u - 2v = (1, 1, 2) - 2(0, -1, 2) = (1 - 2(0), 1 - 2(-1), 2 - 2(2)) \)
\[ = (1, 3, -2) = i + 3j - 2k \]
(c) \( |u| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6} \)
(d) \( ||u|| = ||(1, 1, 2)|| = 3\sqrt{6} \)

**Definition of a Unit Vector**
A vector having length one is called a unit vector.

**Example 5** Let \( v = (4, -3) \). Find \( |v| \), and find a unit vector \( u \) with the same direction as \( v \).

**Solution** In this problem, all vectors are two-dimensional. The length, or magnitude, of \( v \) is \( |v| = \sqrt{4^2 + (-3)^2} = 5 \). To find \( u \), we divide \( v \) by its length \( |v| \); that is,
\[ u = \frac{v}{|v|} = \frac{(4, -3)}{\sqrt{4^2 + (-3)^2}} = \frac{(4, -3)}{5} = \frac{4}{5} \hat{i} - \frac{3}{5} \hat{j} \]
The length of \( u \) is then
\[ |u| = \left| \frac{v}{|v|} \right| = \frac{1}{|v|} |v| = \frac{1}{|v|} |v| = 1 \]

**Concepts Review**
1. Vectors are distinguished from scalars in that vectors have both ______ and ______.
2. Two vectors are considered to be equivalent if ______.
3. If the tail of \( v \) coincides with the head of \( u \), then \( u + v \) is the vector with tail at ______ and head at ______.
4. The vector \( u = (6, 3, 3) \) has length ______ times that of the vector \( u = (2, 1, 1) \).
Problem Set 11.2

In Problems 1–4, draw the vector \( \mathbf{w} \).

1. \( \mathbf{w} = \mathbf{u} + \tfrac{1}{2} \mathbf{v} \)

2. \( \mathbf{w} = 2 \mathbf{u} - 3 \mathbf{v} \)

3. \( \mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \)

4. \( \mathbf{w} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \)

5. Figure 15 is a parallelogram. Express \( \mathbf{w} \) in terms of \( \mathbf{u} \) and \( \mathbf{v} \).

6. In the large triangle of Figure 16, \( \mathbf{m} \) is a median (it bisects the side to which it is drawn). Express \( \mathbf{m} \) and \( \mathbf{n} \) in terms of \( \mathbf{u} \) and \( \mathbf{v} \).

7. In Figure 17, \( \mathbf{w} = - (\mathbf{u} + \mathbf{v}) \) and \( ||\mathbf{w}|| = ||\mathbf{v}|| = 1 \). Find \( ||\mathbf{w}|| \).

8. Do Problem 7 if the top angle is 90° and the two side angles are each 135°.

For the two-dimensional vectors \( \mathbf{u} \) and \( \mathbf{v} \) in Problems 9–12, find the sum \( \mathbf{u} + \mathbf{v} \), the difference \( \mathbf{u} - \mathbf{v} \), and the magnitudes \( ||\mathbf{u}|| \) and \( ||\mathbf{v}|| \).

9. \( \mathbf{u} = (-1, 0), \mathbf{v} = (3, 4) \)

10. \( \mathbf{u} = (0, 0), \mathbf{v} = (-3, 4) \)

11. \( \mathbf{u} = (12, 12), \mathbf{v} = (-2, 2) \)

12. \( \mathbf{u} = (-0.2, 0.8), \mathbf{v} = (-2.1, 1.3) \)

For the three-dimensional vectors \( \mathbf{u} \) and \( \mathbf{v} \) in Problems 13–16, find the sum \( \mathbf{u} + \mathbf{v} \), the difference \( \mathbf{u} - \mathbf{v} \), and the magnitudes \( ||\mathbf{u}|| \) and \( ||\mathbf{v}|| \).

13. \( \mathbf{u} = (-1, 0, 0), \mathbf{v} = (3, 4, 0) \)

14. \( \mathbf{u} = (0, 0, 0), \mathbf{v} = (-3, 3, 1) \)

15. \( \mathbf{u} = (-1, 0, 1), \mathbf{v} = (-5, 0, 0) \)

16. \( \mathbf{u} = (0.3, 0.3, 0.5), \mathbf{v} = (2.2, 1.3, -0.9) \)

- 17. In Figure 18, forces \( \mathbf{u} \) and \( \mathbf{v} \) each have magnitude 50 pounds. Find the magnitude and direction of the force \( \mathbf{w} \) needed to counterbalance \( \mathbf{u} \) and \( \mathbf{v} \).

- 18. Mark pushes on a post in the direction S 30° E (30° east of south) with a force of 60 pounds. Dan pushes on the same post in the direction S 60° W with a force of 80 pounds. What are the magnitude and direction of the resultant force?

- 19. A 300-newton weight rests on a smooth (friction negligible) inclined plane that makes an angle of 30° with the horizontal. What force parallel to the plane will just keep the weight from sliding down the plane? Hint: Consider the downward force of 300 newtons to be the sum of two forces, one parallel to the plane and one perpendicular to it.

- 20. An object weighing 258.5 pounds is held in equilibrium by two ropes that make angles of 27.34° and 39.22° respectively, with the vertical. Find the magnitude of the force exerted on the object by each rope.

- 21. A wind with velocity 45 miles per hour is blowing in the direction N 20° W. An airplane that flies at 425 miles per hour in still air is supposed to fly straight north. How should the airplane be headed and how fast must it then be flying with respect to the ground?

- 22. A ship is sailing due south at 20 miles per hour. A man walks west (i.e., at right angles to the side of the ship) across the deck at 3 miles per hour. What are the magnitude and direction of his velocity relative to the surface of the water?

- 23. Julie, flying in a wind blowing 40 miles per hour due south, discovers that she is heading due east when she points her airplane in the direction N 60° E. Find the airspeed (speed in still air) of the plane.

- 24. What heading and airspeed are required for an airplane to fly 837 miles per hour due north if a wind of 65 miles per hour is blowing in the direction S 11.5° E?

- 25. Prove all parts of Theorem A for the case of two-dimensional vectors.

- 26. Prove parts 1–5 and 7–8 of Theorem A for the case of three-dimensional vectors.

- 27. Prove, using vector methods, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side.

- 28. Prove that the midpoints of the four sides of an arbitrary quadrilateral are the vertices of a parallelogram.
29. Let $v_1, v_2, \ldots, v_n$ be the edges of a polygon arranged in cyclic order as shown for the case $n = 7$ in Figure 19. Show that $v_1 + v_2 + \cdots + v_n = \mathbf{0}$.

30. Let $n$ points be equally spaced on a circle, and let $v_1, v_2, \ldots, v_n$ be the vectors from the center of the circle to these $n$ points. Show that $v_1 + v_2 + \cdots + v_n = \mathbf{0}$.

31. Consider a horizontal triangular table with each vertex angle less than $120^\circ$. At the vertices are frictionless pulleys over which pass strings knotted at $P$, each with a weight $W$ attached as shown in Figure 20. Show that at equilibrium the three angles at $P$ are equal; that is, show that $a + \beta = \alpha + \gamma = \beta + \gamma = 120^\circ$.

32. Show that the point $P$ of the triangle of Problem 31 that minimizes $|AP| + |BP| + |CP|$ is the point where the three angles at $P$ are equal. Hint: Let $A', B'$, and $C'$ be the points where the weights are attached. The center of gravity is then located $\frac{1}{3}(|AA'| + |BB'| + |CC'|)$ units below the plane of the triangle. The system is in equilibrium when the center of gravity of the three weights is lowest.

33. Let the weights at $A$, $B$, and $C$ of Problem 31 be $3w$, $4w$, and $5w$, respectively. Determine the three angles at $P$ at equilibrium. What geometric quantity (as in Problem 32) is now minimized?

34. A company will build a plant to manufacture refrigerators to be sold in cities $A$, $B$, and $C$ in quantities $a$, $b$, and $c$, respectively, each year. Where is the best location for the plant, that is, the location that will minimize delivery costs (see Problem 33)?

35. A 100-pound chandelier is held in place by four wires attached to the ceiling at the four corners of a square. Each wire makes an angle of $45^\circ$ with the horizontal. Find the magnitude of the tension in each wire.

36. Repeat Problem 35 for the case where there are three wires attached to the ceiling at the three corners of an equilateral triangle.

Answers to Concepts Review:

1. magnitude, direction
2. they have the same magnitude and direction
3. the tail of $\mathbf{u}$
4. 3

11.3 The Dot Product

We have discussed scalar multiplication, that is, the multiplication of a vector $\mathbf{u}$ by a scalar $c$. The result $c\mathbf{u}$ is always a vector. Now we introduce a multiplication for two vectors $\mathbf{u}$ and $\mathbf{v}$. It is called the dot product, or scalar product, and is symbolized by $\mathbf{u} \cdot \mathbf{v}$. We define it for two-dimensional vectors as

$$\mathbf{u} \cdot \mathbf{v} = (u_1, u_2) \cdot (v_1, v_2) = u_1v_1 + u_2v_2$$

and for three-dimensional vectors as

$$\mathbf{u} \cdot \mathbf{v} = (u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3$$

**Example 1**: Let $\mathbf{u} = (0, 1, 1)$, $\mathbf{v} = (2, -1, 1)$, and $\mathbf{w} = (6, -3, 3)$. Compute each of the following if they are defined: (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{v} \cdot \mathbf{u}$, (c) $\mathbf{v} \cdot \mathbf{w}$, (d) $\mathbf{u} \cdot \mathbf{u}$, and (e) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$.

**Solution**

(a) $\mathbf{u} \cdot \mathbf{v} = (0, 1, 1) \cdot (2, -1, 1) = (0)(2) + (1)(-1) + (1)(1) = 0$

(b) $\mathbf{v} \cdot \mathbf{u} = (2, -1, 1) \cdot (0, 1, 1) = (2)(0) + (-1)(1) + (1)(1) = 0$

(c) $\mathbf{v} \cdot \mathbf{w} = (2, -1, 1) \cdot (6, -3, 3) = (2)(6) + (-1)(-3) + (1)(3) = 18$

(d) $\mathbf{u} \cdot \mathbf{u} = (0, 1, 1) \cdot (0, 1, 1) = 0^2 + 1^2 + 1^2 = 2$

(e) $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ is not defined. The quantity $\mathbf{u} \cdot \mathbf{v}$ is a scalar. A scalar dotted with a vector doesn’t make sense.
The properties of the dot product are easy to establish. (See Problems 46–51.) Note that this theorem, as well as all others in this section, applies to both two- and three-dimensional vectors.

**Theorem A**  Properties of the Dot Product
If \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors, and \( \alpha \) is a scalar, then
1. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
2. \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \)
3. \( \alpha (\mathbf{u} \cdot \mathbf{v}) = (\alpha \mathbf{u}) \cdot \mathbf{v} \)
4. \( \mathbf{0} \cdot \mathbf{u} = 0 \)
5. \( \mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2 \)

To emphasize the significance of the dot product we offer the following alternative formula for it that involves the geometric properties of the vectors \( \mathbf{u} \) and \( \mathbf{v} \).

**Theorem B**
If \( \theta \) is the smallest nonnegative angle between the nonzero vectors \( \mathbf{u} \) and \( \mathbf{v} \), then
\[
\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta
\]

**Proof**  To prove this result, apply the Law of Cosines to the triangle in Figure 1.
\[
||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}|| ||\mathbf{v}|| \cos \theta
\]
On the other hand, from the properties of the dot product stated in Theorem A,
\[
||\mathbf{u} - \mathbf{v}||^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})
= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}
= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2 \mathbf{u} \cdot \mathbf{v}
\]
Equating the two expressions for \( ||\mathbf{u} - \mathbf{v}||^2 \) gives
\[
||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}|| ||\mathbf{v}|| \cos \theta = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2 \mathbf{u} \cdot \mathbf{v}
-2||\mathbf{u}|| ||\mathbf{v}|| \cos \theta = -2 \mathbf{u} \cdot \mathbf{v}
\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta
\]

**Example 2**  Find the angle between \( \mathbf{u} = (8, 6) \) and \( \mathbf{v} = (5, 12) \) (see Figure 2).

**Solution**  Solving for \( \cos \theta \) in Theorem B gives
\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} = \frac{(8)(5) + (6)(12)}{(10)(13)} = \frac{112}{130} \approx 0.862
\]

Then
\[
\theta = \cos^{-1}(0.862) \approx 0.532 \text{ (or 30.5°)}
\]
An important consequence of Theorem B is the following.

**Theorem C**  Perpendicularity Criterion
Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) are perpendicular if and only if their dot product, \( \mathbf{u} \cdot \mathbf{v} \), is 0.
Proof Two nonzero vectors are perpendicular if and only if the smallest nonnegative angle \( \theta \) between them is \( \pi/2 \); that is, if and only if \( \cos \theta = 0 \). But \( \cos \theta = 0 \) if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \). (This result is valid for zero vectors, provided that we agree that a zero vector is perpendicular to every other vector.)

**Definition** Orthogonal

Vectors that are perpendicular are said to be **orthogonal**.

**Example 3** Find the angles between each of the three pairs of vectors from Example 1. Which pairs are orthogonal?

**Solution** For the vectors \( \mathbf{u} \) and \( \mathbf{v} \), we have

\[
\cos \theta_1 = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{0(2) + 1(1) + 1(1)}{\| (0, 1, 1) \| \| (2, -1, 1) \|} = \frac{0}{\sqrt{2} \sqrt{6}} = 0
\]

For the vectors \( \mathbf{u} \) and \( \mathbf{w} \), we have

\[
\cos \theta_2 = \frac{\mathbf{u} \cdot \mathbf{w}}{\| \mathbf{u} \| \| \mathbf{w} \|} = \frac{0(6) + 1(-3) + 1(3)}{\| (0, 1, 1) \| \| (6, -3, 3) \|} = \frac{0}{\sqrt{2} \sqrt{36}} = 0
\]

Finally, for the vectors \( \mathbf{v} \) and \( \mathbf{w} \), we have

\[
\cos \theta_3 = \frac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{v} \| \| \mathbf{w} \|} = \frac{2(6) + (-1)(-3) + 1(3)}{\| (2, -1, 1) \| \| (6, -3, 3) \|} = \frac{18}{\sqrt{3} \sqrt{36}} = 1
\]

Thus the pair \( \mathbf{u} \) and \( \mathbf{v} \) and the pair \( \mathbf{u} \) and \( \mathbf{w} \) are orthogonal, so \( \theta_1 = \theta_2 = \pi/2 \). Note that for the pair \( \mathbf{v} \) and \( \mathbf{w} \), the cosine of the angle between them is 1, indicating that \( \theta_3 = 0 \); that is, the vectors point the same direction.

Recall that every vector \( \mathbf{u} \) in the plane can be written as \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} \), where \( \mathbf{i} = (1, 0) \) and \( \mathbf{j} = (0, 1) \), and that every vector \( \mathbf{v} \) in three space can be written as \( \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \), where, in this case, \( \mathbf{i} = (1, 0, 0) \), \( \mathbf{j} = (0, 1, 0) \), and \( \mathbf{k} = (0, 0, 1) \).

**Example 4** Find the measure of the angle \( ABC \), where the three points are \( A(4, 3) \), \( B(1, -1) \), and \( C(6, -4) \) as in Figure 3.

**Solution**

\[
\mathbf{u} = \overrightarrow{AB} = (4 - 1)\mathbf{i} + (3 + 1)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j} = \langle 3, 4 \rangle
\]

\[
\mathbf{v} = \overrightarrow{BC} = (6 - 1)\mathbf{i} + (-4 + 1)\mathbf{j} = 5\mathbf{i} - 3\mathbf{j} = \langle 5, -3 \rangle
\]

\[
\| \mathbf{u} \| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5
\]

\[
\| \mathbf{v} \| = \sqrt{5^2 + (-3)^2} = \sqrt{34}
\]

\[
\mathbf{u} \cdot \mathbf{v} = (3)(5) + (-3)(-3) = 18
\]

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{18}{5 \sqrt{34}} = 0.1029
\]

\[
\theta = 1.468 \text{ (about 84.09°)}
\]

**Example 5** Find the measure of angle \( ABC \) if the points are \( A(-1, 2, 3) \), \( B(2, -4, -6) \), and \( C(5, -3, 2) \) (Figure 4).

**Solution** First we determine vectors \( \mathbf{u} \) and \( \mathbf{v} \) (emanating from the origin) equivalent to \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \). This is done by subtracting the coordinates of the initial points from those of the terminal points, that is.

\[
\mathbf{u} = \overrightarrow{BA} = (-1, 2, -4, 3 + 6) = (-1, -6, 9)
\]

\[
\mathbf{v} = \overrightarrow{BC} = (5 - 2, -3 - 4, 2 + 6) = (3, -7, 8)
\]
Thus,
\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||} = \frac{(-1)(3) + (-6)(-7) + (9)(8)}{\sqrt{1 + 36 + 81\sqrt{9} + 49 + 64}} \approx 0.9251
\]
\[
\theta = 0.3894 \quad \text{(about 22.31°)}
\]

**Direction Angles and Cosines** The smallest nonnegative angles between a nonzero three-dimensional vector \( \mathbf{a} \) and the basis vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are called the **direction angles** of \( \mathbf{a} \); they are denoted \( \alpha, \beta \) and \( \gamma \), respectively, as shown in Figure 5. It is often more convenient to work with the **direction cosines** \( \cos \alpha, \cos \beta, \) and \( \cos \gamma \). If \( \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}, \) then

\[
\cos \alpha = \frac{a_1}{||\mathbf{a}||}, \quad \cos \beta = \frac{a_2}{||\mathbf{a}||}, \quad \cos \gamma = \frac{a_3}{||\mathbf{a}||}
\]

Notice that

\[
\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{a_1^2}{||\mathbf{a}||^2} + \frac{a_2^2}{||\mathbf{a}||^2} + \frac{a_3^2}{||\mathbf{a}||^2} = 1
\]

The vector \( (\cos \alpha, \cos \beta, \cos \gamma) \) is a unit vector having the same direction as \( \mathbf{a} \).

**EXAMPLE 6** Find the direction angles for the vector \( \mathbf{a} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} \).

**SOLUTION** Since \( ||\mathbf{a}|| = \sqrt{4^2 + (-5)^2 + 3^2} = 5 \sqrt{2} \),

\[
\cos \alpha = \frac{4}{5 \sqrt{2}} = \frac{2 \sqrt{2}}{5}, \quad \cos \beta = -\frac{5}{5 \sqrt{2}} = -\frac{\sqrt{2}}{2}, \quad \cos \gamma = \frac{3}{5 \sqrt{2}} = \frac{3 \sqrt{2}}{10}
\]

and

\[
\alpha \approx 55.55^\circ, \quad \beta = 135^\circ, \quad \gamma \approx 64.90^\circ
\]

**Projections** Let \( \mathbf{u} \) and \( \mathbf{v} \) be vectors, and let \( \theta \) be the angle between them. For now, we assume that \( 0 \leq \theta \leq \pi/2 \). Let \( \mathbf{w} \) be the vector in the direction of \( \mathbf{v} \) that has magnitude \( ||\mathbf{u}|| \cos \theta \) (see Figure 6). Since \( \mathbf{w} \) has the same direction as \( \mathbf{v} \), we know that \( \mathbf{w} = c \mathbf{v} \) for some nonnegative scalar \( c \). On the other hand, the **magnitude** of \( \mathbf{w} \) must be \( ||\mathbf{u}|| \cos \theta \). Thus,

\[
||\mathbf{u}|| \cos \theta = ||\mathbf{v}|| = ||c \mathbf{v}|| = ||c|| ||\mathbf{v}||
\]

The constant \( c \) is therefore

\[
c = \frac{||\mathbf{u}||}{||\mathbf{v}||} \cos \theta = \frac{||\mathbf{u}||}{||\mathbf{v}||} \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}
\]

Thus,

\[
\mathbf{w} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v}
\]

For \( \pi/2 < \theta < \pi \), we define \( \mathbf{w} \) to be the vector in the line determined by \( \mathbf{v} \), but pointing in the direction opposite \( \mathbf{v} \) (see Figure 7). The magnitude of this vector is

\[
||\mathbf{w}|| = -||\mathbf{u}|| \cos \theta = c ||\mathbf{v}|| \quad \text{for some positive scalar} \ c. \quad \text{Thus,} \quad c = \frac{||\mathbf{u}|| \cos \theta}{||\mathbf{v}||} = \frac{-\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2}.
\]

Since \( \mathbf{w} \) points in the direction opposite \( \mathbf{v} \), we have \( \mathbf{w} = -c \mathbf{v} = \left( -\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v} \). Thus, in both cases we have \( \mathbf{w} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v} \). The vector \( \mathbf{w} \) is called
the vector projection of \( \mathbf{u} \) on \( \mathbf{v} \), or sometimes just the projection of \( \mathbf{u} \) on \( \mathbf{v} \), and is denoted \( \text{pr}_v \mathbf{u} \):

\[
\text{pr}_v \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v}
\]

The scalar projection of \( \mathbf{u} \) on \( \mathbf{v} \) is defined to be \( ||\mathbf{u}|| \cos \theta \). It is positive, zero, or negative, depending on whether \( \theta \) is acute, right, or obtuse. When \( \theta = \pi / 2 \), the scalar projection is equal to the magnitude of \( \text{pr}_v \mathbf{u} \), and when \( \pi / 2 < \theta < \pi \), the scalar projection is equal to the opposite of the magnitude of \( \text{pr}_v \mathbf{u} \).

**EXAMPLE 7** Let \( \mathbf{u} = (-1, 5) \) and \( \mathbf{v} = (3, 3) \). Find the vector projection of \( \mathbf{u} \) on \( \mathbf{v} \) and the scalar projection of \( \mathbf{u} \) on \( \mathbf{v} \).

**SOLUTION** Figure 8 shows the two vectors. The vector projection is

\[
\text{pr}_{(3,3)}(-1,5) = \left( \frac{(-1,5) \cdot (3,3)}{||(3,3)||^2} \right) (3,3) = \frac{-3 + 15}{\sqrt{3^2 + 3^2}} (3,3) = (2,2) - 2i + 2j
\]

and the scalar projection is

\[
||\mathbf{u}|| \cos \theta = \frac{(-1,5) \cdot (3,3)}{||(3,3)||} = \frac{-3 + 15}{\sqrt{3^2 + 3^2}} = 2\sqrt{2}
\]

The work done by a constant force \( \mathbf{F} \) in moving an object along the line from \( P \) to \( Q \) is the magnitude of the force in the direction of the motion, times the distance moved. Thus, if \( \mathbf{D} \) is the vector from \( P \) to \( Q \), the work done is

\[
(\text{Scalar projection of } \mathbf{F} \text{ on } \mathbf{D}) ||\mathbf{D}|| = ||\mathbf{F}|| \cos \theta ||\mathbf{D}||
\]

That is,

\[
\text{Work} = \mathbf{F} \cdot \mathbf{D}
\]

**EXAMPLE 8** A force \( \mathbf{F} = 8i + 5j \) in newtons moves an object from \((1,0)\) to \((7,1)\), where distance is measured in meters (Figure 9). How much work is done?

**SOLUTION** Let \( \mathbf{D} \) be the vector from \((1,0)\) to \((7,1)\); that is, \( \mathbf{D} = 6i + j \). Then

\[
\text{Work} = \mathbf{F} \cdot \mathbf{D} = (8)(6) + (5)(1) = 53 \text{ newton-meters} = 53 \text{ joules}
\]

**Planes** One fruitful way to describe a plane is by using vector language. Let \( \mathbf{n} = (A,B,C) \) be a fixed nonzero vector and \( \mathbf{P}(x_1,y_1,z_1) \) be a fixed point. The set of points \( P(x,y,z) \) satisfying \( \mathbf{P} \cdot \mathbf{n} = 0 \) is the plane through \( P \) perpendicular to \( \mathbf{n} \). Since every plane contains a point and is perpendicular to some vector, a plane can be characterized in this way.

To get the Cartesian equation of the plane, write the vector \( \mathbf{P} \) in component form; that is,

\[
\mathbf{P} = (x-x_1, y-y_1, z-z_1)
\]

Then \( \mathbf{P} \cdot \mathbf{n} = 0 \) is equivalent to

\[
A(x-x_1) + B(y-y_1) + C(z-z_1) = 0
\]

This equation (in which at least one of \( A, B, \) and \( C \) is different from zero) is called the **standard form for the equation of a plane**.

If we remove the parentheses and simplify, the boxed equation takes the form of the general linear equation

\[
Ax + By + Cz = D, \quad A^2 + B^2 + C^2 \neq 0
\]
Thus every plane has a linear equation. Conversely, the graph of a linear equation in three-space is always a plane. To see the latter, let \((x_1, y_1, z_1)\) satisfy the equation; that is,

\[Ax_1 + By_1 + Cz_1 = D\]

When we subtract this equation from the one above, we have the boxed equation, which we know represents a plane.

**Example 9** Find the equation of the plane through \((5, 1, -2)\) perpendicular to \(m = (2, 4, 3)\). Then find the angle between this plane and the one with equation \(3x - 4y + 7z = 5\).

**Solution** To perform the first task, simply apply the standard form for the equation of a plane to the problem at hand, which gives

\[2(x - 5) + 4(y - 1) + 3(z + 2) = 0\]

or, equivalently,

\[2x + 4y + 3z = 8\]

A vector \(m\) perpendicular, or normal, to the second plane is \(m = (3, -4, 7)\). The angle \(\theta\) between two planes is the angle between their normals (Figure 10). Thus,

\[
\cos \theta = \frac{m \cdot n}{|m||n|} = \frac{(3)(2) + (-4)(4) + (7)(3)}{\sqrt{9 + 16 + 49}\sqrt{4 + 16 + 9}} \\
\theta \approx 76.26^\circ
\]

Actually, there are two angles between two planes, but they are supplementary. The process just described will lead to one of them. The other, if desired, is obtained by subtracting the first value from 180°. In our case, it would be 103.74°.

**Example 10** Show that the distance \(L\) from the point \((x_0, y_0, z_0)\) to the plane \(Ax + By + Cz = D\) is given by the formula

\[L = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}\]

**Solution** Let \((x_1, y_1, z_1)\) be a point on the plane, and let \(m = (x_0 - x_1, y_0 - y_1, z_0 - z_1)\) be the vector from \((x_1, y_1, z_1)\) to \((x_0, y_0, z_0)\), as in Figure 11. Now \(m = (A, B, C)\) is a vector perpendicular to the given plane, though it might point in the opposite direction of that in our figure. The number \(L\) that we seek is the length of the projection of \(m\) on \(n\). Thus,

\[L = \frac{|m|\cos \theta}{|n|} = \frac{|m \cdot n|}{|n|} = \frac{|Ax_0 + By_0 + Cz_0 - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}}\]

But \((x_1, y_1, z_1)\) is on the plane, and so

\[Ax_1 + By_1 + Cz_1 = D\]

Substitution of this result in the expression for \(L\) yields the desired formula.
EXAMPLE 11: Find the distance between the parallel planes \(3x - 4y + 5z = 9\) and \(3x - 4y + 5z = 4\).

**SOLUTION** The planes are parallel, since the vector \((3, -4, 5)\) is perpendicular to both of them (Figure 12). The point \((1, 1, 2)\) is easily seen to be on the first plane. We find the distance \(L\) from \((1, 1, 2)\) to the second plane using the formula of Example 10.

\[
L = \frac{|3(1) - 4(1) + 5(2) - 4|}{\sqrt{9 + 16 + 25}} = \frac{5}{5\sqrt{2}} = 0.7071
\]

**Concepts Review**

1. The dot product of \(\mathbf{u} = (u_1, u_2, u_3)\) and \(\mathbf{v} = (v_1, v_2, v_3)\) is defined by \[\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3\]. The corresponding geometric formulation for \(\mathbf{u} \cdot \mathbf{v} = \theta\), where \(\theta\) is the angle between \(\mathbf{u}\) and \(\mathbf{v}\).

2. Two vectors \(\mathbf{u}\) and \(\mathbf{v}\) are orthogonal if and only if their dot product is \(0\).

3. The work done by a constant force \(\mathbf{F}\) in moving an object along the vector \(\mathbf{D}\) is given by \[\mathbf{F} \cdot \mathbf{D}\].

4. A normal vector to the plane \(Ax + By + Cz = D\) is \(\mathbf{a} \times \mathbf{b}\).\(\mathbf{c}\).

**Problem Set 11.3**

1. Let \(\mathbf{a} = -2i + 3j\), \(\mathbf{b} = 2i - 3j\), and \(\mathbf{c} = -5j\). Find each of the following:
   - (a) \(2\mathbf{a} - 4\mathbf{b}\)
   - (b) \(\mathbf{a} \cdot \mathbf{b}\)
   - (c) \(\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})\)
   - (d) \(-2\mathbf{a} + 3\mathbf{b}\) \cdot 5\mathbf{c}\)
   - (e) \(|\mathbf{a}| \cdot \mathbf{a}\)
   - (f) \(\mathbf{b} \cdot \mathbf{b} - |\mathbf{b}|\)

2. Let \(\mathbf{a} = (3, -1, 1)\), \(\mathbf{b} = (1, -1, 1)\), and \(\mathbf{c} = (0, 5)\). Find each of the following:
   - (a) \(-4\mathbf{a} + 3\mathbf{b}\)
   - (b) \(\mathbf{b} \cdot \mathbf{c}\)
   - (c) \((\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}\)
   - (d) \(2\mathbf{c} \cdot (3\mathbf{a} + 4\mathbf{b})\)
   - (e) \(|\mathbf{b}| \cdot \mathbf{a}\)
   - (f) \(|\mathbf{a}|^2 - \mathbf{c} \cdot \mathbf{c}\)

3. Find the cosine of the angle between \(\mathbf{a}\) and \(\mathbf{b}\) and make a sketch.
   - (a) \(\mathbf{a} = (1, -3, 3)\), \(\mathbf{b} = (-1, 2, 0)\)
   - (b) \(\mathbf{a} = (1, -3, 3)\), \(\mathbf{b} = (-1, -2, 0)\)
   - (c) \(\mathbf{a} = (2, -1, 1)\), \(\mathbf{b} = (-2, 2, -4)\)
   - (d) \(\mathbf{a} = (4, -7, 0)\), \(\mathbf{b} = (-8, 0, 0)\)

4. Find the angle between \(\mathbf{a}\) and \(\mathbf{b}\) and make a sketch.
   - (a) \(\mathbf{a} = 12\mathbf{i} + 3\mathbf{j}\), \(\mathbf{b} = -8\mathbf{i} - 6\mathbf{j}\)
   - (b) \(\mathbf{a} = 4\mathbf{i} + 3\mathbf{j}\), \(\mathbf{b} = 2\mathbf{i} - 6\mathbf{j}\)
   - (c) \(\mathbf{a} = \sqrt{3}\mathbf{i} + \mathbf{j}\), \(\mathbf{b} = 3\mathbf{i} + \sqrt{3}\mathbf{j}\)

5. Let \(\mathbf{a} = 1 + 2\mathbf{j} - \mathbf{k}\), \(\mathbf{b} = \mathbf{j} + \mathbf{k}\), and \(\mathbf{c} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}\). Find each of the following:
   - (a) \(\mathbf{a} \cdot \mathbf{b}\)
   - (b) \((\mathbf{a} + \mathbf{c}) \cdot \mathbf{b}\)
   - (c) \(\mathbf{a} \cdot \mathbf{a}\)
   - (d) \((\mathbf{b} - \mathbf{c}) \cdot \mathbf{a}\)
   - (e) \(|\mathbf{a}| \cdot |\mathbf{b}|\)
   - (f) \(\mathbf{b} \cdot \mathbf{b} - |\mathbf{b}|^2\)

6. Let \(\mathbf{a} = (\sqrt{2}, \sqrt{2}, 0)\), \(\mathbf{b} = (1, -1, 1)\), and \(\mathbf{c} = (-2, 2, 1)\). Find each of the following:
   - (a) \(\mathbf{a} \cdot \mathbf{c}\)
   - (b) \((\mathbf{a} - \mathbf{c}) \cdot \mathbf{b}\)

7. For the vectors \(\mathbf{a}, \mathbf{b},\) and \(\mathbf{c}\) from Problem 6, find the angle between each pair of vectors.

8. Let \(\mathbf{a} = (\sqrt{3}, \sqrt{3}, \sqrt{3})\), \(\mathbf{b} = (1, -1, 0)\), and \(\mathbf{c} = (-2, -2, 1)\). Find the angle between each pair of vectors.

9. For the vectors \(\mathbf{a}, \mathbf{b},\) and \(\mathbf{c}\) from Problem 6, find the direction cosines and the direction angles.

10. For the vectors \(\mathbf{a}, \mathbf{b},\) and \(\mathbf{c}\) from Problem 8, find the direction cosines and the direction angles.

11. Show that the vectors \((6, 3)\) and \((-1, 2)\) are orthogonal.

12. Show that the vectors \(\mathbf{a} = (1, 1, 1)\), \(\mathbf{b} = (1, -1, 0)\), and \(\mathbf{c} = (-1, -1, 2)\) are mutually orthogonal, that is, each pair of vectors is orthogonal.

13. Show that the vectors \(\mathbf{a} - \mathbf{i} - \mathbf{j}\), \(\mathbf{b} = \mathbf{i} + \mathbf{j}\), and \(\mathbf{c} = 2\mathbf{k}\) are mutually orthogonal, that is, each pair of vectors is orthogonal.

14. If \(\mathbf{u} + \mathbf{v}\) is orthogonal to \(\mathbf{u} - \mathbf{v}\), what can you say about the relative magnitudes of \(\mathbf{u}\) and \(\mathbf{v}\)?

15. Find two vectors of length 10, each of which is perpendicular to both \(-4\mathbf{i} + 5\mathbf{j} + \mathbf{k}\) and \(4\mathbf{i} + \mathbf{j}\).

16. Find all vectors perpendicular to both \((1, -2, -3)\) and \((-3, 2, 0)\).

17. Find the angle \(ABC\) if the points are \(A(1, 2, 3)\), \(B(-4, 5, 6)\), and \(C(1, 0, 1)\).

18. Show that the triangle \(ABC\) is a right triangle if the vertices are \(A(6, 3, 3)\), \(B(3, 1, -1)\), and \(C(-1, 10, -2.5)\). **Hint:** Check the angle at \(B\).
19. For what numbers $c$ are $(c, 6)$ and $(c, -4)$ orthogonal?
20. For what numbers $c$ are $2a - 8j$ and $3i + cj$ orthogonal?
21. For what numbers $c$ and $d$ are $u = c\mathbf{i} + j + k$ and $v = 2j + dk$ orthogonal?
22. For what values of $a$, $b$, and $c$ are the three vectors 
\((a, 0, 1), (0, 2, b), \) and \((1, c, 1)\) mutually orthogonal?

In Problems 23-28, find each of the given projections if $u = i + 2j$, $v = 2i - j$, and $w = i + j$.
23. $\text{proj}_u u$
24. $\text{proj}_v u$
25. $\text{proj}_w u$
26. $\text{proj}_u (w - v)$
27. $\text{proj}_w u$
28. $\text{proj}_w u$

In Problems 29-34, find each of the given projections if $u = 3i + 2j + k$, $v = 2i - k$, and $w = i + j - 3k$.
29. $\text{proj}_u u$
30. $\text{proj}_v u$
31. $\text{proj}_w w$
32. $\text{proj}_u (w + v)$
33. $\text{proj}_v u$
34. $\text{proj}_w u$

35. Find a simple expression for each of the following for an arbitrary vector $u$.
   \[(a) \text{proj}_u u \quad \quad \quad \quad \quad \quad \quad \quad (b) \text{proj}_{-u} u\]
   36. Find a simple expression for each of the following for an arbitrary vector $u$.
   \[(a) \text{proj}_u (-u) \quad \quad \quad \quad \quad \quad \quad \quad (b) \text{proj}_{-u} (-u)\]
37. Find the scalar projection of $u = i + 2j + 3k$ on $v = -i + j - k$.
38. Find the scalar projection of $u = 2i + 3j + 2k$ on $v = -\sqrt{5}i + \sqrt{5}j + k$.
39. A vector $u = 2i + 3j + k$ emanating from the origin points into the first octant, that part of three-space where all components are positive. If $|u| = 5$, find $c$.
40. If $\alpha = 45^\circ$ and $\beta = 105^\circ$ are direction angles for a vector $u$, find two possible values for the third angle.

In Problems 41-44, find two perpendicular vectors $u$ and $v$ such that each is also perpendicular to $w = (-4, 2, 5)$.
41. Find the vector emanating from the origin whose terminal point is the midpoint of the segment joining $(3, 2, -1)$ and $(5, 7, 2)$.
42. Which of the following do not make sense?
   \[(a) u \cdot (v \cdot w) \quad \quad \quad (b) (u \cdot v) + w \quad \quad \quad (c) |u| \cdot |v| \cdot |w|
43. Which of the following do not make sense?
   \[(a) u \cdot (v + w) \quad \quad \quad (b) (u \cdot w) + u \quad \quad \quad (c) (u \cdot v) \cdot (u \cdot w)
44. Given a proof of the indicated property for two-dimensional vectors. Use $u = (u_1, u_2)$, $v = (v_1, v_2)$, and $w = (w_1, w_2)$.
45. $(a + b)u = au + bu$
46. $u \cdot v = v \cdot u$
47. $c(u \cdot v) = (cu) \cdot v$
48. $u \cdot (v + w) = u \cdot v + u \cdot w$
49. $0 \cdot u = 0$
50. $u \cdot u = |u|^2$

51. Given the two nonparallel vectors $a = 3i - 2j$ and $b = -3i + 4j$ and another vector $r = 7i - 5j$, find scalars $k$ and $m$ such that $r = ka + mb$.
52. Given the two nonparallel vectors $a = -4i + 3j$ and $b = 2i - j$ and another vector $r = 6i - 7j$, find scalars $k$ and $m$ such that $r = ka + mb$.

53. Show that the vector $u = ai + bj$ is perpendicular to the line with equation $ax + by = c$. Hint: Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the line and show that $\mathbf{a} \cdot \mathbf{P}_2 - \mathbf{P}_1 = 0$.
54. Prove that $|u + v|^2 + |u - v|^2 = 2|u|^2 + 2|v|^2$.
55. Prove that $u \cdot v = \frac{1}{2}|u + v|^2 - \frac{1}{4}|u|^2 - \frac{1}{4}|v|^2$.

56. Find the angle between a main diagonal of a cube and one of its faces.
57. Find the smallest angle between the main diagonals of a rectangular box 4 feet by 6 feet by 10 feet.
58. Find the angles formed by the diagonals of a cube.
59. Find the work done by the force $F = 3i + 10j$ newtons in moving an object 10 meters north (i.e., in the $j$ direction).
60. Find the work done by a force of 100 newtons acting in the direction $S 70^\circ$ E in moving an object 30 meters east.
61. Find the work done by the force $F = 6i + 5j$ pounds in moving an object from $(1, 0)$ to $(6, 8)$, where distance is in feet.
62. Find the work done by a force $F = -5i + 3j$ newtons in moving an object 12 meters north.
63. Find the work done by a force $F = -4k$ newtons moving an object from $(0, 0, 8)$ to $(4, 4, 0)$, where distance is in meters.
64. Find the work done by a force $F = 3i - 6j + 7k$ pounds moving an object from $(2, 1, 3)$ to $(9, 4, 6)$, where distance is in feet.

In Problems 65-68, find the equation of the plane having the given normal vector $n$ and passing through the given point $P$.
65. $n = 2i - 4j + 3k; P(1, 2, -3)$
66. $n = 3i - 2j - 1k; P(-2, -3, 4)$
67. $n = (1, 4, 4); P(1, 2, 1)$
68. $n = (0, 0, 1); P(1, 2, -3)$
69. Find the smaller of the angles between the two planes from Problems 65 and 66.
70. Find the equation of the plane through $(-1, 2, -3)$ and parallel to the plane $2x + 4y - z = 6$.
71. Find the equation of the plane passing through $(-\sqrt{2}, -1, 2)$ and parallel to $(a)$ the $xy$-plane $(b)$ the plane $2x - 3y - 4z = 0$
72. Find the equation of the plane passing through the origin and parallel to $(a)$ the $xy$-plane $(b)$ the plane $x + y + z = 1$
73. Find the distance from $(1, -1, 2)$ to the plane $x + 3y + z = 7$.
74. Find the distance from $(2, 6, 3)$ to the plane $-3x + 2y + z = 9$.
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75. Find the distance between the parallel planes $\tau x + 2y + 3z = 9$ and $6x + 4y - 2z = 19$.

76. Find the distance between the parallel planes $5x - 3y - 2z = 5$ and $-5x + 3y + 2z = 7$.

77. Find the distance from the sphere $x^2 + y^2 + z^2 + 2x + 6y - 8z = 0$ to the plane $3x + 4y + z = 15$.

78. Find the equation of the plane each of whose points is equidistant from $(-2, 1, 4)$ and $(6, 1, -2)$.

79. Prove the Cauchy-Schwarz inequality for two-dimensional vectors:

$$ |\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}| |\mathbf{v}| $$

80. Prove the Triangle Inequality (see Figure 13) for two-dimensional vectors:

$$ |\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}| $$

Hint: Use the dot product to compute $|\mathbf{u} + \mathbf{v}|$; then use the Cauchy-Schwarz Inequality from Problem 79.

Figure 13

81. A weight of 30 pounds is suspended by three wires with resulting tensions $A + D = 15B$, $7B - 2 + 10D$, and $a + b + c$. Determine $a$, $b$, and $c$ so that the net force is straight up.

82. Show that the work done by a constant force $\mathbf{F}$ on an object that moves completely around a closed polygonal path is 0.

83. Let $\mathbf{u} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be fixed vectors. Show that $(x - a) \cdot (x - b) = 0$ is the equation of a sphere, and find its center and radius.

84. Refine the method of Example 10 by showing that the distance $L$ between the parallel planes $Ax + By + Cz = D$ and $Ax + By + Cz = E$ is

$$ L = \frac{|D - E|}{\sqrt{A^2 + B^2 + C^2}} $$

85. The medians of a triangle meet at a point $P$ (the centroid by Problem 39 of Section 5.6) that is two-thirds of the way from a vertex to the midpoint of the opposite edge. Show that $P$ is the head of the position vector $(a + b + c)/3$, where $a$, $b$, and $c$ are the position vectors of the vertices, and use this to find $P$ if the vertices are $(2, 6, 5), (4, -1, 2)$, and $(6, 1, 2)$.

86. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be the position vectors of the vertices of a tetrahedron. Show that the lines joining the vertices to the centroids of the opposite faces meet in a point $P$, and give a nice vector formula for it, thus generalizing Problem 85.

87. Suppose that the three coordinate planes bounding the first octant are mirrors. A light ray with direction $\mathbf{a} + b + c\mathbf{k}$ is reflected successively from the $xy$-plane, the $xz$-plane, and the $yz$-plane. Determine the direction of the ray after each reflection, and state a nice conclusion concerning the final reflected ray.

Answers to Concepts Review: 1. $\mathbf{u}_1 \mathbf{v}_1 + \mathbf{u}_2 \mathbf{v}_2 + \mathbf{u}_3 \mathbf{v}_3$; $|\mathbf{u}| |\mathbf{v}| \cos \theta$ 2. 0 3. $\mathbf{F} \cdot \mathbf{D}$ 4. $(A, B, C)$

11.4 The Cross Product

The dot product of two vectors is a scalar. We have explored some of its uses in the previous section. Now we introduce the cross product (or vector product); it will also have many uses. The cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined by

$$ \mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) $$

In this form, the formula is hard to remember and its significance is not obvious. Note the one thing that is obvious: The cross product of two vectors is a vector.

To help us remember the formula for the cross product, we recall a subject from an earlier mathematics course, namely, determinants. First, the value of a $2 \times 2$ determinant is

$$ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc $$

Then the value of a $3 \times 3$ determinant is (expanding along the top row)

$$ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} $$

$$ = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} $$
Using determinants, we can write the definition of \( \mathbf{u} \times \mathbf{v} \) as

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = u_2 v_3 \mathbf{i} - u_3 v_2 \mathbf{j} + u_1 v_3 \mathbf{k}
\]

Note that the components of the left vector \( \mathbf{u} \) go in the second row, and those of the right vector \( \mathbf{v} \) go in the third row. This is important, because if we interchange the positions of \( \mathbf{u} \) and \( \mathbf{v} \), we interchange the second and third rows of the determinant, and this changes the sign of the determinant’s value, as you may check. Thus,

\[
\mathbf{u} \times \mathbf{v} = - (\mathbf{v} \times \mathbf{u})
\]

which is sometimes called the anticommutative law.

**Example 1** Let \( \mathbf{u} = \langle 1, -2, -1 \rangle \) and \( \mathbf{v} = \langle -2, 4, 1 \rangle \). Calculate \( \mathbf{u} \times \mathbf{v} \) and \( \mathbf{v} \times \mathbf{u} \) using the determinant definition.

**Solution**

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ -2 & 4 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -2 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \mathbf{k} = 2\mathbf{i} + \mathbf{j} + 0\mathbf{k}
\]

\[
\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 4 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ -2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ -2 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \mathbf{k} = -2\mathbf{i} - \mathbf{j} + 0\mathbf{k}
\]

**Geometric Interpretation of \( \mathbf{u} \times \mathbf{v} \)** Like the dot product, the cross product gains significance from its geometric interpretation.

**Theorem A**

Let \( \mathbf{u} \) and \( \mathbf{v} \) be vectors in three-space and \( \theta \) be the angle between them. Then

1. \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \), that is, \( \mathbf{u} \times \mathbf{v} \) is perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \);
2. \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{u} \times \mathbf{v} \) form a right-handed triple;
3. \( ||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| ||\mathbf{v}|| \sin \theta \).

**Proof** Let \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) and \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \).

(i) \( \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) \). When we remove parentheses, the six terms cancel in pairs, leaving a sum of 0. A similar event occurs when we expand \( \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) \).

(ii) The meaning of right-handedness for the triple \( \mathbf{u} \), \( \mathbf{v} \), \( \mathbf{u} \times \mathbf{v} \) is illustrated in Figure 1. There \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \), and the fingers of the right hand are curled in the direction of the rotation through \( \theta \) that makes \( \mathbf{u} \) coincide with \( \mathbf{v} \). It is difficult to establish analytically that the indicated triple is right-handed, but you might check it with a few examples. Note in particular that \( \mathbf{i} \times \mathbf{j} = \mathbf{k} \), and by definition we know that the triple \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) is right-handed.

(iii) We need Lagrange’s Identity,

\[
||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2 ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2
\]
whose proof is a simple algebraic exercise (Problem 31). Using this identity, we may write

\[ \| \mathbf{u} \times \mathbf{v} \|^2 = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 (1 - \cos^2 \theta) = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \sin^2 \theta \]

Since \( 0 \leq \theta \leq \pi \), \( \sin \theta \geq 0 \). Taking the principal square root yields

\[ \| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta \]

It is important that we have geometric interpretations of both \( \mathbf{u} \cdot \mathbf{v} \) and \( \mathbf{u} \times \mathbf{v} \). While both products were originally defined in terms of components that depend on a choice of coordinate system, they are actually independent of coordinate systems. They are intrinsic geometric quantities, and you will get the same results for \( \mathbf{u} \cdot \mathbf{v} \) and \( \mathbf{u} \times \mathbf{v} \) no matter how you introduce the coordinates used to compute them.

Here is a simple consequence of Theorem A (part 3) and the fact that vectors are parallel if and only if the angle \( \theta \) between them is either 0° or 180°.

**Theorem B.**

Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in three-space are parallel if and only if \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \).

**Applications**

Our first application is to find the equation of the plane through three noncollinear points.

**EXAMPLE 2** Find the equation of the plane (Figure 2) through the three points \( P_1(1, -2, 3), P_2(4, 1, -2) \), and \( P_3(-2, -3, 0) \).

**SOLUTION** Let \( \mathbf{u} = \mathbf{P}_2 \mathbf{P}_1 = (-3, -3, 5) \) and \( \mathbf{v} = \mathbf{P}_3 \mathbf{P}_1 = (-6, -4, 2) \). From the first part of Theorem A we know that

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
1 & j & k \\
-3 & -3 & 5 \\
-6 & -4 & 2 \\
\end{vmatrix} = 14\mathbf{i} - 24\mathbf{j} - 6\mathbf{k}
\]

is perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \) and thus to the plane containing them. The plane through \((4, 1, -2)\) with normal \(14\mathbf{i} - 24\mathbf{j} - 6\mathbf{k}\) has equation (see Section 11.3)

\[ 14(x - 4) - 24(y - 1) - 6(z + 2) = 0 \]

or

\[ 14x - 24y - 6z = 44 \]

**EXAMPLE 3** Show that the area of a parallelogram with \( \mathbf{a} \) and \( \mathbf{b} \) as adjacent sides is \( \| \mathbf{a} \times \mathbf{b} \| \).

**SOLUTION** Recall that the area of a parallelogram is the product of the base times the height. Now look at Figure 3 and use the fact that \( \| \mathbf{a} \times \mathbf{b} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \theta \).

**EXAMPLE 4** Show that the volume of the parallelepiped determined by the vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is

\[ V = | \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) | = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3 \\
\end{vmatrix} \]
SOLUTION  Refer to Figure 4 and regard the parallelogram determined by \( \mathbf{b} \) and \( \mathbf{c} \) as the base of the parallelepiped. The area of this base is \( |\mathbf{b} \times \mathbf{c}| \) by Example 3; the height \( h \) of the parallelepiped is the absolute value of the scalar projection of \( \mathbf{a} \) on \( \mathbf{b} \times \mathbf{c} \). Thus,
\[
h = |\mathbf{a}| \cos \theta = \frac{|\mathbf{a}| |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|} = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{b} \times \mathbf{c}|}
\]
and
\[
V = h|\mathbf{b} \times \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.
\]

Suppose that the vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) from the previous example are in the same plane. In this case, the parallelepiped has height zero, so the volume should be zero. Does the formula for the volume yield \( V = 0 \)? If \( \mathbf{a} \) is in the plane determined by \( \mathbf{b} \) and \( \mathbf{c} \), then any vector perpendicular to \( \mathbf{b} \) and \( \mathbf{c} \) will be perpendicular to \( \mathbf{a} \) as well. The vector \( \mathbf{b} \times \mathbf{c} \) is perpendicular to both \( \mathbf{b} \) and \( \mathbf{c} \); hence \( \mathbf{b} \times \mathbf{c} \) is perpendicular to \( \mathbf{a} \). Thus, \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0 \).

Algebraic Properties  The rules for calculating with cross products are summarized in the following theorem. Proving this theorem is a matter of writing everything out in terms of components and will be left as an exercise.

**Theorem C**
If \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are vectors in three-space and \( k \) is a scalar, then
1. \( \mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \) (anticommutative law);
2. \( \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w}) \) (left distributive law);
3. \( k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) \);
4. \( \mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0} ; \mathbf{u} \times \mathbf{u} = \mathbf{0} ; \)
5. \( (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) ; \)
6. \( \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} . \)

Once the rules in Theorem C are mastered, complicated calculations with vectors can be done with ease. We illustrate by calculating a cross product in a new way. We will need the following simple but important products.
\[
\begin{align*}
\mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{i} &= \mathbf{j}.
\end{align*}
\]
These results have a cyclic order, which can be remembered by appealing to Figure 5.

**EXAMPLE 5**  Calculate \( \mathbf{u} \times \mathbf{v} \) if \( \mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{v} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \).

**SOLUTION**  We appeal to Theorem C, especially the distributive law and the anticommutative law.
\[
\begin{align*}
\mathbf{u} \times \mathbf{v} &= (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \\
&= 12(\mathbf{i} \times \mathbf{j}) + 6(\mathbf{j} \times \mathbf{k}) - 9(\mathbf{k} \times \mathbf{i}) - 8(\mathbf{j} \times \mathbf{i}) - 4(\mathbf{j} \times \mathbf{j}) \\
&\quad + 8(\mathbf{j} \times \mathbf{k}) + 4(\mathbf{k} \times \mathbf{i}) + 2(\mathbf{k} \times \mathbf{j}) - 3(\mathbf{k} \times \mathbf{k}) \\
&= 12(\mathbf{0}) + 6(\mathbf{k}) - 9(-\mathbf{j}) - 8(-\mathbf{k}) - 4(\mathbf{0}) \\
&\quad + 8(\mathbf{i}) + 4(\mathbf{j}) + 2(-\mathbf{i}) - 3(\mathbf{0}) \\
&= 4\mathbf{i} + 13\mathbf{j} + 14\mathbf{k}
\end{align*}
\]
Experts would do most of this in their heads; novices might find the determinant method easier.
Chapter 11 Geometry in Space and Vectors

Concepts Review
1. The cross product of \( \mathbf{u} = (-1, 2, 1) \) and \( \mathbf{v} = (3, 1, -1) \) is given by a specific determinant; evaluation of this determinant gives \( \mathbf{u} \times \mathbf{v} = \).  
2. Geometrically, \( \mathbf{u} \times \mathbf{v} \) is a vector perpendicular to the plane of \( \mathbf{u} \) and \( \mathbf{v} \) and has length \( |\mathbf{u} \times \mathbf{v}| = \).  
3. The cross product is anticommutative; that is, \( \mathbf{u} \times \mathbf{v} = \).  
4. Two vectors are if and only if their cross product is \( \mathbf{0} \).

Problem Set 11.4

1. Let \( \mathbf{a} = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \mathbf{b} = -4 + 2\mathbf{j} - 4\mathbf{k} \), and \( \mathbf{c} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \). Find each of the following: 
   (a) \( \mathbf{a} \times \mathbf{b} \) 
   (b) \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) \) 
   (c) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) 
   (d) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) 
2. If \( \mathbf{a} = (3, 3, 1), \mathbf{b} = (-2, -1, 0) \), and \( \mathbf{c} = (-2, -3, -1) \), find each of the following: 
   (a) \( \mathbf{a} \times \mathbf{b} \) 
   (b) \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) \) 
   (c) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) 
   (d) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) 
3. Find all vectors perpendicular to both the vectors \( \mathbf{a} = -i + 2j + 3k \) and \( \mathbf{b} = -2i + 2j - 4k \). 
4. Find all vectors perpendicular to both the vectors \( \mathbf{a} = 2i + 2j - k \) and \( \mathbf{b} = i + j - k \) as the adjacent sides. 
5. Find the unit vectors perpendicular to the plane determined by the three points \((1, 3, 5), (3, -1, 2), \) and \((4, 0, 1)\). 
6. Find the unit vectors perpendicular to the plane determined by the three points \((-1, 3, 0), (5, 1, 2), \) and \((4, -3, -1)\). 
7. Find the area of the parallelogram with \( \mathbf{a} = -i + j - 3k \) and \( \mathbf{b} = 4i + 2j - 4k \) as the adjacent sides. 
8. Find the area of the parallelogram with \( \mathbf{a} = 2i + 2j - k \) and \( \mathbf{b} = -i + j - 4k \) as the adjacent sides. 
9. Find the area of the triangle with \((3, 2, 1), (2, 4, 6), \) and \((-1, 2, 5)\) as vertices. 
10. Find the area of the triangle with \((1, 2, 3), (3, 1, 5), \) and \((4, 5, 0)\) as vertices. 

In Problems 11–14, find the equation of the plane through the given points.
11. \((1.3, 2), (0.3, 0), \) and \((2, 4, 3)\) 
12. \((1.1, 2), (0.1, 1), \) and \((-2, 3, 0)\) 
13. \((3, 5, 0), (0.3, 0), \) and \((0, 5, 0)\) 
14. \((a, 0, 0), (0, b, 0), \) and \((0, 0, c)\). (None of \(a, b, \) and \(c\) is zero.) 
15. Find the equation of the plane through \((2, 5, 1)\) that is parallel to the plane \(x - y + 2z = 4\). 
16. Find the equation of the plane through \((0, 0, 2)\) that is parallel to the plane \(x + y + z = 1\). 
17. Find the equation of the plane through \((-1, -2, 3)\) and perpendicular to both the planes \(x + 3y + 2z = 7\) and \(2x - 2y - z = -3\). 
18. Find the equation of the plane through \((2, -1, 4)\) that is perpendicular to both the planes \(x + y + z = 2\) and \(x - y - z = 4\). 
19. Find the equation of the plane through \((-2, -3, 2)\) and parallel to the plane of the vectors \(4i + 3j - k \) and \(2i - 5j + 6k\). 
20. Find the equation of the plane through the origin that is perpendicular to the \(xy\)-plane and the plane \(3x - 2y + z = 4\). 
21. Find the equation of the plane through \((6, 2, -1)\) and perpendicular to the line of intersection of the planes \(4x - 3y + 2z + 5 = 0\) and \(3x + 2y - z + 11 = 0\). 
22. Let \(\mathbf{a} = \mathbf{b}\) be nonparallel vectors, and let \(\mathbf{c}\) be any nonzero vector. Show that \((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\) is a vector in the plane of \(\mathbf{a}\) and \(\mathbf{b}\). 
23. Find the volume of the parallelepiped with edges \((2, 3, 4), (0, 4, -1), \) and \((5, 1, 3)\) (see Example 4). 
24. Find the volume of the parallelepiped with edges \(3i - 4j + 2k, -i + 2j + k, \) and \(3j - 2j + 5k\). 
25. Let \(K\) be the parallelepiped determined by \(\mathbf{u} = (3, 2, 1), \mathbf{v} = (1, 1, 2), \) and \(\mathbf{w} = (1, 3, 3)\). 
   (a) Find the volume of \(K\). 
   (b) Find the area of the face determined by \(\mathbf{u}\) and \(\mathbf{v}\). 
   (c) Find the angle between \(\mathbf{u}\) and the plane containing the face determined by \(\mathbf{v}\) and \(\mathbf{w}\). 
26. The formula for the volume of a parallelepiped derived in Example 4 should not depend on the choice of which one of the three vectors we call \(\mathbf{a}\), which one we call \(\mathbf{b}\), and which one we call \(\mathbf{c}\). Use this result to explain why \(|\mathbf{b} \times (\mathbf{a} \times \mathbf{c})| = |\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})| = |\mathbf{a} \times (\mathbf{b} \times \mathbf{c})|\). 
27. Which of the following do not make sense?
   (a) \(\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\) 
   (b) \(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})\) 
   (c) \(\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\) 
   (d) \(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\) 
   (e) \(\mathbf{a} \cdot (\mathbf{b} + c)\) 
   (f) \((\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d})\) 
   (g) \((\mathbf{u} \times \mathbf{v}) \times \mathbf{w}\) 
   (h) \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\) 
28. Show that if \(\mathbf{a}, \mathbf{b}, \mathbf{c},\) and \(\mathbf{d}\) all lie in the same plane then \((\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 0\). 
29. The volume of a tetrahedron is known to be \(\frac{1}{6}\) of the area of base times the height. From this show that the volume of the tetrahedron with edges \(\mathbf{a}, \mathbf{b}, \) and \(\mathbf{c}\) is \(\frac{1}{6} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|\). 
30. Find the volume of the tetrahedron with vertices \((-1, 2, 3), (4, -1, 2), (5, 6, 3),\) and \((1, 1, -2)\) (see Problem 29). 
31. Prove Lagrange's identity, \(|\mathbf{a} \times \mathbf{v}|^2 = |\mathbf{a}|^2 |\mathbf{v}|^2 - (\mathbf{a} \cdot \mathbf{v})^2\) without using Theorem A. 
32. Prove the left distributive law, \(\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\) 
33. Use Problem 32 and the anticommutative law to prove the right distributive law.
34. If both \( \mathbf{u} \times \mathbf{v} = \mathbf{0} \) and \( \mathbf{u} \cdot \mathbf{v} = 0 \), what can you conclude about \( \mathbf{u} \) or \( \mathbf{v} \)?

35. Use Example 3 to develop a formula for the area of the triangle with vertices \( P(a, 0, 0) \), \( Q(0, b, 0) \), and \( R(0, 0, c) \) shown in the left half of Figure 6.

36. Show that the triangle in the plane with vertices \( (x_1, y_1) \), \( (x_2, y_2) \), and \( (x_3, y_3) \) has area equal to one-half the absolute value of the determinant

\[
\begin{vmatrix}
1 & y_1 & 1 \\
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1 \\
\end{vmatrix}
\]

Figure 6

37. A Pythagorean Theorem in Three-Space As in Figure 6, let \( P, Q, R, \) and \( O \) be the vertices of a (right-handed) tetrahedron, and let \( A, B, C, \) and \( D \) be the areas of the opposite faces, respectively. Show that \( A^2 + B^2 + C^2 = D^2 \).

38. Let vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) with common initial point determine a tetrahedron, and let \( \mathbf{m}, \mathbf{n}, \mathbf{p}, \) and \( \mathbf{q} \) be vectors perpendicular to the four faces, pointing outward, and having length equal to the area of the corresponding face. Show that \( \mathbf{m} + \mathbf{n} + \mathbf{p} + \mathbf{q} = \mathbf{0} \).

39. Let \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) denote the three edges of a triangle with lengths \( a, b, \) and \( c \), respectively. Use Lagrange’s Identity together with \( 2 \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2 \) to prove Heron’s Formula for the area \( A \) of a triangle.

\[
A = \sqrt{s(s-a)(s-b)(s-c)}
\]

where \( s \) is the semiperimeter \( (a + b + c)/2 \).

40. Use the method of Example 5 to show directly that, if \( u = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \) and \( v = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k} \), then

\[
\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}
\]

**Answers to Concepts Review:**
1. \((-3, 2, -7) \) or \( -3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k} \)
2. \(|\mathbf{u}||\mathbf{v}| \sin \theta \)
3. \(-(\mathbf{v} \times \mathbf{u}) \)
4. parallel

---

### 11.5 Vector-Valued Functions and Curvilinear Motion

Recall that a function \( f \) is a rule that associates with each member \( i \) of one set (the domain) a unique value \( f(i) \) from a second set (Figure 1). The set of values so obtained is the range of the function. So far in this book, our functions have been real-valued functions (scalar-valued functions) of a real variable; that is, both the domain and range have been sets of real numbers. A typical example is \( f(t) = t^2 \), which associates with each real number \( t \) the real number \( t^2 \).

Now we offer the first of many generalizations (Figure 2). A **vector-valued function** \( \mathbf{F} \) of a real variable \( t \) associates with each real number \( t \) a vector \( \mathbf{F}(t) \). Thus,

\[
\mathbf{F}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} + h(t) \mathbf{k} = (f(t), g(t), h(t))
\]

where \( f, g, \) and \( h \) are ordinary real-valued functions. A typical example is

\[
\mathbf{F}(t) = t \mathbf{i} + c \mathbf{j} + 2 \mathbf{k} = (t, c, 2)
\]

Note our use of a boldface letter; this helps us to distinguish between vector functions and scalar functions.

**Calculus for Vector Functions** The most fundamental notion in calculus is that of limit. Intuitively, \( \lim_{t \to c} \mathbf{F}(t) = \mathbf{L} \) means that the vector \( \mathbf{F}(t) \) tends toward the vector \( \mathbf{L} \) as \( t \) tends toward \( c \). Alternatively, it means that the vector \( \mathbf{F}(t) - \mathbf{L} \) approaches \( \mathbf{0} \) as \( t \to c \) (Figure 3). The precise \( \epsilon-\delta \) definition is nearly identical with that given for real-valued functions in Section 1.2.

**Definition** Limit of a Vector-Valued Function

To say that \( \lim_{t \to c} \mathbf{F}(t) = \mathbf{L} \) means that, for each given \( \epsilon > 0 \) (no matter how small), there is a corresponding \( \delta > 0 \) such that \( |\mathbf{F}(t) - \mathbf{L}| < \epsilon \), provided that \( 0 < |t - c| < \delta \); that is,

\[
0 < |t - c| < \delta \Rightarrow |\mathbf{F}(t) - \mathbf{L}| < \epsilon
\]
The definition of \( \lim_{t \to c} F(t) \) is nearly the same as our definition of the limit from Chapter 1, once we interpret \(|F(t) - L|\) as the length of the vector \( F(t) - L \). Our definition says that we can make \( F(t) \) as close as we like (within \( \varepsilon \)) to \( L \) (here distance is measured in three-dimensional space), as long as we take \( t \) to be close enough (within \( \delta \)) of \( c \). The next theorem, which is proved for two-dimensional vectors in Appendix A.2, Theorem D, gives the relationship between the limit of \( F(t) \) and the limits of the components of \( F(t) \).

**Theorem A**

Let \( F(t) = f(t)i + g(t)j + h(t)k \). Then \( F \) has a limit at \( c \) if and only if \( f, g, \) and \( h \) have limits at \( c \). In this case,

\[
\lim_{t \to c} F(t) = \left[ \lim_{t \to c} f(t) \right] i + \left[ \lim_{t \to c} g(t) \right] j + \left[ \lim_{t \to c} h(t) \right] k
\]

As you would expect, all the standard limit theorems hold. Also, continuity has its usual meaning; that is, \( F \) is continuous at \( c \) if \( \lim_{t \to c} F(t) = F(c) \). From Theorem A, it is clear that \( F \) is continuous at \( c \) if and only if \( f, g, \) and \( h \) are all continuous there. Finally, the derivative \( F'(t) \) is defined just as for real-valued functions by

\[
F'(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}
\]

This can also be written in terms of components.

\[
F'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t)i + g(t + \Delta t)j + h(t + \Delta t)k - [f(t)i + g(t)j + h(t)k]}{\Delta t}
\]

\[
= \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} i + \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} j + \lim_{\Delta t \to 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} k
\]

\[
= f'(t)i + g'(t)j + h'(t)k
\]

In summary, if \( F(t) = f(t)i + g(t)j + h(t)k \), then

\[
F'(t) = f'(t)i + g'(t)j + h'(t)k = (f'(t), g'(t), h'(t))
\]

**EXAMPLE 1** If \( F(t) = (t^2 + t)i + e^tj + 2tk \), find \( F'(t) \), \( F''(t) \), and the angle \( \theta \) between \( F'(0) \) and \( F''(0) \).

**SOLUTION** \( F'(t) = (2t + 1)i + e^tj \) and \( F''(t) = 2i + e^tj \). Thus, \( F'(0) = i + j \), \( F''(0) = 2i + j \), and

\[
\cos \theta = \frac{\langle F'(0), F''(0) \rangle}{\|F'(0)\| \cdot \|F''(0)\|} = \frac{(1)(2) + (1)(1) + (0)(0)}{\sqrt{1^2 + 1^2 + 0^2} \cdot \sqrt{2^2 + 1^2 + 0^2}} = \frac{3}{\sqrt{2} \sqrt{5}}
\]

\[
\theta \approx 0.3218 \quad \text{(about 18.43°)}
\]

Here are the rules for differentiation.
Theorem B  Differentiation Formulas

Let \( F \) and \( G \) be differentiable, vector-valued functions, \( p \) a differentiable, real-valued function, and \( c \) a scalar. Then

1. \( D_t[F(t) + G(t)] = F'(t) + G'(t) \)
2. \( D_t[cF(t)] = cF'(t) \)
3. \( D_t[p(t)F(t)] = p(t)F'(t) + p'(t)F(t) \)
4. \( D_t[F(t) \cdot G(t)] = F'(t) \cdot G(t) + G'(t) \cdot F(t) \)
5. \( D_t[F(t) \times G(t)] = F(t) \times G'(t) + F'(t) \times G(t) \)
6. \( D_t[F(p(t))] = F'(p(t))p'(t) \) (Chain Rule)

Proof  We prove formula 4 and leave the other parts to the reader. Let

\[
F(t) = f_1(t)i + f_2(t)j + f_3(t)k, \\
G(t) = g_1(t)i + g_2(t)j + g_3(t)k.
\]

Then

\[
D_t[F(t) \cdot G(t)] = D_t[ f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t) ]
\]

\[
= f_1(t)g_2'(t) + f_1'(t)g_2(t) + f_2(t)g_3'(t) + f_2'(t)g_3(t) + f_3(t)g_1'(t) + f_3'(t)g_1(t)
\]

Since derivatives of vector-valued functions are found by differentiating components, it is natural to define integration in terms of components; that is, if \( \mathbf{F}(t) = f(t)i + g(t)j + h(t)k \),

\[
\int \mathbf{F}(t) \, dt = \left[ \int f(t) \, dt \right] i + \left[ \int g(t) \, dt \right] j + \left[ \int h(t) \, dt \right] k
\]

\[
\int_0^b \mathbf{F}(t) \, dt = \left[ \int_a^b f(t) \, dt \right] i + \left[ \int_a^b g(t) \, dt \right] j + \left[ \int_a^b h(t) \, dt \right] k
\]

Example 2  If \( \mathbf{F}(t) = r^2i + e^{-t}j - 2k \), find

(a) \( D_t[r^2\mathbf{F}(t)] \)
(b) \( \int_0^1 \mathbf{F}(t) \, dt \)

Solution

(a) \( D_t[r^2\mathbf{F}(t)] = r^2(2ri - e^{-t}j) + 2r(e^2r^2i - e^{-t}j - 2k) \)

\[
= 5r^2i + (3r^2 - r^2)e^{-t}j - 6r^2k
\]

(b) \( \int_0^1 \mathbf{F}(t) \, dt = \left[ \int_0^1 r^2 \, dt \right] i + \left[ \int_0^1 e^{-t} \, dt \right] j + \left[ \int_0^1 (-2) \, dt \right] k \)

\[
= \frac{1}{2}i + (1 - e^{-1})j - 2k
\]

Curvilinear Motion  We are going to use the theory developed above for vector-valued functions to study the motion of a point in space. Let \( t \) measure time, and suppose that the coordinates of a moving point \( P \) are given by the parametric equations \( x = f(t), y = g(t), z = h(t) \). Then the vector

\[
\mathbf{r}(t) = f(t)i + g(t)j + h(t)k
\]
assumed to emanate from the origin, is called the **position vector** of the point. As $t$ varies, the head of $\mathbf{r}(t)$ traces the path of the moving point $P$ (Figure 4). This is a curve, and we call the corresponding motion **curvilinear motion**.

In analogy with linear (straight line) motion, we define the velocity $\mathbf{v}(t)$ and the acceleration $\mathbf{a}(t)$ of the moving point $P$ by

$$
\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} \\
\mathbf{a}(t) = \mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}
$$

Since

$$
\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}
$$

it is clear (from Figure 5) that $\mathbf{v}(t)$ has the direction of the tangent line. The acceleration vector $\mathbf{a}(t)$ points to the **concave** side of the curve (i.e., the side toward which the curve is bending).

If $\mathbf{r}(t)$ is the position vector of an object, then the arc length of the path that it traces from time $t = a$ to time $t = b$ is

$$
L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt = \int_a^b \|\mathbf{r}'(t)\| \, dt
$$

The accumulated arc length from time $t = a$ to an arbitrary time $t$ is thus

$$
\gamma = \int_a^t \sqrt{[f'(u)]^2 + [g'(u)]^2 + [h'(u)]^2} \, du = \int_a^t \|\mathbf{r}'(u)\| \, du
$$

By the First Fundamental Theorem of Calculus, the derivative of the accumulated arc length, $ds/dt$, is

$$
\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} = \|\mathbf{r}'(t)\|
$$

But the derivative (i.e., rate of change) of accumulated arc length is what we think of as speed. Thus, the **speed** of an object is

$$
\text{speed} = \frac{ds}{dt} = \|\mathbf{r}'(t)\| = \|\mathbf{v}(t)\|
$$

Note that the speed of an object is a scalar quantity, whereas its velocity is a vector.

One of the most important applications of curvilinear motion, **uniform circular motion**, occurs in two dimensions. Suppose that an object moves in the $xy$-plane...
Section 11.5 Vector-Valued Functions and Curvilinear Motion

A point moves in the plane with constant speed in a circle of radius $a$ and center $(0,0)$. If its initial position is $(a, 0)$, then its position vector is

$$\mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$$

**Example 3**

Find the velocity, acceleration, and speed for uniform circular motion.

**Solution**

We differentiate the position vector $\mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j}$ to get $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = -a \omega \sin \omega t \mathbf{i} + a \omega \cos \omega t \mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -a \omega^2 \cos \omega t \mathbf{i} - a \omega^2 \sin \omega t \mathbf{j}$$

The speed is

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{(-a \omega \sin \omega t)^2 + (a \omega \cos \omega t)^2}$$

$$= a \omega \sqrt{\sin^2 \omega t + \cos^2 \omega t} = a \omega$$

Note that if we think of $a$ as being based at the object’s location at point $P$, then $\mathbf{a}$ points directly toward the origin and is perpendicular to the velocity vector $\mathbf{v}$ (Figure 6).

We saw a particular case of a helix in Example 6 of Section 11.1. Here we generalize that concept a bit and say that the path traced out by an object whose position vector is given by

$$\mathbf{r}(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j} + ct \mathbf{k}$$

is a helix. If we look at just the $x$- and $y$-components of motion, we see uniform circular motion, and if we look at just the $z$-component of motion, we see uniform straight line motion. When we put these two together, we see that the object spirals around and around as it moves higher and higher (assuming $c > 0$).

**Example 4**

Find the velocity, acceleration, and speed for motion along a helix.

**Solution**

The velocity and acceleration vectors are

$$\mathbf{v}(t) = \mathbf{r}'(t) = -a \omega \sin \omega t \mathbf{i} + a \omega \cos \omega t \mathbf{j} + c \mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = -a \omega^2 \cos \omega t \mathbf{i} - a \omega^2 \sin \omega t \mathbf{j}$$

The speed is

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{(-a \omega \sin \omega t)^2 + (a \omega \cos \omega t)^2 + c^2} = \sqrt{a^2 \omega^2 + c^2}$$

**Example 5**

Parametric equations for an object moving in the plane are $x = 3 \cos t$ and $y = 2 \sin t$, where $t$ represents time and $0 \leq t \leq 2\pi$. Let $P$ denote the object’s position.

(a) Graph the path of $P$.

(b) Find the expressions for the velocity $\mathbf{v}(t)$, speed $|\mathbf{v}(t)|$, and acceleration $\mathbf{a}(t)$.

(c) Find the maximum and minimum values of the speed and where they occur.

(d) Show that the acceleration vector based at $P$ always points to the origin.

**Solution**

(a) Since $x^2/9 + y^2/4 = 1$, the path is the ellipse shown in Figure 7.

(b) The position vector is

$$\mathbf{r}(t) = 3 \cos t \mathbf{i} + 2 \sin t \mathbf{j}$$

and so
\[ v(t) = -3 \sin t \mathbf{i} + 2 \cos t \mathbf{j} \]

\[ |v(t)| = \sqrt{9 \sin^2 t + 4 \cos^2 t} = \sqrt{9 \sin^2 t + 4} \]

\[ a(t) = -3 \cos t \mathbf{i} - 2 \sin t \mathbf{j} \]

(c) Since the speed is given by \( \sqrt{9 \sin^2 t + 4} \), the maximum speed of 3 occurs when \( \sin t = \pm 1 \), that is, when \( t = \pm \pi/2 \). This corresponds to the points \((0, \pm 2)\) on the ellipse. Similarly, the minimum speed of 2 occurs when \( \sin t = 0 \), which corresponds to the points \((\pm 3, 0)\).

(d) Note that \( a(t) = -r(t) \). Thus, if we base \( a(t) \) at \( P \), this vector will point to and exactly reach the origin. We conclude that \( |a(t)| \) is largest at \((\pm 3, 0)\) and smallest at \((0, \pm 2)\).

[EXEMPLARY 6] A projectile is shot from the origin at an angle \( \theta \) from the positive \( x \)-axis with an initial speed of \( v_0 \) feet per second (Figure 8). Neglecting friction, find expressions for the velocity \( v(t) \) and position \( r(t) \), and show that the path is a parabola.

**SOLUTION** The acceleration due to gravity is \( a(t) = -32 \mathbf{j} \) feet per second per second. The initial conditions are \( r(0) = \mathbf{0} \) and \( v(0) = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} \). Starting with \( a(t) = -32 \mathbf{j} \), we integrate twice.

\[ v(t) = \int a(t) \, dt = \int (-32) \, dt \mathbf{j} = -32t \mathbf{j} + C_1 \]

The condition \( v(0) = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} \) allows us to evaluate \( C_1 \) and gives \( C_1 = v_0 \cos \theta \mathbf{i} + v_0 \sin \theta \mathbf{j} \). Thus,

\[ v(t) = (v_0 \cos \theta \mathbf{i} + (v_0 \sin \theta - 32t) \mathbf{j} \]

and

\[ r(t) = \int v(t) \, dt = (tv_0 \cos \theta \mathbf{i}) + (tv_0 \sin \theta - 16t^2) \mathbf{j} + C_2 \]

The condition \( r(0) = \mathbf{0} \) implies that \( C_2 = \mathbf{0} \), so

\[ r(t) = (tv_0 \cos \theta \mathbf{i} + (tv_0 \sin \theta - 16t^2) \mathbf{j} \]

To find the equation of the path, we eliminate the parameter \( t \) in the equations

\[ x = (v_0 \cos \theta) t, \quad y = (v_0 \sin \theta) t - 16t^2 \]

Specifically, we solve the first equation for \( t \) and substitute in the second, giving

\[ y = (\tan \theta) x - \left( \frac{4}{v_0 \cos \theta} \right)^2 x^2 \]

This is the equation of a parabola.

[EXEMPLARY 7] A baseball is thrown with an initial velocity of 75 miles per hour (110 feet per second) 1 degree above horizontal in the direction of the positive \( x \)-axis from an initial height of 8 feet. The initial position is \( r(0) = 8 \mathbf{k} \). In addition to acceleration due to gravity, the spin on the ball causes an acceleration of 2 feet per second per second in the positive \( y \) direction. What is the position of the ball when its \( x \)-component is 60.5 feet?

**SOLUTION** The initial position vector is \( r(0) = 8 \mathbf{k} \), and the initial velocity vector is \( v(0) = 110 \cos 1 \mathbf{i} + 110 \sin 1 \mathbf{k} \). The acceleration vector is \( a(t) = 2 \mathbf{j} - 32 \mathbf{k} \). Proceeding as in the previous example, we have

\[ v(t) = \int a(t) \, dt = \int (2 \mathbf{j} - 32 \mathbf{k}) \, dt = 2t \mathbf{j} - 32t \mathbf{k} + C_1 \]

Since \( 110 \cos 1 \mathbf{i} + 110 \sin 1 \mathbf{k} = v(0) = C_1 \), we have

\[ v(t) = 110 \cos 1 \mathbf{i} + 2t \mathbf{j} + (110 \sin 1 - 32t) \mathbf{k} \]
Integrating the velocity vector gives the position:

\[ r(t) = \int v(t) \, dt = \int [110 \cos 1^\circ \mathbf{i} + 2t \mathbf{j} + (110 \sin 1^\circ - 32)t] \mathbf{k} \, dt \]

\[ = 110(\cos 1^\circ)d\mathbf{i} + 2t \mathbf{j} + [110(\sin 1^\circ)t - 16t^2] \mathbf{k} + C_2 \]

The initial position \( r(0) = 8\mathbf{k} \) implies that \( C_2 = 8\mathbf{k} \). Thus

\[ r(t) = 110(\cos 1^\circ)\mathbf{i} + 2t \mathbf{j} + [8 + 110(\sin 1^\circ)t - 16t^2] \mathbf{k} \]

Next, we must find the value of \( t \) for which the \( x \)-component is 60.5 feet. Setting \( 110(\cos 1^\circ)t = 60.5 \) yields \( t = 60.5/(110 \cos 1^\circ) \approx 0.55008 \) second. The position of the ball at this time is

\[ r(0.55008) = 110(\cos 1^\circ)0.55008\mathbf{i} + (0.55008)^2 \mathbf{j} \]

\[ + [8 + 110(\sin 1^\circ)0.55008 - 16(0.55008)^2] \mathbf{k} \]

\[ \approx 60.5\mathbf{i} + 0.303\mathbf{j} + 4.21\mathbf{k} \]

If this pitch were thrown by a major league pitcher to a major league batter, the ball would be just above the waist (4.21 feet off the ground) and about 4 inches (0.303 feet) from the center of home plate.

**Kepler’s Laws of Planetary Motion (Optional)** In the early part of the 17th century, Johannes Kepler inherited a collection of planetary data from the Danish nobleman Tycho Brahe. Kepler spent years studying the data and through trial and error, and a little luck, he formulated his three laws of planetary motion:

1. Planets move in elliptical orbits with the sun at one focus.
2. A line from the sun to the planet sweeps out equal areas in equal times.
3. The square of a planet’s orbital period is proportional to the cube of its mean distance from the sun.

Only later was it discovered that Kepler’s Laws of Planetary Motion are a consequence of Newton’s Laws of Motion. Kepler’s First Law can be stated as

\[ r(\theta) = \frac{r_{0}(1 + e)}{1 + e \cos \theta} \]

which is the polar equation of an ellipse. Here \( r(\theta) \) is the planet’s distance from the sun for the angle \( \theta \), and \( e \) is the eccentricity of the ellipse. Problem 48, which guides the reader through the derivation of Kepler’s First Law, shows that

\[ e = \frac{r_{0}v_{0}^2}{GM} = 1 - \frac{\left(\frac{dA}{dt}\right)^2}{\frac{GM}{r_0}} \]

where \( M \) is the sun’s mass, \( G \) is the gravitational constant, \( r_0 \) is the shortest distance from the sun to the planet, \( v_0 \) is the planet’s speed when it is closest to the sun, and \( dA/dt \) is the rate of change in the area swept out by a line segment joining the sun and planet (a constant by Kepler’s Second Law). We will assume Kepler’s First Law.

**EXAMPLE 8** Derive Kepler’s Second Law.

**SOLUTION** Let \( r(t) \) denote the position vector of a planet at time \( t \), and let \( r(t + \Delta t) \) be its position \( \Delta t \) time units later (Figure 9). The area \( \Delta A \) swept out in time \( \Delta t \) is approximately half the area of the parallelogram formed by \( r(t) \) and \( r(t + \Delta t) \). Using the fact from the previous section that the area of a triangle formed by two vectors is half the magnitude of the cross product of the vectors, we have

\[ \Delta A \approx \frac{1}{2} |r(t) \times \Delta r| \]

Thus

\[ \Delta A \approx \frac{1}{2} \frac{1}{\Delta t} |r(t) \times \Delta r| \frac{\Delta t}{\Delta t} \]

\[ = \frac{1}{2} |r(t) \times \frac{\Delta r}{\Delta t}| \]

\[ \mathbf{Figure 9} \]
so, letting $\Delta t \to 0$, we get
\[
\frac{dA}{dt} = \frac{1}{2} \left| \vec{r}(t) \times \vec{r}'(t) \right|
\]

The only force acting on the planet is the gravitational attraction of the sun which acts along the line from the sun to the planet and has magnitude $GMm/|\vec{r}(t)|^3$, where $m$ is the planet’s mass. Newton’s Second Law ($F = ma$) implies
\[
-\frac{GMm}{|\vec{r}(t)|^3} \vec{r}(t) = ma(t) = m\vec{r}''(t)
\]

Dividing both sides by $m$ gives $\vec{r}''(t) = -\left(\frac{GM}{|\vec{r}(t)|^3}\right)\vec{r}(t)$.

In light of this, consider the vector $\vec{r}(t) \times \vec{r}'(t)$ in the above expression for $dA/dt$. Differentiating this vector using Property 5 of Theorem B gives
\[
\frac{d}{dt} (\vec{r}(t) \times \vec{r}'(t)) = \vec{r}(t) \times \vec{r}'(t) + \vec{r}'(t) \times \vec{r}'(t)
\]
\[
= \vec{r}(t) \times \left( -\frac{GM}{|\vec{r}(t)|^3} \vec{r}(t) \right) + \vec{0}
\]
\[
= \left( -\frac{GM}{|\vec{r}(t)|^3} \right) \vec{r}(t) \times \vec{r}(t) = \vec{0}
\]

This tells us that the vector $\vec{r}(t) \times \vec{r}'(t)$ is a constant and as a result, its magnitude $|\vec{r}(t) \times \vec{r}'(t)|$ is constant. Thus, $dA/dt$ is a constant.

**EXAMPLE 9** Derive Kepler’s Third Law.

**SOLUTION** Place the sun at the origin, and the $x$-axis so that the planet’s perihelion (point on the orbit closest to the sun) lies along the $x$-axis. The perihelion occurs at point $A$ in Figure 10. Let $C$ denote the point on the orbit that lies on the minor axis, and let $B$ denote the point on the orbit that lies on a line perpendicular to the major axis at the origin as shown in Figure 10. Let $a$ and $b$ denote half the lengths of the major and minor axes of the ellipse, respectively, and let $c$ denote the distance from the center of the two foci to a focus. The string property for ellipses says that the sum of the distances from the focus to any point on the ellipse is $2a$. Thus $FC + CF = 2a$, and since $FC = CF$, we conclude that $FC = CF = a$. Another application of the string property to point $B$ gives $FB + BF = 2a$.

Using the Pythagorean Theorem we conclude $a^2 = b^2 + c^2$ and $(F'B)^2 = h^2 + (2c)^2$ (see Figure 11). From above, $F'B = 2a - BF = 2a - h$. Putting these results together gives
\[
h^2 + (2c)^2 = (F'B)^2 = (2a - h)^2 = 4a^2 - 4ah + h^2
\]
so we conclude that $4c^2 - 4a^2 - 4ah$, hence $c^2 = a^2 - ah$. Since $a^2 - b^2 + c^2$, we conclude that
\[
a^2 - b^2 = c^2 = a^2 - ah
\]
Thus, $b^2 = ah$.

The point $B$ also occurs when the angle $\theta$ is $\pi/2$. Using Kepler’s First Law,
\[
h = r(\pi/2) = \frac{r_0(1 + e)}{1 + e \cos(\pi/2)} = \frac{1}{GM} \left( \frac{2dA}{dt} \right)^2
\]
Let $T$ denote the planet’s period. Over one orbit about the sun, the area $\pi ab$ is swept out. The average rate at which area is swept out is thus $\pi ab/T$ but since $dA/dt$ is constant (Kepler’s Second Law), $dA/dt = \pi ab/T$. Thus
\[
T = \frac{\pi ab}{dA/dt}
\]
Now it all comes together. Using the relationships $b^2 = ah$ and $h = \left( \frac{2dA}{dt} \right)^2 / GM$ from above, we have

---

**Figure 10**

**Figure 11**
\[ T^2 = \left( \frac{\pi ab}{(dA/dt)^2} \right)^2 = \frac{\pi^2 a^2}{(dA/dt)^2} \cdot \frac{2}{GM} \cdot \frac{\pi^2 a^3}{GM} = \frac{4\pi^4 a^5}{GM^2} \]

The closest the planet gets to the sun is \( a - c \) and the farthest is \( a + c \). Kepler called the average of these two values, \( (a - c + a + c)/2 = a \), the mean distance from the sun. The last formula thus implies that the square of the period is proportional to the cube of the mean distance from the sun.

\[ \text{Concepts Review} \]

1. A function that associates with each real number a single vector is called a ______.

2. The function \( \mathbf{F}(t) = f(t) \mathbf{i} + g(t) \mathbf{j} \) is continuous at \( t = c \) if and only if ______. The derivative of \( \mathbf{F} \) is given in terms of \( f \) and \( g \) by \( \mathbf{F}'(t) = \) ______.

3. If a point moves along a curve so that it is at point \( P \) at time \( t \), then the vector \( \mathbf{r}(t) \) from the origin to \( P \) is called the ______ vector of \( P \).

4. In terms of \( \mathbf{r}(t) \), the velocity is ______ and the acceleration is ______. The velocity vector at \( t \) is ______ to the curve, whereas the acceleration vector points to the ______ side of the curve.

\[ \text{Problem Set 11.5} \]

In Problems 1-8, find the required limit or indicate that it does not exist.

1. \[ \lim_{t \to 1} \frac{2(1 - t^2)}{1 - t} \]

2. \[ \lim_{t \to 3} \frac{2(t - 3) - (1 - t^3)}{t - 3} \]

3. \[ \lim_{t \to 0} \frac{t - 1}{t^2 - 1} \]

4. \[ \lim_{t \to 2} \frac{2t^2 + 10t - 28}{t + 2} \]

5. \[ \lim_{t \to 0} \frac{\sin t}{t^2} \]

6. \[ \lim_{t \to 0} \frac{\tan t}{t} \]

7. \[ \lim_{t \to 0} \frac{\ln(1 + t)}{t} \]

8. \[ \lim_{t \to 0} \frac{e^{5t} - 1}{t} \]

9. When no domain is given in the definition of a vector-valued function, it is to be understood that the domain is the set of all (real) scalars for which the rule for the function makes sense and gives real vectors (i.e., vectors with real components). Find the domain of each of the following vector-valued functions:

(a) \( \mathbf{r}(t) = \frac{2}{t - 4} \mathbf{i} + \sqrt{3 - 4} \mathbf{j} + \mathbf{k} \)

(b) \( \mathbf{r}(t) = \left( \frac{2}{t^2} \right)^3 \mathbf{i} + \sqrt{20 - t} \mathbf{j} + 3 \mathbf{k} \)

(c) \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sqrt{9 - t^2} \mathbf{k} \)

10. State the domain of each of the following vector-valued functions:

(a) \( \mathbf{r}(t) = \ln(t - 1) \mathbf{i} + \sqrt{20 - t} \mathbf{j} \)

(b) \( \mathbf{r}(t) = \ln(t^3) \mathbf{k} + \tan^{-1} t \mathbf{j} + t \mathbf{k} \)

(c) \( \mathbf{r}(t) = \frac{1}{\sqrt{1 - t^2}} \mathbf{j} + \frac{1}{\sqrt{9 - t^2}} \mathbf{k} \)

11. For what values of \( t \) is each function in Problem 9 continuous?

12. For what values of \( t \) is each function in Problem 10 continuous?

13. Find \( D_1 \mathbf{r}(t) \) and \( D_2 \mathbf{r}(t) \) for each of the following:

(a) \( \mathbf{r}(t) = (3t + 1) \mathbf{i} + \mathbf{j} + 2t^3 \mathbf{k} \)

(b) \( \mathbf{r}(t) = \sin^2 t \mathbf{i} + \cos 3t \mathbf{j} + t^2 \mathbf{k} \)

14. Find \( \mathbf{r}'(t) \) and \( \mathbf{r}''(t) \) for each of the following:

(a) \( \mathbf{r}(t) = (e^t + e^{-t}) \mathbf{i} + 2t \mathbf{j} + t^2 \mathbf{k} \)

(b) \( \mathbf{r}(t) = \tan 2t \mathbf{i} + \sec 2t \mathbf{j} \)

15. If \( \mathbf{r}(t) = e^{3t} \mathbf{i} - \ln(2t^2) \mathbf{j} \), find \( D_1 [\mathbf{r}(t) \cdot \mathbf{r}''(t)] \).

16. If \( \mathbf{r}(t) = \sin t \mathbf{i} - \cos 3t \mathbf{j} \), find \( D_1 [\mathbf{r}(t) \cdot \mathbf{r}''(t)] \).

17. If \( \mathbf{r}(t) = \sqrt{t^2 - 1} \mathbf{i} + \ln(2t^2) \mathbf{j} \) and \( h(t) = e^{3t} \), find \( D_1 [h(t) \mathbf{r}(t)] \).

18. If \( \mathbf{r}(t) = \sin 2t \mathbf{i} + \cosh t \mathbf{j} \) and \( h(t) = \ln(3t^2 - 2) \), find \( D_1 [h(t) \mathbf{r}(t)] \).

In Problems 19-30, find the velocity \( v \), acceleration \( a \), and speed \( s \) at the indicated time \( t = t_i \).

19. \( \mathbf{r}(t) = 4 \mathbf{i} + 5(t^2 - 1) \mathbf{j} + 2 \mathbf{k} ; t_i = 1 \)

20. \( \mathbf{r}(t) = \mathbf{a} + (t - 1)^2 \mathbf{j} + (t - 3)^3 \mathbf{k} ; t_i = 0 \)

21. \( \mathbf{r}(t) = (1/t) \mathbf{i} + (1/t^2 - 1) \mathbf{j} + t \mathbf{k} ; t_i = 2 \)

22. \( \mathbf{r}(t) = t^2 \mathbf{i} + (t^3 - 3t^2) \mathbf{j} + t \mathbf{k} ; t_i = 1 \)

23. \( \mathbf{r}(t) = \mathbf{i} + \left( \int_0^t x^2 \, dx \right) \mathbf{j} + t^2 \mathbf{k} ; t_i = 2 \)

24. \( \mathbf{r}(t) = \mathbf{i} + 5(t - 1) \mathbf{j} + (t^2 + t + 1)^2 \mathbf{k} \)

25. \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} ; t_i = \pi \)

26. \( \mathbf{r}(t) = \sin 2t \mathbf{i} + \cos 4t \mathbf{j} + \cos 4t \mathbf{k} ; t_i = \frac{\pi}{2} \)

27. \( \mathbf{r}(t) = \tan t \mathbf{i} + \mathbf{j} + \cos 4t \mathbf{k} ; t_i = \frac{\pi}{4} \)

28. \( \mathbf{r}(t) = \left( \int_0^t e^x \, dx \right) \mathbf{i} + \left( \int_0^t \sin x \, dx \right) \mathbf{j} + t^2 \mathbf{k} ; t_i = 2 \)
29. \( r(t) = t \sin \pi t + t \cos \pi t + e^{-t} k, t = 0 \)
30. \( r(t) = \ln t \mathbf{i} + \ln r \mathbf{j} + \ln t \mathbf{k}, t = 2 \)
31. Show that if the speed of a moving particle is constant its acceleration vector is always perpendicular to its velocity vector.
32. Prove that \(|r'(t)| = \text{constant}\) if and only if \(r(t) \cdot r''(t) = 0\).

In Problems 33-38, find the length of the curve with the given vector equation.
33. \( r(t) = t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}, 0 \leq t \leq 2 \)
34. \( r(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j} + \sqrt{2} t \mathbf{k}, 0 \leq t \leq 2 \)
35. \( r(t) = \sqrt{2} t \mathbf{i} + \frac{1}{2} t \mathbf{j} + 6 t \mathbf{k}, 3 \leq t \leq 6 \)
36. \( r(t) = t \mathbf{i} - 2 t \mathbf{j} + 6 t \mathbf{k}, 0 \leq t \leq 1 \)
37. \( r(t) = t \mathbf{i} - 2 t \mathbf{j} + 6 t \mathbf{k}, 0 \leq t \leq 1 \)
38. \( r(t) = \sqrt{2} t \mathbf{i} - \sqrt{2} t \mathbf{j} + 6 t \mathbf{k}, 0 \leq t \leq 1 \)

In Problems 39 and 40, \( \mathbf{F}(t) = \mathbf{f}(u(t)) \). Find \( \mathbf{F}'(t) \) in terms of \( t \).
39. \( \mathbf{f}(u) = \cos u \mathbf{i} + e^u \mathbf{j} \) and \( u(t) = 3 t^2 - 4 \)
40. \( \mathbf{f}(u) = u^3 \mathbf{i} + \sin^2 u \mathbf{j} \) and \( u(t) = \tan t \)

Evaluate the integrals in Problems 41 and 42.
41. \( \int_0^1 (e^t + e^{-t}) \, dt \)
42. \( \int_0^1 [(1 + t^2)^{1/2} + (1 - t)^{1/2}] \, dt \)

A point moves around the circle \( x^2 + y^2 = 25 \) at constant angular speed of 6 radians per second starting at \((5,0)\). Find expressions for \( r(t), \mathbf{v}(t), |\mathbf{r}(t)|, \) and \( \mathbf{u}(t) \) (see Example 3).

44. Consider the motion of a particle along a helix given by \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + (t^2 - 3 t + 2) \mathbf{k} \), where the \( \mathbf{k} \) component measures the height in meters above the ground and \( t = 0 \).
   (a) Does the particle ever move downward?
   (b) Does the particle ever stop moving?
   (c) At what times does it reach a position 12 meters above the ground?
   (d) What is the velocity of the particle when it is 12 meters above the ground?

EXPL. 45. In many places in the solar system, a moon orbits a planet, which in turn orbits the sun. In some cases the orbits are very close to circular. We will assume that these orbits are circular with the sun at the center of the planet’s orbit and the planet at the center of the moon’s orbit. We will further assume that all motion is in a single \( xy \)-plane. Suppose that in the time the planet orbits the sun once the moon orbits the planet ten times.
   (a) If the radius of the moon’s orbit is \( R_m \) and the radius of the planet’s orbit about the sun is \( R_p \), show that the motion of the moon with respect to the sun at the origin could be given by \( x = R_m \cos t + R_p \cos 10t, \quad y = R_m \sin t + R_p \sin 10t \)
   (b) For \( R_p = 1 \) and \( R_m = 0.1 \), plot the path traced by the moon as the planet makes one revolution around the sun.
   (c) Find one set of values for \( R_p, R_m \) and so that at time \( t \) the moon is motionless with respect to the sun.

46. Assuming that the orbits of the earth about the sun and the moon about the earth lie in the same plane and are circular, we can represent the motion of the moon by
   \( r(t) = [93 \cos(2\pi t) + 0.24 \cos(26\pi t)] \mathbf{i} + [93 \sin(2\pi t) + 0.24 \sin(26\pi t)] \mathbf{j} \)
   where \( r(t) \) is measured in millions of miles.
   (a) What are the proper units for \( r(t) \)?
   (b) Plot the path traced by the moon as the earth makes one revolution around the sun.
   (c) What is the period of each of the two motions?
   (d) What is the maximum distance that the moon is from the sun?
   (e) What is the minimum distance that the moon is from the sun?
   (f) Is there ever a time that the moon is stationary with respect to the sun?
   (g) What are the velocity, speed, and acceleration of the moon when \( t = 1/2 \)?

47. Describe in general terms the following “helical” type motions:
   (a) \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k} \)
   (b) \( r(t) = \sin r \mathbf{i} + \cos r \mathbf{j} + r \mathbf{k} \)
   (c) \( r(t) = \sin(\sqrt{t^2 + \pi}) \mathbf{i} + t \mathbf{j} + \cos(\sqrt{t^2 + \pi}) \mathbf{k} \)
   (d) \( r(t) = \sqrt{t} \mathbf{i} + t \cos \mathbf{j} + t \mathbf{k} \)
   (e) \( r(t) = t^2 \sin t \mathbf{i} + t \cos t \mathbf{j} + t \mathbf{k}, t > 0 \)
   (f) \( r(t) = t^2 \sin(\sqrt{t}) \mathbf{i} + t \mathbf{j} + t^2 \cos(\sqrt{t}) \mathbf{k} + t > 1 \)

**Figure 12**

**EXPL.** 48. In this exercise you will derive Kepler’s First Law, that planets travel in elliptical orbits. We begin with the notation. Place the coordinate system so that the sun is at the origin and the planet’s closest approach to the sun (the perihelion) is on the positive \( x \)-axis and occurs at time \( t = 0 \). Let \( r(t) \) denote the position vector and let \( r(t) = \sqrt{r^2(t)} \) denote the distance from the sun at time \( t \). Also, let \( \theta(t) \) denote the angle that the vector \( r(t) \) makes with the positive \( x \)-axis at time \( t \). Thus, \( r(t), \theta(t) \) is the polar coordinate representation of the planet’s position. Let \( \mathbf{u}_1 = \frac{\mathbf{r}}{r} = (\cos \theta) \mathbf{i} + (\sin \theta) \mathbf{j} \) and \( \mathbf{u}_2 = (-\sin \theta) \mathbf{i} + (\cos \theta) \mathbf{j} \). Vectors \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are orthogonal unit vectors pointing in the directions of increasing \( r \) and increasing \( \theta \), respectively. Figure 12 summarizes this notation. We will often omit the argument \( t \), but keep in mind that \( r, \theta, \mathbf{u}_1, \) and \( \mathbf{u}_2 \) are all functions of \( t \). A prime indicates differentiation with respect to time \( t \).
   (a) Show that \( \dot{u}_1 = \dot{\theta} \mathbf{u}_2 \) and \( \dot{\mathbf{u}}_1 = -\dot{\theta} \mathbf{u}_2 \).
   (b) Show that the velocity and acceleration vectors satisfy \( \mathbf{v} = r \dot{r} \mathbf{u}_1 + \dot{r} \mathbf{u}_2 \) and \( \mathbf{a} = (r^2 - r \dot{\theta}^2) \mathbf{u}_1 + (2r \dot{r} \dot{\theta} + r \ddot{\theta}) \mathbf{u}_2 \).
   (c) Use the fact that the only force acting on the planet is the gravity of the sun to express \( \mathbf{a} \) as a multiple of \( \mathbf{u}_2 \), then explain how we can conclude that
Chapter 11 Geometry in Space and Vectors

29. \( \mathbf{r}(t) = t \sin \pi t + t \cos \pi t + e^{t^2} \mathbf{k} \), \( t = 2 \)
30. \( \mathbf{r}(t) = \ln t \mathbf{i} + \ln t^2 \mathbf{j} + \ln t \mathbf{k}, t = 2 \)

31. Show that if the speed of a moving particle is constant its acceleration vector is always perpendicular to its velocity vector.

32. Prove that \( |\mathbf{r}(t)| \) is constant if and only if \( \mathbf{r}(t) \times \dot{\mathbf{r}}(t) = 0 \).

In Problems 33–38, find the length of the curve with the given vector equation.

33. \( \mathbf{r}(t) = t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}; 0 \leq t \leq 2 \)
34. \( \mathbf{r}(t) = t \cos t \mathbf{i} + t \sin t \mathbf{j} + \sqrt{2t} \mathbf{k}; 0 \leq t \leq 2 \)
35. \( \mathbf{r}(t) = \sqrt{6t^3 + 1} \mathbf{i} + 6t \mathbf{k}; \frac{1}{3} \leq t \leq 6 \)
36. \( \mathbf{r}(t) = t^2 \mathbf{i} - 2t \mathbf{j} + 6t \mathbf{k}; 0 \leq t \leq 1 \)
37. \( \mathbf{r}(t) = t \mathbf{i} - 2t \mathbf{j} + 6t \mathbf{k}; 0 \leq t \leq 1 \)
38. \( \mathbf{r}(t) = \sqrt{7t^3} - \sqrt{2t^3} + 6t^2 \mathbf{k}; 0 \leq t \leq 1 \)

In Problems 39 and 40, \( \mathbf{F}(t) = \mathbf{f}(u(t)) \). Find \( \mathbf{F}'(t) \) in terms of \( t \).

39. \( \mathbf{f}(t) = \cos u \mathbf{i} + e^u \mathbf{j} \) and \( u(t) = 3t^2 - 4 \)
40. \( \mathbf{f}(t) = a^2 \mathbf{i} + \sin^2 u \mathbf{j} \) and \( u(t) = \tan t \)

Evaluate the integrals in Problems 41 and 42.

41. \( \int_0^9 (e^t + e^{-t}) \ dt \)
42. \( \int_0^1 [(x^2 + 3)^{1/2} + (1-x^2)] \ dx \)

43. A point moves around the circle \( x^2 + y^2 = 25 \) at constant angular speed of \( \theta \) radians per second starting at \((5,0)\). Find \( \mathbf{r}(t) \), \( \mathbf{v}(t) \), \( |\mathbf{r}(t)| \), and \( u(t) \) (see Example 3).

44. Consider the motion of a particle along a helix given by \( \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + (t^2 - 3t + 2) \mathbf{k} \), where the \( \mathbf{k} \) component measures the height in meters above the ground and \( t \geq 0 \).
   (a) Does the particle ever move downward?
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45. In many places in the solar system, a moon orbits a planet, which in turn orbits the sun. In some cases the orbits are very close to circular. We will assume that these orbits are circular with the sun at the center of the planet’s orbit and the planet at the center of the moon’s orbit. We will further assume that all motion is in a single \( xy \)-plane. Suppose that in the time the planet orbits the sun once the moon orbits the planet ten times.
   (a) If the radius of the moon’s orbit is \( R_0 \) and the radius of the planet’s orbit about the sun is \( R_0 \), show that the motion of the moon with respect to the sun at the origin could be given by \( x = R_0 \cos t + R_0 \cos 10t, \ y = R_0 \sin t + R_0 \sin 10t \).
   (b) For \( R_0 = 1 \) and \( R_0 = 0.1 \), plot the path traced by the moon as it makes one revolution around the sun.
   (c) Find one set of values for \( R_0, R_0 \) and \( \alpha \) so that at time \( t \) the moon is motionless with respect to the sun.

46. Assuming that the orbits of the earth about the sun and the moon about the earth lie in the same plane and are circular, we can represent the motion of the moon by \( \mathbf{r}(t) = [93 \cos(2\pi t) + 0.24 \cos(26\pi t)] \mathbf{i} + [93 \sin(2\pi t) + 0.24 \sin(26\pi t)] \mathbf{j} + e^{t^2} \mathbf{k} \).

where \( x(t) \) is measured in millions of miles.

(a) What are the proper units for \( t \)?
(b) Plot the path traced by the moon as the earth makes its first revolution around the sun.
(c) What is the period of each of the two motions?
(d) What is the maximum distance that the moon is from the sun?
(e) What is the minimum distance that the moon is from the sun?
(f) Is there ever a time that the moon is stationary with respect to the sun?
(g) What are the velocity, speed, and acceleration of the moon when \( t = 1/2 \)?

47. Describe in general terms the following “helical” type motions:
   (a) \( \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k} \)
   (b) \( \mathbf{r}(t) = \sin t^2 \mathbf{i} + \cos t^2 \mathbf{j} + t \mathbf{k} \)
   (c) \( \mathbf{r}(t) = \sin(t^2 - \pi) \mathbf{i} + t \mathbf{j} + \cos(t^3 + \pi) \mathbf{k} \)
   (d) \( \mathbf{r}(t) = \sin t \mathbf{i} + t \cos t \mathbf{j} + t \mathbf{k} \)
   (e) \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}, t > 0 \)
   (f) \( \mathbf{r}(t) = t^2 \sin(\ln t) \mathbf{i} + \ln t \mathbf{j} + \cos(\ln t) \mathbf{k}, t > 1 \)

![Figure 12](image_url)

EXPL. 48. In this exercise you will derive Kepler’s First Law, that planets travel in elliptical orbits. We begin with the notation. Place the coordinate system so that the sun is at the origin and the planet’s closest approach to the sun (the perihelion) is on the positive \( x \)-axis and occurs at time \( t = 0 \). Let \( \mathbf{r}(t) \) denote the position vector and let \( \mathbf{r}(t) = |\mathbf{r}(t)| \) denote the distance from the sun at time \( t \). Also, let \( \theta(t) \) denote the angle that the vector \( \mathbf{r}(t) \) makes with the positive \( x \)-axis at time \( t \). Thus, \( \mathbf{r}(t), \theta(t) \) is the polar coordinate representation of the planet’s position. Let \( \mathbf{u}_1 = \mathbf{r}/|\mathbf{r}| \) and \( \mathbf{u}_2 = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \).

Vectors \( \mathbf{u}_1, \mathbf{u}_2 \) are orthogonal unit vectors pointing in the directions of increasing \( r \) and increasing \( \theta \), respectively. Figure 12 summarizes this notation. We will often omit the argument \( t \), but keep in mind that \( r, \theta, \mathbf{u}_1, \mathbf{u}_2 \) are all functions of \( t \). A prime indicates differentiation with respect to time \( t \).

(a) Show that \( \dot{\mathbf{u}}_1 = \theta' \mathbf{u}_2 \) and \( \mathbf{u}_2 = -\theta' \mathbf{u}_1 \).

(b) Show that the velocity and acceleration vectors satisfy \( \mathbf{v} = \mathbf{r}' \mathbf{u}_1 + \theta' \mathbf{u}_2 \) and \( \mathbf{a} = (\theta'^2 + \theta') \mathbf{u}_1 + (2\theta' \theta'' + \theta'^2) \mathbf{u}_2 \).

(c) Use the fact that the only force acting on the planet is the gravity of the sun to express \( \mathbf{a} \) as a multiple of \( \mathbf{u}_1 \), then explain how we can conclude that
If we solve each of the parametric equations for \( t \) (assuming that \( a, b, \) and \( c \) are all different from zero) and equate the results, we obtain the symmetric equations for the line through \( (x_0, y_0, z_0) \) with direction numbers \( a, b, c \); that is,

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

This is the conjunction of the two equations

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}
\]

both of which are the equations of planes (Figure 3); and, of course, the intersection of two planes is a line.

**EXAMPLE 2.** Find the symmetric equations of the line that is parallel to the vector \( (4, -3, 2) \) and goes through \( (2, 5, -1) \).

**SOLUTION**

\[
\frac{x - 2}{4} = \frac{y - 5}{-3} = \frac{z + 1}{2}
\]

**EXAMPLE 3.** Find the symmetric equations of the line of intersection of the planes

\[
2x - y - 5z = -14 \quad \text{and} \quad 4x + 5y + 4z = 28
\]

**SOLUTION** We begin by finding two points on the line. Any two points would do, but we choose to find the points where the line pierces the \( yz \)-plane and the \( xz \)-plane (Figure 4). The former is obtained by setting \( x = 0 \) and solving the resulting equations \(-y - 5z = -14 \) and \( 5y + 4z = 28 \) simultaneously. This yields the point \((0, 4, 2)\). A similar procedure with \( y = 0 \) gives the point \((3, 0, 4)\). Consequently, a vector parallel to the required line is

\[
(3 - 0, 0 - 4, 4 - 2) = (3, -4, 2)
\]

Using \((3, 0, 4)\) for \((x_0, y_0, z_0)\), we get

\[
\frac{x - 3}{3} = \frac{y - 0}{-4} = \frac{z - 4}{2}
\]

An alternative solution is based on the fact that the line of intersection of two planes is perpendicular to both of their normals. The vector \( \mathbf{u} = (2, -1, -5) \) is normal to the first plane; \( \mathbf{v} = (4, 5, 4) \) is normal to the second. Since

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 2 & -1 & -5 \\ 4 & 5 & 4 \end{vmatrix} = 21i - 28j + 14k
\]

the vector \( \mathbf{w} = (21, -28, 14) \) is parallel to the required line. This implies that \( \frac{1}{3} \mathbf{w} = (3, -4, 2) \) also has this property. Next, find any point on the line of intersection, for example, \((3, 0, 4)\), and proceed as in the earlier solution.

**EXAMPLE 4.** Find parametric equations of the line through \((1, -2, 3)\) that is perpendicular to both the \( x \)-axis and the line

\[
\frac{x - 4}{2} = \frac{y - 3}{-1} = \frac{z}{5}
\]
**SOLUTION**  The x-axis and the given line have directions \( \mathbf{u} = (1, 0, 0) \) and \( \mathbf{v} = (2, -1, 5) \), respectively. A vector perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \) is

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 2 & -1 & 5 \end{vmatrix} = 0i - 5j - k
\]

The required line is parallel to \((0, -5, -1)\) and so also to \((0, 5, 1)\). Since the first direction number is zero, the line does not have symmetric equations. Its parametric equations are

\[
x = 1, \quad y = -2 + 5t, \quad z = 3 + t
\]

**Tangent Line to a Curve**  Let

\[
\mathbf{r} = \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = (f(t), g(t), h(t))
\]

be the position vector determining a curve in three-space (Figure 5). The tangent line to the curve has direction vector

\[
\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} = (f'(t), g'(t), h'(t))
\]

**EXAMPLE 5**  Find the parametric equations and symmetric equations for the tangent line to the curve determined by

\[
\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + \frac{1}{3}t^3\mathbf{k}
\]

at \( P(2) = (2, 2, \frac{8}{3}) \).

**SOLUTION**

\[
\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}
\]

and

\[
\mathbf{r}'(2) = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}
\]

so the tangent line has direction vector \((1, 2, 4)\). Its symmetric equations are

\[
\frac{x - 2}{1} = \frac{y - 2}{2} = \frac{z - \frac{8}{3}}{4}
\]

The parametric equations are

\[
x = 2 + t, \quad y = 2 + 2t, \quad z = \frac{8}{3} + 4t
\]

There is exactly one plane perpendicular to a smooth curve at a given point \( P \). If we have a direction vector for the tangent line to the curve at \( P \), then it is a normal vector for the plane (Figure 6). This, together with the given point, is enough to obtain the equation of the desired plane.

**EXAMPLE 6**  Find the equation of the plane perpendicular to the curve \( \mathbf{r}(t) = 2\cos\pi t\mathbf{i} + \sin\pi t\mathbf{j} + t\mathbf{k} \) at \( P(2, 0, 8) \).

**SOLUTION**  The first issue to address is the value of \( t \) that yields the given point. Equating the \( z \) components gives \( t^3 = 8 \), leading to \( t = 2 \). A quick check verifies that \( t = 2 \) also yields the \( x \)- and \( y \)-components of \( P \). Since \( \mathbf{r}'(t) = -2\pi \sin \pi t \mathbf{i} + \pi \cos \pi t \mathbf{j} + \mathbf{k} \), we see that the direction vector for the tangent line at \( P \), which is also a normal vector for the desired plane, is \( \mathbf{r}'(2) = \pi \mathbf{j} + 12\mathbf{k} = (0, \pi, 12) \). The equation of the plane is therefore

\[
0x + \pi y + 12z = D
\]

To determine \( D \), we substitute \( x = 2, y = 0, z = 8 \):

\[
D = 0(2) + \pi(0) + 12(8) = 96
\]

The equation of the desired plane is \( \pi y + 12z = 96 \).
Concepts Review

1. The parametric equations for a line through (1, -2, 3) parallel to the vector (4, -2, -1) are \( x = \ldots \), \( y = \ldots \), \( z = \ldots \).

2. The symmetric equations for the line of Question 1 are ___.

Problem Set 11.6

In Problems 1–4, find the parametric equations of the line through the given point and the given vector.

1. \((1, -2, 3), \langle 4, 5, 6 \rangle\)  
2. \((2, -1, -5), \langle 7, -2, 3 \rangle\)  
3. \((4, 2, 3), \langle 6, 2, -1 \rangle\)  
4. \((5, -3, -3), \langle 5, 4, 2 \rangle\)

In Problems 5–8, write both the parametric equations and the symmetric equations for the line through the given point parallel to the given vector.

5. \((4, 5, -3), \langle 3, 2, 1 \rangle\)  
6. \((-1, 3, -6), \langle -2, 0, 5 \rangle\)  
7. \((1, 1, -1), \langle -10, -100, -1000 \rangle\)  
8. \((-2, -2, -2), \langle 7, -6, 3 \rangle\)

In Problems 9–12, find the parametric equations of the line of intersection of the given pair of planes.

9. \(4x + 3y - 7z = 1, 10x + 6y - 5z = 10\)  
10. \(x + y - z = 2, 3x - 2y + z = 3\)  
11. \(x + 4y - 2z = 13, 2x - y - 2z = 5\)  
12. \(x - 3y + z = -1, 6x - 5y + 4z = 9\)

13. Find the symmetric equations of the line through \((4, 0, 6)\) and perpendicular to the plane \(x - 5y + 2z = 10\).

14. Find the symmetric equations of the line through \((-5, 7, -2)\) and parallel to the line \((2, 1, -3)\) and \((5, 4, -1)\).

15. Find the symmetric equations of the line through \((-5, -3, 4)\) that intersects the \(z\)-axis at a right angle.

16. Find the symmetric equations of the line through \((2, 4, 5)\) that is parallel to the plane \(3x + y - 2z = 5\) and perpendicular to the line \(\frac{x + 8}{2} = \frac{y - 5}{3} = \frac{z - 1}{1}\).

17. Find the equation of the plane that contains the parallel lines:
\[
\begin{align*}
x &= -2 + 2t \\
y &= 1 + 4t \\
z &= 2 - t
\end{align*}
\]
and
\[
\begin{align*}
x &= 2 - 2t \\
y &= 3 - 4t \\
z &= 1 + 4t
\end{align*}
\]

18. Show that the lines
\[
\frac{x - 1}{-4} = \frac{y - 2}{3} = \frac{z - 4}{-2}
\]
and
\[
\frac{x - 2}{-1} = \frac{y - 1}{1} = \frac{z + 2}{6}
\]
intersect, and find the equation of the plane that they determine.

19. Find the equation of the plane containing the line \(x = 1 + 2t, y = -1 + 3t, z = 4 + 4t\).

20. Find the equation of the plane containing the line \(x = 3t, y = 1 + t, z = 2 + t\) parallel to the intersection of the planes \(2x - y + z = 0\) and \(y + z + 1 = 0\).

21. Find the distance between the skew (nonintersecting and nonparallel) lines \(x = 2 - t, y = 3 + 4t, z = 2 + 2t\) and \(x = -1 + t, y = 2, z = -1 + 2t\) by using the following steps.

(a) Note by setting \(t = 0\) that \((2, 3, 0)\) is on the first line.

(b) Find the equation of the plane \(\pi\) through \((2, 3, 0)\) parallel to both given lines (i.e., with normal perpendicular to both).

(c) Find a point \(Q\) on the second line.

(d) Find the distance from \(Q\) to the plane \(\pi\). (See Example 10 of Section 11.3.)

22. Find the distance between the skew lines \(x = 1 + 2t, y = -3 + 4t, z = -1 - t\) and \(x = 4 - 2t, y = 1 + 3t, z = 2t\) (see Problem 21).

23. Find the symmetric equations of the tangent line to the curve with equation
\[
r(t) = 2 \cos t \mathbf{i} + 6 \sin t \mathbf{j} + t \mathbf{k}
\]
at \(t = \pi/3\).

24. Find the parametric equations of the tangent line to the curve \(x = 2t^2, y = 4t, z = t^3\) at \(t = 1\).

25. Find the equation of the plane perpendicular to the curve \(x = 3t, y = 2t^2, z = t^3\) at \(t = -1\).

26. Find the equation of the plane perpendicular to the curve \(r(t) = t \sin t \mathbf{i} + 3t \mathbf{j} + 2t \cos t \mathbf{k}\)
at \(t = \pi/2\).

27. Consider the curve
\[
r(t) = 2t \mathbf{i} + \sqrt{7} t \mathbf{j} + \sqrt{9 - 7t - 4t^2} \mathbf{k}, 0 \leq t \leq \frac{1}{2}
\]
(a) Show that the curve lies on a sphere centered at the origin.

(b) Where does the tangent line at \(t = \frac{1}{4}\) intersect the \(x\)-axis?

28. Consider the curve \(r(t) = \sin t \mathbf{i} + \sin^2 t \mathbf{j} + \cos t \mathbf{k}\), \(0 \leq t \leq 2\pi\).

(a) Show that the curve lies on a sphere centered at the origin.

(b) Where does the tangent line at \(t = \pi/6\) intersect the \(xy\)-plane?

29. Consider the curve \(r(t) = 2t \mathbf{i} + t \mathbf{j} + (1 - t)^2 \mathbf{k}\)

(a) Show that this curve lies on a plane and find the equation of this plane.

(b) Where does the tangent line at \(t = 2\) intersect the \(xy\)-plane?

30. Let \(P\) be a point on a plane with normal vector \(\mathbf{n}\) and let \(Q\) be a point off the plane. Show that the result of Example 10 of
Section 11.3, the distance \( d \) between the point \( Q \) and the plane, can be expressed as
\[
d = \frac{|PQ \cdot n|}{|n|}
\]
and use this result to find the distance from \((4, -2, 3)\) to the plane \(4x - 4y + 2z = 2\).

31. **Point to Line** Let \( P \) be a point on a line with direction \( n \) and \( Q \) a point off the line (Figure 7). Show that the distance \( d \) from \( Q \) to the line is given by
\[
d = \frac{|PQ \times n|}{|n|}
\]
and use this result to find each distance in parts (a) and (b).

(a) From \( Q(1, 0, -4) \) to the line \( \frac{x - 3}{2} = \frac{y + 2}{-2} = \frac{z - 1}{1} \)

(b) From \( Q(2, -1, 3) \) to the line \( x = 1 + 2t, \ y = -1 + 3t, \ z = -6t \)

![Figure 7](image)

32. **Line to Line** Let \( P \) and \( Q \) be points on nonintersecting skew lines with directions \( n_1 \) and \( n_2 \), and let \( n = n_1 \times n_2 \) (Figure 8). Show that the distance \( d \) between these lines is given by
\[
d = \frac{|PQ \cdot n|}{|n|}
\]
and use this result to find the distance between each pair of lines in parts (a) and (b).

(a) \( \frac{x - 3}{1} = \frac{y + 2}{1} = \frac{z - 1}{2} \) and \( \frac{x + 4}{3} = \frac{y + 5}{4} = \frac{z}{5} \)

(b) \( x = 1 + 2t, \ y = -2 + 3t, \ z = -4t \) and \( x = 3t, \ y = 1 + t, \ z = -5t \)

**Answers to Concepts Review:**
1. \( 1 + 4t; -3 - 2t; 2 - t \)
2. \( \frac{x - 1}{4} = \frac{y + 3}{-2} = \frac{z - 1}{-1} \)
3. \( 2t; 3j + 3r^2k \)
4. \( (2, -3, 3); \frac{x - 1}{2} = \frac{y + 3}{-3} = \frac{z - 1}{3} \)

### 11.7 Curvature and Components of Acceleration

We want to introduce a number, called the curvature, that measures how sharply a curve bends at a given point. A line should have curvature zero, and a curve that is turning sharply should have a large curvature (Figure 1).

Let \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \) denote the position of an object at time \( t \). We will assume that \( \mathbf{r}'(t) \) is continuous and that \( \mathbf{r}'(t) \) is never equal to the zero vector. This last condition assures that the accumulated arc length \( s(t) \) increases as \( t \) increases. Our measure of curvature is going to involve how fast the tangent vector is changing. Rather than working with the tangent vector \( \mathbf{r}'(t) \) we choose to work with the unit tangent vector (Figure 2)

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}
\]

To accomplish the task of defining curvature, we consider the rate of change in the unit tangent vector. Figures 3 and 4 illustrate this concept for a given curve. As the object moves from points \( A \) to \( B \) (Figure 3) in time \( \Delta t \), the unit tangent vector
changed very little; in other words, the magnitude of \( \mathbf{T}(t + \Delta t) - \mathbf{T}(t) \) is small. On the other hand, as the object moves from points \( C \) to \( D \) (figure 4), also in time \( \Delta t \), the unit tangent vector changed quite a bit; in other words, the magnitude of \( \mathbf{T}(t + \Delta t) - \mathbf{T}(t) \) is large. Our definition of curvature \( \kappa \) is therefore the magnitude of the rate of change of the unit tangent vector with respect to arc length \( s \); that is,

\[
\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|
\]

We differentiate with respect to arc length \( s \) rather than with respect to \( t \) because we want the curvature to be an intrinsic property of the curve, not how fast the object moves along the curve. (Imagine circular motion; the curvature of the circle should not depend on how fast the object travels around the curve.)

The definition of curvature given above does not help us to actually evaluate the curvature of a particular curve. To find a workable formula, we proceed as follows. In Section 11.5 we saw that the speed of an object could be expressed as

\[
\text{speed} = |\mathbf{v}(t)| = \frac{ds}{dt}
\]

Since \( s \) increases as \( t \) increases we can apply the Inverse Function Theorem (Theorem 6.2B) to conclude that the inverse of \( s(t) \) exists and

\[
\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\mathbf{v}(t)|}
\]

This allows us to write

\[
\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds} \right\| = \frac{1}{|\mathbf{v}(t)|} \left\| \frac{d\mathbf{T}}{dt} \right\| = \frac{\left\| \frac{d\mathbf{T}}{dt} \right\|}{|\mathbf{v}(t)|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|}
\]

**Some Important Examples**  To convince you that our definition of curvature is sensible, we illustrate with some familiar curves.

**EXAMPLE 1**  Show that the curvature of a line is identically zero.

**SOLUTION**  For a line, the unit tangent vector is a constant, so its derivative is \( \mathbf{0} \). But to illustrate vector methods, we give an algebraic demonstration. If motion is along the line whose parametric equation is given by

\[
\begin{align*}
  x &= x_0 + at \\
  y &= y_0 + bt \\
  z &= z_0 + ct
\end{align*}
\]

then the position vector can be written as

\[
\mathbf{r}(t) = (x_0, y_0, z_0) + t(a, b, c)
\]

Thus

\[
\mathbf{v}(t) = \mathbf{r}'(t) = (a, b, c)
\]

\[
\mathbf{T}(t) = \frac{(a, b, c)}{\sqrt{a^2 + b^2 + c^2}}
\]

\[
\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|} = \frac{0}{\sqrt{a^2 + b^2 + c^2}} = 0
\]

**EXAMPLE 2**  Find the curvature of a circle of radius \( a \).

**SOLUTION**  We assume that the circle lies in the xy-plane and is centered at the origin so that the position vector is

\[
\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}
\]

Curvature is a relatively simple concept. However, the computations required for computing the curvature are often long and messy.
Thus,

\[ \mathbf{v}(t) = \mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} \]

\[ |\mathbf{v}(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a \]

\[ \mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{-a \sin t \mathbf{i} + a \cos t \mathbf{j}}{a} = -\sin t \mathbf{i} + \cos t \mathbf{j} \]

\[ \kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|} = \frac{|-\cos t \mathbf{i} - \sin t \mathbf{j}|}{a} = \frac{1}{a} \]

Since \( \kappa \) is the reciprocal of the radius, small circles have large curvature, and large circles have small curvature. See Figure 5.

**EXAMPLE 3** Find the curvature for the helix \( \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j} + ct \mathbf{k} \).

**SOLUTION**

\[ \mathbf{v}(t) = \mathbf{r}'(t) = -a \sin t \mathbf{i} + a \cos t \mathbf{j} + c \mathbf{k} \]

\[ |\mathbf{v}(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + c^2} = \sqrt{a^2 + c^2} \]

\[ \mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{-a \sin t \mathbf{i} + a \cos t \mathbf{j} + c \mathbf{k}}{\sqrt{a^2 + c^2}} \]

\[ \mathbf{T}'(t) = \frac{-a \cos t \mathbf{i} - a \sin t \mathbf{j}}{\sqrt{a^2 + c^2}} \]

\[ \kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|} = \frac{|(-a \cos t \mathbf{i} - a \sin t \mathbf{j})/\sqrt{a^2 + c^2}|}{\sqrt{a^2 + c^2}} = \frac{a}{a^2 + c^2} \]

For the three curves discussed so far, the line, circle, and helix, the curvature is a constant. This phenomenon occurs only for special curves. Normally the curvature is a function of \( t \).

**Radius and Center of Curvature for a Plane Curve** Let \( P \) be a point on a plane curve (i.e., a curve lying entirely in the xy-plane) where the curvature is nonzero. Consider the circle that is tangent to the curve at \( P \) which has the same curvature there. Its center will lie on the concave side of the curve. This circle is called the **circle of curvature** or **osculating circle**. Its radius \( R = 1/\kappa \) is called the **radius of curvature** and its center is the **center of curvature**. (See Figure 6.) These notions are illustrated in the next example.

**EXAMPLE 4** Find the curvature and the radius of curvature of the curve traced by the position vector

\[ \mathbf{r}(t) = 2t \mathbf{i} + t^2 \mathbf{j} \]

at the points \((0,0)\) and \((2,1)\).

**SOLUTION**

\[ \mathbf{v}(t) = \mathbf{r}'(t) = 2 \mathbf{i} + 2t \mathbf{j} \]

\[ |\mathbf{v}(t)| = \sqrt{2^2 + (2t)^2} = 2\sqrt{1 + t^2} \]

\[ \mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{2 \mathbf{i} + 2t \mathbf{j}}{2\sqrt{1 + t^2}} = \frac{1}{\sqrt{1 + t^2}} \mathbf{i} + \frac{t}{\sqrt{1 + t^2}} \mathbf{j} \]

\[ \mathbf{T}'(t) = \frac{-t}{(1 + t^2)^{3/2}} \mathbf{i} + \frac{1}{(1 + t^2)^{3/2}} \mathbf{j} \]

\[ \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{v}(t)|} = \frac{\sqrt{t^2 + 1}}{2(1 + t^2)^{3/2}} = \frac{1}{2(1 + t^2)^{3/2}} \]
The points (0,0) and (2,1) occur when \( t = 0 \) and \( t = 1 \), respectively. Thus, the values of the curvature at these points are
\[
\kappa(0) = \frac{1}{2(1 + 0^2)^{3/2}} = \frac{1}{2} \\
\kappa(1) = \frac{1}{2(1 + 1^2)^{3/2}} = \frac{\sqrt{2}}{8}
\]
The two values for the radius of curvature are thus \( 1/\kappa(0) = 2 \) and \( 1/\kappa(1) = 8/\sqrt{2} = 4\sqrt{2} \). The circles of curvature are shown in Figure 7.

**Other Formulas for Curvature of a Plane Curve** Let \( \phi \) denote the angle measured counterclockwise from \( \mathbf{i} \) to \( \mathbf{T} \) (Figure 8). Then,
\[
\mathbf{T} = \cos \phi \mathbf{i} + \sin \phi \mathbf{j}
\]
and so
\[
\frac{d\mathbf{T}}{d\phi} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}
\]
Now \( d\mathbf{T}/d\phi \) is a unit vector (length 1) and \( \mathbf{T} \cdot d\mathbf{T}/d\phi = 0 \). Moreover,
\[
\kappa = \frac{d\mathbf{T}}{ds} \cdot \frac{d\mathbf{T}}{d\phi} = \frac{d\mathbf{T}}{ds} \cdot \frac{d\mathbf{T}}{d\phi} = \frac{d\mathbf{T}}{ds} \cdot \frac{d\mathbf{T}}{d\phi} = \frac{d\mathbf{T}}{ds}
\]
This formula for \( \kappa \) helps our intuitive understanding of curvature (it measures the rate of change of \( \phi \) with respect to \( s \)), and also helps us to give a fairly simple proof of the following important theorem.

**Theorem A**

Consider a curve with vector equation \( \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \), that is, with parametric equations \( x = f(t) \) and \( y = g(t) \). Then
\[
\kappa = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}}
\]
In particular, if the curve is the graph of \( y = g(x) \), then
\[
\kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}
\]
Primes indicate differentiation with respect to \( t \) in the first formula and with respect to \( x \) in the second formula.

**Proof** We might calculate \( \kappa \) directly from the formula \( \kappa = \frac{|\mathbf{T}'(t)|}{||\mathbf{r}'(t)||} \), a task we propose in Problem 78. It is a good (but painful) exercise in differentiation and algebraic manipulation. Rather, we choose to use the formula \( \kappa = |d\phi/ds| \) derived above. Refer to Figure 8, from which we see that
\[
\tan \phi = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'}
\]
Differentiate both sides of this equation with respect to \( t \) to obtain
\[
\sec^2 \phi \frac{d\phi}{dt} = \frac{x'y'' - y'x''}{(x')^2}
\]
Then
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\[ \frac{d\phi}{dt} = \frac{x'y'' - y'x''}{(x')^2 \sec^2 \phi} = \frac{x'y'' - y'x''}{(x')^2 (1 + \frac{1}{x'^2})} = \frac{x'y'' - y'x''}{(x')^2 + (y')^2} \]

But

\[ \kappa = \frac{d\phi}{ds} \]

When we put these two results together, we obtain

\[ \kappa = \frac{|x'y'' - y'x'|}{[(x')^2 + (y')^2]^{3/2}} \]

which is the first assertion of the theorem.

To obtain the second assertion, simply regard \( y = g(x) \) as corresponding to the parametric equations \( x = t, y = g(t) \) so that \( x' = 1 \) and \( x'' = 0 \). The conclusion follows.

**Example 5** Find the curvature of the ellipse

\[ x = 3 \cos t, \quad y = 2 \sin t \]

at the points corresponding to \( t = 0 \) and \( t = \pi/2 \), that is, at \((3, 0)\) and \((0, 2)\). Sketch the ellipse showing the corresponding circles of curvature.

**Solution** From the given equations,

\[ x' = -3 \sin t, \quad y' = 2 \cos t \]
\[ x'' = -3 \cos t \quad y'' = -2 \sin t \]

Thus,

\[ \kappa = \kappa(t) = \frac{|x'y'' - y'x'|}{[(x')^2 + (y')^2]^{3/2}} = \frac{6 \sin^2 t + 6 \cos^2 t}{9 \sin^2 t + 4 \cos^2 t} = \frac{5 \sin^2 t + 4}{9 \sin^2 t + 4}^{1/2} \]

Consequently,

\[ \kappa(0) = \frac{6}{4^{3/2}} = \frac{3}{4} \]
\[ \kappa \left( \frac{\pi}{2} \right) = \frac{6}{9^{3/2}} = \frac{2}{9} \]

Note that \( \kappa(0) \) is larger than \( \kappa(\pi/2) \), as it should be. Figure 9 shows the circle of curvature at \((3, 0)\), which has radius \( \frac{3}{4} \), and the one at \((0, 2)\), which has radius \( \frac{2}{9} \).

**Example 6** Find the curvature of \( y = \ln|\cos x| \) at \( x = \pi/3 \).

**Solution** We employ the second formula of Theorem A, noting that the primes now indicate differentiation with respect to \( x \). Since \( y' = -\tan x \) and \( y'' = -\sec^2 x \),

\[ \kappa = \frac{|-\sec^2 x|}{(1 + \tan^2 x)^{3/2}} = \frac{\sec^2 x}{(\sec^2 x)^{3/2}} = |\cos x| \]

At \( x = \pi/3 \), \( \kappa = \frac{1}{2} \).
**Components of Acceleration** For motion along the curve with position vector \( r(t) \), the unit tangent vector is \( T(t) = \frac{r'(t)}{\|r'(t)\|} \). This vector satisfies
\[
T(t) \cdot T(t) = 1
\]
for all \( t \). Differentiating both sides with respect to \( t \), and using the Product Rule on the left side, gives
\[
T(t) \cdot T'(t) + T(t) \cdot T'(t) = 0
\]
This reduces to \( T(t) \cdot T'(t) = 0 \) telling us that \( T(t) \) and \( T'(t) \) are perpendicular for all \( t \). In general, \( T' \) is not a unit vector, so we define the **principal unit normal vector** to be
\[
N(t) = \frac{T'(t)}{\|T'(t)\|}
\]
Now, imagine that you are riding in a car on a winding road. As the car accelerates you feel pushed in the opposite direction. If the car speeds up, you feel a push backwards, and when you are turning left, you feel a push to the right. These two kinds of acceleration are called the **tangential** and **normal components of acceleration**, respectively. What we would like to do is to express the acceleration vector \( a(t) = r''(t) \) in terms of these two components, that is, in terms of the unit tangent vector \( T(t) \) and the unit normal vector \( N(t) \). Specifically, we would like to find scalars \( a_T \) and \( a_N \) so that
\[
a = a_T T + a_N N
\]
To accomplish this we note that
\[
T = \frac{v}{|v|} = \frac{v}{ds/dt}
\]
so
\[
v = \frac{ds}{dt} T
\]
Differentiating both sides with respect to \( t \), and using the Product Rule, gives
\[
v' = \frac{ds}{dt} T' + T \frac{d^2 s}{dt^2}
\]
Using the facts that \( a = v' \), \( T' = |T'||N| \), and \( |T'| = \kappa \frac{ds}{dt} \) we have
\[
a = \frac{d^2 s}{dt^2} T + \frac{ds}{dt} |T'||N| = \frac{d^2 s}{dt^2} T + \left( \frac{ds}{dt} \right)^2 \kappa N
\]
The tangential and normal components of acceleration are
\[
a_T = \frac{d^2 s}{dt^2}
\]
and
\[
a_N = \left( \frac{ds}{dt} \right)^2 \kappa
\]
These results make sense from a physical point of view. If you are speeding up on a straight road, then \( a_T = \frac{d^2 s}{dt^2} > 0 \), and \( \kappa = 0 \) so \( a_N = 0 \). Thus, in this case you would feel a push backward and no push to either side. On the other hand, if you are going around a curve at a constant speed (i.e., \( ds/dt \) is constant) then
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\[ a_T = \frac{d^2 s}{dt^2} = 0 \text{ and } \kappa > 0, \text{ making } a_N \text{ positive. Finally, imagine going around a curve while speeding up. In this case both } a_T \text{ and } a_N \text{ will be positive, and } a \text{ will point inward and forward as shown in Figure 10. You would feel thrown back and to the right.} \]

To calculate \( a_N \), it appears that we must calculate the curvature \( \kappa \). However, this can be avoided by noting that since \( T \) and \( N \) are orthogonal,

\[ |a|^2 = a_T^2 + a_N^2 \]

so we can compute

\[ a_N = \sqrt{|a|^2 - a_T^2} \]

The vector \( N \) can be computed indirectly from

\[ N = \frac{a - a_T T}{a_N} \]

**Vector Forms for the Components of Acceleration**

We can write the formulas for the components of acceleration in terms of the position vector \( r \). We begin with

\[ a = a_T T + a_N N \]

and dot both sides by \( T \) to get

\[ T \cdot a = T \cdot (a_T T + a_N N) = a_T T \cdot T + a_N T \cdot N = a_T (T) + a_N (0) = a_T \]

Here we have used the facts that \( T \) is a unit vector and that \( T \) and \( N \) are orthogonal. Thus,

\[ a_T = T \cdot a = \frac{r' \cdot r''}{|r'|} \]

We can find a similar formula for \( a_N \) by crossing both sides by \( T \):

\[ T \times a = a_T T \times T + a_N T \times N = a_T 0 + a_N (T \times N) = a_N (T \times N) \]

Taking the magnitude of both sides gives

\[ |T \times a| = |a_N||T \times N| = a_N|T||N| \sin \frac{\pi}{2} = a_N (1)(1) = a_N \]

Notice that \( a_N = (ds/dt)^2 \kappa > 0 \), so the absolute value bars are not needed for \( a_N \). Thus,

\[ a_N = |T \times a| = \frac{r' \times r''}{|r'|} \]

Finally, we can find a formula for the curvature \( \kappa \):

\[ \kappa = \frac{a_N}{(ds/dt)^2} = \frac{|r' \times r''|}{|r'|} = \frac{|r' \times r''|}{|r'|^3} \]

**Binormal at \( P \) (Optional)**

Given a curve \( C \) and the unit tangent vector \( T \) at \( P \), there are, of course, infinitely many unit vectors perpendicular to \( T \) at \( P \) (Figure 11). We picked one of them, \( N = T / |T| \), and called it the principal normal. The vector

\[ B = T \times N \]
is called the **binormal**. It, too, is a unit vector and it is perpendicular to both \( T \) and \( N \). (Why?)

If the unit tangent vector \( T \), the principal normal \( N \), and the binormal \( B \) have their initial points at \( P \), they form a right-handed, mutually perpendicular triple of unit vectors known as the **trihedral** at \( P \) (Figure 12). This moving trihedral plays a crucial role in a subject called differential geometry: The plane of \( T \) and \( N \) is called the **osculating plane** at \( P \).

**EXAMPLE 7** Find \( T \), \( N \), and \( B \), and the normal and tangential components of acceleration for uniform circular motion \( r(t) = a \cos \omega t \mathbf{i} + a \sin \omega t \mathbf{j} \).

**SOLUTION**

\[
T = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{-a \omega \sin \omega t \mathbf{i} + a \omega \cos \omega t \mathbf{j}}{\sqrt{(-a \omega \sin \omega t)^2 + (a \omega \cos \omega t)^2}} = -\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j}
\]

\[
N = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{-\omega \cos \omega t \mathbf{i} - \omega \sin \omega t \mathbf{j}}{\sqrt{(-\omega \cos \omega t)^2 + (-\omega \sin \omega t)^2}} = -\cos \omega t \mathbf{i} - \sin \omega t \mathbf{j}
\]

\[
B = T \times N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \omega t & \cos \omega t & 0 \\ -\cos \omega t & -\sin \omega t & 0 \end{vmatrix} = \mathbf{k}
\]

\[
\mathbf{a}_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|^2} = \frac{(-a \omega \sin \omega t \mathbf{i} + a \omega \cos \omega t \mathbf{j}) \cdot (-a \omega^2 \cos \omega t \mathbf{i} + a \omega^2 \sin \omega t \mathbf{j})}{a \omega^2} = 0
\]

\[
\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \\ -\omega \cos \omega t & -\omega \sin \omega t & 0 \end{vmatrix} = a^2 \omega^3 \mathbf{k}
\]

\[
\mathbf{a}_N = \frac{\mathbf{r}' \times \mathbf{r}''}{\|\mathbf{r}'\|} = \frac{a^2 \omega^3 \mathbf{k}}{a \omega} = a \omega^2 \mathbf{k}
\]

The tangential component of acceleration is 0 since the object is moving at uniform speed. The normal component of acceleration is equal to the magnitude of the acceleration vector. Figure 13 shows the vectors \( T \), \( N \), and \( B \). □

**EXAMPLE 8** At the point \( (1, 1, \frac{1}{3}) \), find \( T \), \( N \), \( B \), \( a_T \), \( a_N \), and \( \kappa \) for the curvilinear motion

\[
r(t) = t \mathbf{i} + t^2 \mathbf{j} + \frac{1}{3} t^3 \mathbf{k}
\]

**SOLUTION**

\[
r'(t) = \mathbf{i} + 2t \mathbf{j} + t^2 \mathbf{k}
\]

\[
r''(t) = 2 \mathbf{j} + 2t \mathbf{k}
\]

At \( t = 1 \), which gives the point \( (1, 1, \frac{1}{3}) \), we have

\[
r' = \mathbf{i} + 2 \mathbf{j} + \mathbf{k}
\]

\[
r'' = 2 \mathbf{j} + 2 \mathbf{k}
\]

\[
\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|} = \frac{\mathbf{i} + 2 \mathbf{j} + \mathbf{k}}{\sqrt{6}}
\]

\[
\mathbf{a}_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{\|\mathbf{r}'\|} = \frac{6}{\sqrt{6}}
\]
\[
\alpha_N = \frac{[r' \times r'']}{|r'|^3} = \frac{1}{\sqrt{6}} \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \frac{1}{\sqrt{6}} |2i - 2j + 2k| = \sqrt{2}
\]

\[
\mathbf{N} = \frac{\mathbf{a} - \alpha_N \mathbf{T}}{\alpha_N} = \frac{(2\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + \mathbf{k})}{\sqrt{2}} = -i + k \sqrt{2}
\]

\[
\mathbf{B} = \mathbf{T} \times \mathbf{N} = \begin{vmatrix} i & j & k \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 1/\sqrt{3} \end{vmatrix} = \frac{1}{\sqrt{3}} i - \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k
\]

\[
\kappa = \frac{|r' \times r''|}{|r'|^3} = \frac{\alpha_N}{|r'|^3} = \frac{\sqrt{2}}{6}
\]

Concepts Review

1. Curvature is defined to be the magnitude of the vector _____.

2. The curvature of a circle of radius \(a\) is constant and has value \(\kappa = \underline{\text{constant}}\); the curvature of a line is _____.

3. The acceleration vector \(\mathbf{a}\) can be expressed as

\[
\mathbf{a} = \underline{\mathbf{T}} + \underline{\mathbf{N}}.
\]

4. For uniform circular motion in the plane, the tangential component of acceleration is _____.

Problem Set 11.7

In Problems 1–6, sketch the curve over the indicated domain for \(t\).

Find \(\mathbf{v}\), \(\mathbf{a}\), \(\mathbf{T}\), and \(\mathbf{N}\) at the point where \(t = t_1\).

1. \(\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}; \quad 0 \leq t \leq 2; t_1 = 1\)

2. \(\mathbf{r}(t) = t^2 \mathbf{i} + (2t + 1) \mathbf{j}; \quad 0 \leq t \leq 2; t_1 = 1\)

3. \(\mathbf{r}(t) = t \mathbf{i} + 2 \cos t \mathbf{j} + 2 \sin t \mathbf{k}; \quad 0 \leq t \leq 4\pi; t_1 = \pi\)

4. \(\mathbf{r}(t) = 5 \cos t \mathbf{i} + 2t \mathbf{j} + 5 \sin t \mathbf{k}; \quad 0 \leq t \leq 4\pi; t_1 = \pi\)

5. \(\mathbf{r}(t) = \frac{t^2}{8} \mathbf{i} + 5 \cos t \mathbf{j} + 5 \sin t \mathbf{k}; \quad 0 \leq t \leq 4\pi; t_1 = \pi\)

6. \(\mathbf{r}(t) = \frac{t^2}{4} \mathbf{i} + 2 \cos t \mathbf{j} + 2 \sin t \mathbf{k}; \quad 0 \leq t \leq 4\pi; t_1 = \pi\)

In Problems 7–14, find the unit tangent vector \(\mathbf{T}(t)\) and the curvature \(\kappa(t)\) at the point where \(t = t_1\). For calculating \(\kappa\), we suggest using Theorem A, as in Example 5.

7. \(\mathbf{u}(t) = 4t \mathbf{i} + 4t \mathbf{j} + \frac{t}{2} \mathbf{k}; t_1 = \frac{1}{2}\)

8. \(\mathbf{r}(t) = \frac{t^2}{3} \mathbf{i} + \frac{t^2}{3} \mathbf{j}; t_1 = 1\)

9. \(\mathbf{r}(t) = 3 \cos t \mathbf{i} + 4 \sin t \mathbf{j}; t_1 = \pi/4\)

10. \(\mathbf{r}(t) = e^t \mathbf{i} + e^t \mathbf{j}; t_1 = \ln 2\)

11. \(x(t) = 1 - t^2, y(t) = 1 - t^2; t_1 = 1\)

12. \(x(t) = \sin t, y(t) = \cos t; t_1 = \pi\)

13. \(x(t) = e^{-t} \cos t, y(t) = e^{-t} \sin t; t_1 = 0\)

14. \(\mathbf{r}(t) = t \mathbf{i} + t \mathbf{j} + t \mathbf{k}; t_1 = 1\)

In Problems 15–26, sketch the curve in the xy-plane. Then, for the given point, find the curvature and the radius of curvature. Finally, draw the circle of curvature at the point. Hint: For the curvature, you will use the second formula in Theorem A, as in Example 6.

15. \(y = 2x^2, (1, 2)\)

16. \(y = x - 4, (4, 0)\)

17. \(y = \sin x, \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)\)

18. \(y = x - 1, (1, 0)\)

19. \(y^2 - 4x^2 = 20, (2, 6)\)

20. \(y^2 - 4x^2 = 20, (2, -6)\)

21. \(y = \cos 2x, \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\)

22. \(y = e^x, (1, 1/e)\)

23. \(y = \tan x, \left(\pi/4, 1\right)\)

24. \(y = \sqrt{x}, (1, 1)\)

25. \(y = \sqrt{x}, (1, 1)\)

26. \(y = \tan h x, (\ln 2, e)\)

In Problems 27–34, find the curvature \(\kappa\), the unit tangent vector \(\mathbf{T}\), the unit normal vector \(\mathbf{N}\), and the binormal vector \(\mathbf{B}\) at \(t = t_1\).

27. \(\mathbf{r}(t) = \frac{1}{2}t \mathbf{i} + t \mathbf{j} + \frac{1}{2}t \mathbf{k}; t_1 = 2\)

28. \(x = \sin 3t, y = \cos 3t, z = t; t_1 = \pi/9\)

29. \(x = 7 \sin 3t, y = 7 \cos 3t, z = 14t; t_1 = \pi/3\)

30. \(\mathbf{r}(t) = \cos^3 t \mathbf{i} + \sin^3 t \mathbf{k}; t_1 = \pi/2\)

31. \(\mathbf{r}(t) = 3 \cosh(t/3) \mathbf{i} + \frac{t}{3} \mathbf{j}; t_1 = 1\)

32. \(\mathbf{r}(t) = e^t \cos 2t \mathbf{i} + e^t \sin 2t \mathbf{j} + e^t \mathbf{k}; t_1 = \pi/3\)

33. \(\mathbf{r}(t) = e^{-t} \mathbf{i} + e^{-t} \mathbf{j} + 2\sqrt{2} \mathbf{k}; t_1 = 0\)

34. \(x = \ln t, y = 3t, z = t^2; t_1 = 2\)

In Problems 35–40, find the point of the curve at which the curvature is a maximum.

35. \(y = \ln x\)

36. \(y = \sin x, -\pi \leq x \leq \pi\)

37. \(y = \cosh x\)

38. \(y = \sinh x\)
39. \( y = e^x \)
40. \( y = \ln \cos x \) for \(-\pi/2 < x < \pi/2\)

In Problems 41–52, find the tangential and normal components \((a_T\) and \(a_N)\) of the acceleration vector at \(t\). Then evaluate at \(t = t_1\).

See Examples 7 and 8.

41. \( \mathbf{r}(t) = 2t \mathbf{i} + 3x^2 \mathbf{j}; t_1 = 1 \)
42. \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{k}; t_1 = 1 \)
43. \( \mathbf{r}(t) = (2t + 1) \mathbf{i} + (t^2 - 2) \mathbf{j}; t_1 = -1 \)
44. \( \mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}; t_1 = \pi/6 \)
45. \( \mathbf{r}(t) = a \cosh t \mathbf{i} + a \sinh t \mathbf{j}; t_1 = \ln 3 \)
46. \( x(t) = 1 + 3t, y(t) = 2 - 6t, t_1 = 2 \)
47. \( \mathbf{r}(t) = (t + 1) \mathbf{i} + 3t \mathbf{j} + 3t \mathbf{k}; t_1 = 1 \)
48. \( x = t, y = t^2, z = t^3; t_1 = 2 \)
49. \( x = e^t, y = 2t, z = e^t; t_1 = 0 \)
50. \( \mathbf{r}(t) = (t - 2) \mathbf{i} - t^2 \mathbf{j} + r \mathbf{k}; t_1 = 2 \)
51. \( \mathbf{r}(t) = (t - \frac{1}{2}t^2) \mathbf{i} - (t + \frac{1}{2}t^2) \mathbf{j} + r \mathbf{k}; t_1 = 3 \)
52. \( \mathbf{r}(t) = t \mathbf{i} + \frac{1}{2}t^2 \mathbf{j} + r \mathbf{k}; t_1 > 0; t_1 = 1 \)

Sketch the path for a particle if its position vector is \( r = \sin t \mathbf{i} + \sin 2t \mathbf{j}, 0 \leq t \leq 2\pi \) (you should get a figure eight). Where is the acceleration zero? Where does the acceleration vector point to the origin?

54. The position vector of a particle at time \( t \geq 0 \) is \( \mathbf{r}(t) = (\cos t + \sin t) \mathbf{i} + (\sin t - \cos t) \mathbf{j} \)
   (a) Show that the speed \( ds/dt = t \).
   (b) Show that \( a_T = a \) and \( a_N = t \).

55. If, for a particle, \( a_T = 0 \) for all \( t \), what can you conclude about its speed? If \( a_N = 0 \) for all \( t \), what can you conclude about its curvature?

56. Find \( \mathbf{N} \) for the ellipse \( \mathbf{r}(t) = a \cos wt \mathbf{i} + b \sin wt \mathbf{j} \).

57. Consider the motion of a particle along a helix given by \( \mathbf{r}(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + (t^2 - 3t + 2) \mathbf{k} \), where the \( \mathbf{k} \) component measures the height in meters above the ground and \( t \geq 0 \). If the particle leaves the helix and moves along the line tangent to the helix when it is 12 meters above the ground, give the direction vector for the line.

58. An object moves along the curve \( y = \sin 2x \). Without doing any calculations, decide where \( a_N = 0 \).

59. A dog is running counterclockwise around the circle \( x^2 + y^2 = 400 \) (distances in feet). At the point \((-12, 16)\), it is running at 10 feet per second and is speeding up at 5 feet per second per second. Express its acceleration \( \mathbf{a} \) at the point first in terms of \( \mathbf{T} \) and \( \mathbf{N} \), and then in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

60. An object moves along the parabola \( y = x^2 \) with constant speed of 4. Express \( \mathbf{a} \) at the point \((x, x^2)\) in terms of \( \mathbf{T} \) and \( \mathbf{N} \).

61. A car traveling at constant speed \( v \) rounds a level curve, which we take to be a circle of radius \( R \). If the car is to avoid sliding outward, the horizontal frictional force \( F \) exerted by the road on the tires must at least balance the centrifugal force pulling outward. The force \( F \) satisfies \( F = \mu mg \), where \( \mu \) is the coefficient of friction, \( m \) is the mass of the car, and \( g \) is the acceleration of gravity. Thus, \( \mu mg \geq mv^2/ R \). Show that \( v_B \), the speed beyond which skidding will occur, satisfies
\[ v_B = \sqrt{\mu g R} \]
and use this to determine \( v_B \) for a curve with \( R = 400 \) feet and \( \mu = 0.4 \). Use \( g = 32 \) feet per second per second.

62. Consider again the car of Problem 61. Suppose that the curve is icy at its worst spot (\( \mu = 0 \)), but is banked at angle \( \theta \) from the horizontal (Figure 14). Let \( \mathbf{F} \) be the force exerted by the road on the car. Then, at the critical speed \( v_C \), \( mg = |\mathbf{F}| \cos \theta \) and \( mv_C^2/R = |\mathbf{F}| \sin \theta \).
   (a) Show that \( v_C = \sqrt{RG \tan \theta} \).
   (b) Find \( v_C \) for a curve with \( R = 400 \) feet and \( \theta = 10^\circ \).

![Figure 14](image)

63. Demonstrate that the second formula in Theorem A can also be written as \( \kappa = |y' \cos \phi| \), where \( \phi \) is the angle of inclination of the tangent line to the graph of \( y = f(x) \).

64. Show that for a plane curve \( \mathbf{N} \) points to the concave side of the curve. Hint: One method is to show that
\[ \mathbf{N} = ( \sin \phi \mathbf{i} + \cos \phi \mathbf{j} ) \frac{dB}{ds} \]
Then consider the cases \( dB/ds > 0 \) (curve bends to the left) and \( dB/ds < 0 \) (curve bends to the right).

65. Prove that \( \mathbf{N} = \mathbf{B} \times \mathbf{T} \). Derive a similar result for \( \mathbf{T} \) in terms of \( \mathbf{N} \) and \( \mathbf{B} \).

66. A curve has the equation \( y = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } x > 0 \end{cases} \) has continuous first derivatives and curvature at all points.

67. Find a curve given by a polynomial \( P(x) \) that provides a smooth transition between two horizontal lines. That is, assume a function of the form \( P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \), which provides a smooth transition between \( y = 0 \) for \( x < 0 \) and \( y = 1 \) for \( x > 1 \) in such a way that the function, its derivative, and curvature are all continuous for all values of \( x \).
\[ P(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ y & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases} \]

\text{Hint: } P_0(x) = 0, P'_1(0) = 0, P''_0(0) = 0, P_1(1) = 1, P'_2(1) = 0, and P''_2(1) = 0. Use these
six conditions to determine \( a_0, \ldots, a_6 \) uniquely and thus find \( P_5(x) \).

68. Find a curve given by a polynomial \( P_n(x) \) that provides a smooth transition between \( y = 0 \) for \( x \leq 0 \) and \( y = x \) for \( x \geq 1 \).

69. Derive the polar coordinate curvature formula

\[
\kappa = \frac{|r^2 + 2(r')^2 - rr''|}{(r^2 + (r')^2)^{3/2}}
\]

where the derivatives are with respect to \( \theta \).

In Problems 70–75, use the formula in Problem 69 to find the curvature \( \kappa \) of the following:

70. Circle: \( r = 4 \cos \theta \)

71. Cardioid: \( r = 1 + \cos \theta \) at \( \theta = 0 \)

72. \( r = \theta \) at \( \theta = 1 \)

73. \( r = 4(1 + \cos \theta) \) at \( \theta = \pi/2 \)

74. \( r = e^\theta \) at \( \theta = 1 \)

75. \( r = 4(1 + \sin \theta) \) at \( \theta = \pi/2 \)

76. Show that the curvature of the polar curve \( r = e^\theta \) is proportional to \( 1/r \).

77. Show that the curvature of the polar curve \( r^2 = \cos 2\theta \) is directly proportional to \( r \) for \( r > 0 \).

78. Derive the first curvature formula in Theorem A by working directly with \( \kappa = \|T'(s)/\|T'(s)\| \).

79. Draw the graph of \( x = 4 \cos t, y = 3 \sin(t + 0.5), \) \( 0 \leq t \leq 2\pi \). Estimate its maximum and minimum curvature by looking at the graph (curvature is the reciprocal of the radius of curvature). Then use a graphing calculator or a CAS to approximate these two numbers to four decimal places.

80. Show that the unit binormal vector \( B = T \times N \) has the property that \( \frac{dB}{ds} \) is perpendicular to \( B \).

81. Show that the unit binormal vector \( B = T \times N \) has the property that \( \frac{dB}{ds} \) is perpendicular to \( T \).

82. Using the results obtained in Problems 80 and 81, show that \( \frac{dB}{ds} \) must be parallel to \( N \) and, consequently, there must be a number \( \sigma \) depending on \( s \) such that \( \frac{dB}{ds} = -\sigma(x)N \). The function \( r(s) \) is called the torsion of the curve and measures the twist of the curve from the plane determined by \( T \) and \( N \).

83. Show that for a plane curve the torsion is \( \tau(s) = 0 \).

84. Show that for a straight line \( r(t) = r_0 + at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \) both \( \kappa \) and \( \tau \) are zero.

85. A fly is crawling along a wire helix so that its position vector is \( r(t) = 6 \cos \pi t \mathbf{i} + 6 \sin \pi t \mathbf{j} + 2t \mathbf{k}, t \geq 0 \). At what point will the fly hit the sphere \( x^2 + y^2 + z^2 = 100 \), and how far did it travel in getting there (assuming that it started when \( t = 0 \))?

86. The DNA molecule in humans is a double helix, each with about \( 2.9 \times 10^8 \) complete turns. Each helix has radius about 10 angstroms and rises about 34 angstroms on each complete turn (an angstrom is \( 10^{-10} \) centimeter). What is the total length of such a helix?

Answers to Concepts Review:

1. \( \frac{dT}{ds} \)
2. \( 1/\pi, 0 \)
3. \( \frac{d^2}{ds^2} \left( \frac{ds}{dt} \right)^3 \times 4 \times 0 \)

---

### 11.8 Surfaces in Three-Space

The graph of an equation in three variables is normally a surface. We have met two examples already. The graph of \( Ax + By + Cz = D \) is a plane; the graph of \( (x - h)^2 + (y - k)^2 + (z - l)^2 = r^2 \) is a sphere. Graphing surfaces is best accomplished by finding the intersections of the surface with well-chosen planes. These intersections are called cross sections (Figure 1); those with the coordinate planes are also called traces.

**Example 1** Sketch the graph of

\[
\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{9} = 1
\]

**Solution** To find the trace in the \( xy \)-plane, we set \( z = 0 \) in the given equation. The graph of the resulting equation

\[
\frac{x^2}{16} + \frac{y^2}{25} = 1
\]
is an ellipse. The traces in the $xz$-plane and the $yz$-plane (obtained by setting $y = 0$ and $x = 0$, respectively) are also ellipses. These three traces are shown in Figure 2 and help to provide a good visual image of the required surface (called an ellipsoid).

Figure 2

If the surface is very complicated, it may be useful to show the cross sections with many planes parallel to the coordinate planes. Here a computer with graphics capability can be very helpful. In Figure 3 we show a typical computer-generated graph, the graph of the “monkey saddle” $z = x^2 - 3xy^2$. We will have more to say about computer-generated graphs in the next chapter.

Cylinders You should be familiar with right circular cylinders from high school geometry. Here the word cylinder will denote a much more extensive class of surfaces.

Let $C$ be a plane curve, and let $l$ be a line intersecting $C$ that is not in the plane of $C$. The set of all points on lines that are parallel to $l$ and that intersect $C$ is called a cylinder (Figure 4).

Cylinders occur naturally when we graph an equation in three-space that involves just two variables. Consider as a first example

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

in which the variable $z$ is missing. This equation determines a curve $C$ in the $xy$-plane, a hyperbola. Moreover, if $(x_1, y_1, 0)$ satisfies the equation, so does $(x_1, y_1, z)$. As $z$ runs through all real values, the point $(x_1, y_1, z)$ traces out a line parallel to the $z$-axis. We conclude that the graph of the given equation is a cylinder, a hyperbolic cylinder (Figure 5).

A second example is the graph of $z = \sin y$ (Figure 6).
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Quadric Surfaces. If a surface is the graph in three-space of an equation of second degree, it is called a quadric surface. Plane sections of a quadric surface are conics.

The general second-degree equation has the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

It can be shown that any such equation can be reduced, by rotation and translation of coordinate axes, to one of the two forms

$$Ax^2 + By^2 + Cz^2 = 0$$

or

$$Ax^2 + By^2 + Iz = 0$$

The quadric surfaces represented by the first of these equations are symmetric with respect to the coordinate planes and the origin. They are called central quadrics.

In Figures 7 through 12, we show six general types of quadric surfaces. Study them carefully: The graphs were drawn by a technical artist; we do not expect that most of our readers will be able to duplicate them in doing the problems. A more reasonable drawing for most people to make is like the one that is shown in Figure 13 with our next example.

<table>
<thead>
<tr>
<th>QUADRIC SURFACES</th>
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<tr>
<td><strong>ELLIPSOID</strong>: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$</td>
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</table>

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
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</thead>
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<td>Ellipse</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Ellipse</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Ellipse</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
</tbody>
</table>

| HYPERBOLOID OF ONE SHEET: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ |

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Ellipse</td>
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<tr>
<td>xz-plane</td>
<td>Hyperbola</td>
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<tr>
<td>Parallel to xz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Hyperbola</td>
</tr>
</tbody>
</table>
QUADRIC SURFACES (continued)

**HYPERBOLOID OF TWO SHEETS:** \( \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \)

<table>
<thead>
<tr>
<th>Plane</th>
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</tr>
</thead>
<tbody>
<tr>
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<td>yz-plane</td>
<td>Empty set</td>
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<td>Parallel to xy-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Hyperbola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
</tbody>
</table>

**ELLIPTIC PARABOLOID:** \( z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \)

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Point</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Ellipse, point, or empty set</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Parabola</td>
</tr>
</tbody>
</table>

**HYPERBOLIC PARABOLOID:** \( z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \)

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Intersecting straight lines</td>
</tr>
<tr>
<td>xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>yz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to xy-plane</td>
<td>Hyperbola or intersecting straight lines</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Parabola</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Parabola</td>
</tr>
</tbody>
</table>

**ELLIPTIC CONE:** \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \)

<table>
<thead>
<tr>
<th>Plane</th>
<th>Cross Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy-plane</td>
<td>Point</td>
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<tr>
<td>xz-plane</td>
<td>Intersecting straight lines</td>
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</tr>
<tr>
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<td>Ellipse or point</td>
</tr>
<tr>
<td>Parallel to xz-plane</td>
<td>Hyperbola or intersecting straight lines</td>
</tr>
<tr>
<td>Parallel to yz-plane</td>
<td>Hyperbola or intersecting straight lines</td>
</tr>
</tbody>
</table>
**Example 2** Analyze the equation

\[ \frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1 \]

and sketch its graph.

**Solution** The traces in the three coordinate planes are obtained by setting \( z = 0 \), \( y = 0 \), and \( x = 0 \), respectively.

- \( xy\)-plane: \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \), an ellipse
- \( xz\)-plane: \( \frac{x^2}{4} - \frac{z^2}{16} = 1 \), a hyperbola
- \( yz\)-plane: \( \frac{y^2}{9} - \frac{z^2}{16} = 1 \), a hyperbola

These traces are graphed in Figure 13. We have also shown the cross sections in the planes \( z = 4 \) and \( z = -4 \). Note that when we substitute \( z = \pm 4 \) in the original equation we get

\[ \frac{x^2}{4} + \frac{y^2}{9} - \frac{16}{16} = 1 \]

which is equivalent to

\[ \frac{x^2}{8} + \frac{y^2}{18} = 1 \]

an ellipse.

**Example 3** Name the graph of each of the following equations:

(a) \( 4x^2 + 4y^2 - 25z^2 + 100 = 0 \)  
(b) \( y^2 + z^2 - 12y = 0 \)  
(c) \( x^2 - z^2 = 0 \)  
(d) \( 9x^2 + 4z^2 - 36y = 0 \)

**Solution**

(a) Dividing both sides of this equation by \(-100\) gives the form

\[ -\frac{x^2}{25} - \frac{y^2}{25} + \frac{z^2}{4} = 1 \]

Its graph is a hyperboloid of two sheets. It does not intersect the \( xy\)-plane, but cross sections in planes parallel to this plane (and at least 2 units away) are circles.

(b) The variable \( x \) does not appear, so the graph is a cylinder parallel to the \( x \)-axis. Since the equation can be written in the form \((y - 6)^2 + z^2 = 36\), its graph is a circular cylinder.

(c) Since the variable \( y \) is missing, the graph is a cylinder. The given equation can be written \((x - z)(x + z) = 0\); so its graph consists of the two planes \( x = z \) and \( x = -z \).

(d) The equation can be rewritten as

\[ \frac{x^2}{4} + \frac{z^2}{9} = y \]

which has an elliptic paraboloid as its graph. It is symmetric with respect to the \( y \)-axis.
Concepts Review

1. The intersections of a surface with the coordinate planes are called ______. More generally, intersections with any plane are called ______.

2. Equations involving just two variables when graphed in three-space generate surfaces called ______. In particular, the graph of \( x^2 + y^2 = 1 \) is an ordinary right circular cylinder whose axis is the ______.

3. The graph of \( 3x^2 + 2y^2 + 4z^2 = 12 \) is a surface called a(n) ______.

4. The graph of \( 4z = x^2 + 2y^2 \) is a surface called a(n) ______.

Problem Set 11.8

In Problems 1–20, name and sketch the graph of each of the following equations in three-space.

1. \( 4x^2 + 36y^2 = 144 \)
2. \( y^2 + z^2 = 15 \)
3. \( 3x + 2z = 10 \)
4. \( x^2 = 3y \)
5. \( x^2 + y^2 - 8x + 4y + 13 = 0 \)
6. \( 2x^2 - 16z^2 = 0 \)
7. \( 4x^2 + 9y^2 + 49z^2 = 1764 \)
8. \( 9x^2 - y^2 + 9z^2 = 9 \)
9. \( 4x^2 + 16y^2 - 32z = 0 \)
10. \( -x^2 + y^2 + z^2 = 0 \)
11. \( y = e^x \)
12. \( 6x - 3y = x \)
13. \( x^2 - z^2 + y = 0 \)
14. \( x^2 + y^2 - 4z^2 + 4 = 0 \)
15. \( 9x^2 + 4y^2 - 36y = 0 \)
16. \( 9x^2 + 25y^2 + 9z^2 = 225 \)
17. \( 5x + 8y - 2z = 10 \)
18. \( y = \cos x \)
19. \( z = \sqrt{16 - x^2 - y^2} \)
20. \( z = \sqrt{x^2 + y^2 + 1} \)

21. The graph of an equation in \( x, y, \) and \( z \) is symmetric with respect to the \( xy \)-plane if replacing \( z \) by \(-z\) results in an equivalent equation. What condition leads to a graph that is symmetric with respect to each of the following?
   (a) \( yz \)-plane 
   (b) \( z \)-axis 
   (c) origin

22. What condition leads to a graph that is symmetric with respect to the following?
   (a) \( xz \)-plane 
   (b) \( y \)-axis 
   (c) \( x \)-axis

23. Find the general equation of a central ellipsoid that is symmetric with respect to the following:
   (a) origin 
   (b) \( x \)-axis 
   (c) \( y \)-axis 

24. Find the general equation of a central hyperboloid of one sheet that is symmetric with respect to the following:
   (a) origin 
   (b) \( y \)-axis 
   (c) \( xy \)-plane

25. Find the general equation of a central hyperboloid of two sheets that is symmetric with respect to the following:
   (a) origin 
   (b) \( z \)-axis 
   (c) \( yz \)-plane

26. Which of the equations in Problems 1–20 has a graph that is symmetric with respect to each of the following?
   (a) \( xz \)-plane 
   (b) \( z \)-axis

27. If the curve \( z = x^2 \) in the \( xz \)-plane is revolved about the \( z \)-axis, the resulting surface has equation \( z = x^2 + y^2 \), obtained as a result of replacing \( x \) by \( \sqrt{x^2 + y^2} \). If \( y = 2x^2 \) in the \( xy \)-plane is revolved about the \( y \)-axis, what is the equation of the resulting surface?

28. Find the equation of the surface that results when the curve \( z = 2y \) in the \( yz \)-plane is revolved about the \( z \)-axis.

29. Find the equation of the surface that results when the curve \( 4x^2 + 3y^2 = 12 \) in the \( xy \)-plane is revolved about the \( y \)-axis.

30. Find the equation of the surface that results when the curve \( 4x^2 - 3y^2 = 12 \) in the \( xy \)-plane is revolved about the \( x \)-axis.

31. Find the coordinates of the foci of the ellipse that is the intersection of \( z = x^2/4 + y^2/9 \) with the plane \( z = 4 \).

32. Find the coordinates of the focus of the parabola that is the intersection of \( z = x^2/4 + y^2/9 \) with \( z = 4 \).

33. Find the area of the elliptical cross section cut from the surface \( x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \) by the plane \( z = h \), \(-c < h < c\). Recall: The area of the ellipse \( x^2/a^2 + y^2/b^2 = 1 \) is \( \pi a b \). Note: Use the method of slab. Section 5.2.

34. Show that the volume of the solid bounded by the elliptic paraboloid \( x^2/a^2 + y^2/b^2 = h - z \), \( h > 0 \), and the \( xy \)-plane is \( \pi a b h/2 \); that is, the volume is one-half the area of the base times the height. Hint: Use the method of slab. Section 5.2.

35. Show that the projection of the \( xz \)-plane of the curve that is the intersection of the surfaces \( y = 4 - x^2 \) and \( y = x^2 + z^2 \) is an ellipse, and find its major and minor diameters.

36. Sketch the triangle in the plane \( y = x \) that is above the plane \( z = y^2 \), below the plane \( z = 2y \), and inside the cylinder \( x^2 + y^2 = 4 \). Then find the area of this triangle.

37. Show that the spiral \( r = t \cos ti + t \sin tj + tk \) lies on the circular cone \( x^2 + y^2 - z^2 = 0 \). On what surface does the spiral \( r = 3t \cos ti + t \sin tj + t \) lie?

38. Show that the curve determined by \( r = t \cos ti + t \sin tj + t^2 \) is a parabola, and find the coordinates of its focus.

Answers to Concepts Review: 1. traces; cross sections 2. cylinders; \( z \)-axis 3. ellipsoid 4. elliptic paraboloid
11.9 Cylindrical and Spherical Coordinates

Giving the Cartesian (rectangular) coordinates \((x, y, z)\) is just one of many ways of specifying the position of a point in three-space. Two other kinds of coordinates that play a significant role in calculus are cylindrical coordinates \((r, \theta, z)\) and spherical coordinates \((\rho, \theta, \phi)\). The meaning of the three kinds of coordinates is illustrated for the same point \(P\) in Figure 1.

The **cylindrical coordinate system** uses the polar coordinates \(r\) and \(\theta\) (Section 10.5) in place of Cartesian coordinates \(x\) and \(y\) in the plane. The \(z\)-coordinate is the same as in Cartesian coordinates. We will usually require that \(r \geq 0\), and we will restrict \(\theta\) so that \(0 \leq \theta < 2\pi\).

![Figure 1](image1)

A point \(P\) has **spherical coordinates** \((\rho, \theta, \phi)\) if \(\rho\) (rho) is the distance \(|OP|\) from the origin to \(P\), \(\theta\) is the polar angle associated with the projection \(P'\) of \(P\) onto the xy-plane, and \(\phi\) is the angle between the positive z-axis and the line segment \(OP\). We require that

\[
\rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \pi
\]

**Cylindrical Coordinates** If a solid or a surface has an axis of symmetry, it is often wise to orient it so that this axis is the z-axis and then use cylindrical coordinates. Note in particular the simplicity of the equation of a circular cylinder with z-axis symmetry (Figure 2) and also of a plane containing the z-axis (Figure 3). In Figure 3, we have allowed \(r < 0\).

Cylindrical and Cartesian coordinates are related by the following equations:

**Cylindrical to Cartesian**
\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

**Cartesian to Cylindrical**
\[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\tan \theta &= y/x \\
z &= z
\end{align*}
\]

With these relationships, we can go back and forth between the two coordinate systems.

**EXAMPLE 1** Find
(a) the Cartesian coordinates of the point with cylindrical coordinates \((4, 2\pi/3, 5)\)
(b) the cylindrical coordinates of the point with Cartesian coordinates \((-5, -5, 2)\).

**SOLUTION**
(a) \(x = 4 \cos \frac{2\pi}{3} = 4 \cdot (-1/2) = -2\)
\(y = 4 \sin \frac{2\pi}{3} = 4 \cdot \left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}\)
\(z = 5\)
Thus, the Cartesian coordinates of \((4, 2\pi/3, 5)\) are \((-2, 2\sqrt{3}, 5)\).

\[(b) \quad r = \sqrt{(-5)^2 + (-5)^2} = 5\sqrt{2} \]
\[
\tan \theta = \frac{-5}{-5} = 1 \quad z = 2
\]

Figure 4 indicates that \(\theta\) is between \(\pi/2\) and \(\pi\). Since \(\tan \theta = 1\), we must have \(\theta = 5\pi/4\). The cylindrical coordinates of \((-5, -5, 2)\) are \((5\sqrt{2}, 5\pi/4, 2)\).

**Example 2** Find the equations in cylindrical coordinates of the paraboloid and cylinder whose Cartesian equations are \(x^2 + y^2 = 4 - z\) and \(x^2 + y^2 = 2x\).

**Solution**

Paraboloid: \(r^2 = 4 - z\)

Cylinder: \(r^2 = 2r \cos \theta\) or (equivalently) \(r = 2 \cos \theta\)

Division of an equation by a variable creates the potential for losing a solution. For example, dividing \(x^3 = x\) by \(x\) gives \(x = 1\) and loses the solution \(x = 0\). Similarly, dividing \(r^2 = 2r \cos \theta\) by \(r\) gives \(r = 2 \cos \theta\) and appears to lose the solution \(r = 0\) (the origin). However, the origin satisfies the equation \(r = 2 \cos \theta\) with coordinates \((0, \pi/2)\). Thus, \(r^2 = 2r \cos \theta\) and \(r = 2 \cos \theta\) have identical polar graphs (see caution in the margin of Section 10.5).

**Example 3** Find the Cartesian equations of the surfaces whose equations in cylindrical coordinates are \(r^2 + 4z^2 = 16\) and \(r^2 \cos 2\theta = z\).

**Solution** Since \(r^2 = x^2 + y^2\), the surface \(r^2 + 4z^2 = 16\) has the Cartesian equation \(x^2 + y^2 + 4z^2 = 16\) or \(x^2/16 + y^2/16 + z^2/4 = 1\). Its graph is an ellipsoid.

Since \(\cos 2\theta = \cos^2 \theta - \sin^2 \theta\), the second equation can be written \(r^2 \cos^2 \theta - r^2 \sin^2 \theta = z\). In Cartesian coordinates it becomes \(x^2 - y^2 = z\), the graph of which is a hyperbolic paraboloid.

**Spherical Coordinates** When a solid or a surface is symmetric with respect to a point, spherical coordinates are likely to play a simplifying role. In particular, a sphere centered at the origin (Figure 5) has the simple equation \(\rho = p_0\). Also note that the equation of a cone with axis along the \(z\)-axis and vertex at the origin (Figure 6) is \(\phi = \phi_0\).

It is easy to determine the relationships between spherical and cylindrical coordinates and between spherical and Cartesian coordinates. The following table shows some of these relationships.

<table>
<thead>
<tr>
<th>Spherical to Cartesian</th>
<th>Cartesian to Spherical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x = \rho \sin \phi \cos \theta)</td>
<td>(\rho = \sqrt{x^2 + y^2 + z^2})</td>
</tr>
<tr>
<td>(y = \rho \sin \phi \sin \theta)</td>
<td>(\tan \theta = y/x)</td>
</tr>
<tr>
<td>(z = \rho \cos \phi)</td>
<td>(\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}})</td>
</tr>
</tbody>
</table>

**Example 4** Find the Cartesian coordinates of the point \(P\) with spherical coordinates \((8, \pi/3, 2\pi/3)\).
SOLUTION We have plotted the point $P$ in Figure 7.

\[
x = 8 \sin \frac{2\pi}{3} \cos \frac{\pi}{3} = 8 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 2\sqrt{3}
\]
\[
y = 8 \sin \frac{2\pi}{3} \sin \frac{\pi}{3} = 8 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = 6
\]
\[
z = 8 \cos \frac{2\pi}{3} = 8 \left( -\frac{1}{2} \right) = -4
\]

Thus, $P$ has Cartesian coordinates $(2\sqrt{3}, 6, -4)$.

EXAMPLE 5 Describe the graph of $\rho = 2 \cos \phi$.

SOLUTION We change to Cartesian coordinates. Multiply both sides by $\rho$ to obtain

\[
\rho^2 = 2\rho \cos \phi
\]

\[
x^2 + y^2 + z^2 = 2z
\]

\[
x^2 + y^2 + (z - 1)^2 = 1
\]

The graph is a sphere of radius 1 centered at the point with Cartesian coordinates $(0, 0, 1)$.

EXAMPLE 6 Find the equation of the paraboloid $z = x^2 + y^2$ in spherical coordinates.

SOLUTION Substituting for $x, y,$ and $z$ yields

\[
\rho \cos \phi = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta
\]

\[
\rho \cos \phi = \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta)
\]

\[
\rho \cos \phi = \rho^2 \sin^2 \phi
\]

\[
\cos \phi = \rho \sin^2 \phi
\]

\[
\rho = \cos \phi \csc^2 \phi
\]

Note that $\phi = \pi/2$ yields $\rho = 0$, which shows that we did not lose the origin when we canceled $\rho$ at the fourth step.

Spherical Coordinates in Geography Geographers and navigators use a coordinate system very closely related to spherical coordinates, the longitude–latitude system. Suppose that the earth is a sphere with center at the origin, that the positive $z$-axis passes through the North Pole, and that the positive $x$-axis passes through the prime meridian (Figure 8). By convention, longitudes are specified in degrees east or west of the prime meridian and latitudes in degrees north or south of the equator. It is a simple matter to determine spherical coordinates from such data.

EXAMPLE 7 Assuming the earth to be a sphere of radius 3960 miles, find the great-circle distance from Paris (longitude $2.2^\circ$ E, latitude $48.4^\circ$ N) to Calcutta (longitude $88.2^\circ$ E, latitude $22.3^\circ$ N).

SOLUTION We first calculate the spherical angles $\theta$ and $\phi$ for the two cities.

Paris:

\[
\theta = 2.2^\circ \approx 0.0384 \text{ radians}
\]

\[
\phi = 90^\circ - 48.4^\circ = 41.6^\circ \approx 0.7261 \text{ radians}
\]

Calcutta:

\[
\theta = 88.2^\circ \approx 1.5394 \text{ radians}
\]

\[
\phi = 90^\circ - 22.3^\circ = 67.7^\circ \approx 1.1816 \text{ radians}
\]
From these data and \( \rho = 3960 \) miles, we determine the Cartesian coordinates, as illustrated in Example 4.

Paris: \( P_1(2627.2, 100.9, 2961.3) \)

Calcutta: \( P_2(115.1, 3662.0, 1502.6) \)

Next, referring to Figure 9, we determine \( \gamma \), the angle between \( \overline{OP}_1 \) and \( \overline{OP}_2 \).

\[
\cos \gamma = \frac{\overline{OP}_1 \cdot \overline{OP}_2}{||\overline{OP}_1|| ||\overline{OP}_2||} = \frac{(2627.2)(115.1) + (100.9)(3662) + (2961.3)(1502.6)}{(3960)(3960)} \\
= 0.3266
\]

Thus, \( \gamma \approx 1.2381 \) radians and the great-circle distance \( d \) is

\[
d = py \approx (3960)(1.2381) \approx 4903 \text{ miles}
\]

Concepts Review

1. In cylindrical coordinates, the graph of \( r = 6 \) is a(n) _____; in spherical coordinates, the graph of \( \rho = 6 \) is a(n) _____.

2. In cylindrical coordinates, the graph of \( \theta = \pi/6 \) is a(n) _____; in spherical coordinates, the graph of \( \phi = \pi/6 \) is a(n) _____.

3. The equation _____ connects \( \rho \) with \( r \) and \( z \).

4. The equation \( \rho^2 = 4p \cos \phi \) in spherical coordinates becomes the equation _____ when written in rectangular coordinates.

Problem Set 11.9

1. Make a table, like the one just before Example 4, that gives the relationships between cylindrical and spherical coordinates.

2. Change the following from cylindrical to spherical coordinates.
   (a) \((1, \pi/2, 1)\)
   (b) \((-2, \pi/4, 2)\)

3. Change the following from cylindrical to Cartesian (rectangular) coordinates.
   (a) \((6, \pi/6, -2)\)
   (b) \((4, 4\pi/3, -8)\)

4. Change the following from spherical to Cartesian coordinates.
   (a) \((8, \pi/4, \pi/6)\)
   (b) \((4, \pi/3, 3\pi/4)\)

5. Change the following from Cartesian to spherical coordinates.
   (a) \((2, -2\sqrt{3}, 4)\)
   (b) \((-\sqrt{2}, \sqrt{2}, 2\sqrt{3})\)

6. Change the following from Cartesian to cylindrical coordinates.
   (a) \((2, 2, 3)\)
   (b) \((4\sqrt{3}, -\sqrt{4}, 6)\)

In Problems 7–16, sketch the graph of the given cylindrical or spherical equation.

7. \( r = 5 \)
8. \( \rho = 5 \)
9. \( \phi = \pi/6 \)
10. \( \theta = \pi/6 \)
11. \( r = 3 \cos \theta \)
12. \( r = 2 \sin 2\theta \)
13. \( \rho = 3 \cos \phi \)
14. \( \rho = \csc \phi \)
15. \( r^2 + z^2 = 9 \)
16. \( r^2 \cos^2 \theta + z^2 = 4 \)

In Problems 17–30, make the required change in the given equation.

17. \( x^2 + y^2 = 9 \) to cylindrical coordinates
18. \( x^2 - y^2 = 25 \) to cylindrical coordinates
19. \( x^2 + y^2 + 4z^2 = 10 \) to cylindrical coordinates
20. \( x^2 + y^2 + 4z^2 = 10 \) to spherical coordinates
21. \( 2x^2 + 2y^2 - 4z^2 = 0 \) to spherical coordinates
22. \( x^2 - y^2 - z^2 = 1 \) to spherical coordinates
23. \( r^2 + 2z^2 = 4 \) to spherical coordinates
24. \( \rho = 2 \cos \phi \) to cylindrical coordinates
25. \( x + y = 4 \) to cylindrical coordinates
26. \( x + y + z = 1 \) to spherical coordinates
27. \( x^2 + y^2 - 9 \) to spherical coordinates
28. \( r = 2 \sin \theta \) to Cartesian coordinates
29. \( r^2 \cos 2\theta = z \) to Cartesian coordinates
30. \( \rho \sin \phi = 1 \) to Cartesian coordinates

31. The parabola \( z = 2x^2 \) in the \( xz \)-plane is revolved about the \( z \)-axis. Write the equation of the resulting surface in cylindrical coordinates.
32. The hyperbola $2x^2 - 2y^2 = 1$ in the $xz$-plane is revolved about the $z$-axis. Write the equation of the resulting surface in cylindrical coordinates.

33. Find the great-circle distance from St. Paul (longitude 93.1°W, latitude 45° N) to Oslo (longitude 10.5°E, latitude 59.6° N). See Example 7.

34. Find the great-circle distance from New York (longitude 74° W, latitude 40.4° N) to Greenwich (longitude 0°, latitude 51.3° N).

35. Find the great-circle distance from St. Paul (longitude 93.1°W, latitude 45° N) to Turin, Italy (longitude 7.4° E, latitude 45° N).

36. What is the distance along the 45° parallel between St. Paul and Turin? See Problem 35.

37. How close does the great-circle route from St. Paul to Turin get to the North Pole? See Problem 35.

38. Let $(ρ_1, θ_1, φ_1)$ and $(ρ_2, θ_2, φ_2)$ be the spherical coordinates of two points, and let $d$ be the straight-line distance between them. Show that

$$d = \sqrt{(ρ_1 - ρ_2)^2 + 2ρ_1ρ_2(1 - \cos(θ_1 - θ_2)) \sin φ_1 \sin φ_2 + \cos φ_1 \cos φ_2}$$

39. Let $(a, α, β, φ_1)$ and $(a, α, β, φ_2)$ be two points on the sphere $ρ = a$. Show (using Problem 38) that the great-circle distance between these points is $aγ$, where $0 ≤ γ ≤ π$ and

$$\cos γ = \cos(α_1 - α_2) \cos β_1 \cos β_2 + \sin β_1 \sin β_2$$

40. As you may have guessed, there is a simple formula for expressing great-circle distance directly in terms of longitude and latitude. Let $(α_1, β_1)$ and $(α_2, β_2)$ be the longitude–latitude coordinates of two points on the surface of the earth, where we interpret $N$ and $E$ as positive and $S$ and $W$ as negative. Show that the great-circle distance between these points is $3960γ$ miles, where $0 ≤ γ ≤ π$ and

$$\cos γ = \cos(α_1 - α_2) \cos β_1 \cos β_2 + \sin β_1 \sin β_2$$

41. Use Problem 40 to find the great-circle distance between each pair of places.

(a) New York and Greenwich (see Problem 34)

(b) St. Paul and Turin (see Problem 35)

(c) Turin and the South Pole (use $α_1 = α_2$)

(d) New York and Cape Town (18.4° E, 33.9° S)

(e) Two points on the equator with longitudes 100° E and 80° W, respectively

42. It is easy to see that the graph of $ρ = 2a \cos φ$ is a sphere of radius $a$ sitting on the $xy$-plane at the origin. But what is the graph of $ρ = 2a \sin φ$?

Answers to Concepts Review: 1. circular cylinder; sphere
2. plane; cone
3. $ρ^2 = x^2 + y^2$
4. $x^2 + y^2 + (z - 2)^2 = 4$

11.10 Chapter Review

Concepts Test

Respond with true or false to each of the following assertions. Be prepared to justify your answer.

1. Each point in three-space has a unique set of Cartesian coordinates.

2. The equation $x^2 + y^2 + z^2 - 4x + 9 = 0$ represents a sphere.

3. The linear equation $Ax + By + Cz = D$ represents a plane in three-space provided that $A, B, C$ are not all zero.

4. In three-space, the equation $Ax + By = C$ represents a line.

5. The planes $3x - 2y + 4z - 12$ and $3x - 2y + 4z = -12$ are parallel and 24 units apart.

6. The vector $(1, -2, 3)$ is parallel to the plane $2x - 4y + 6z = 5$.

7. The line $x = 2t + 1, y = 4t + 2, z = 6t - 5$ goes through the point $(0, 4, -2)$.

8. If $u = ai + bj + c k$ is a unit vector, then $a$, $b$, and $c$ are direction cosines of $u$.

9. The vectors $2i - 3j$ and $4i + 4j$ are perpendicular.

10. If $u$ and $v$ are unit vectors, then the angle $θ$ between them satisfies $\cos θ = \mathbf{u} \cdot \mathbf{v}$.

11. The dot product for vectors satisfies the associative law.

12. If $u$ and $v$ are any two vectors, then $|\mathbf{u} \cdot \mathbf{v}| ≤ |\mathbf{u}||\mathbf{v}|$.

13. $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$ for nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ if and only if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$.

14. If $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{u} + \mathbf{v}|$, then $\mathbf{u} = \mathbf{v} = 0$.

15. If $\mathbf{u} + \mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{v}$, then $|\mathbf{u}| = |\mathbf{v}|$.

16. For any two vectors $\mathbf{u}$ and $\mathbf{v}$,

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}$$

17. The vector-valued function $(f(t), g(t), h(t))$ is continuous at $t = a$ if and only if $f, g,$ and $h$ are continuous at $t = a$.

18. $\nabla[F(t) \cdot \mathbf{F}(t)] = 2\mathbf{F}(t) \cdot \mathbf{F}'(t)$.

19. For every vector $\mathbf{u}$, $|\mathbf{u} \cdot \mathbf{u}| = |\mathbf{u}|^2$.

20. For every vector $\mathbf{u}$, $|\mathbf{u}||\mathbf{u}| = \mathbf{u} \cdot \mathbf{u}$.

21. For all vectors $\mathbf{u}$ and $\mathbf{v}$, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{v} \times \mathbf{u}|$.

22. If $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$, then $\mathbf{u} \times \mathbf{v} = 0$.

23. The cross product of two unit vectors is a unit vector.

24. Multiplying each component of a vector $\mathbf{v}$ by the scalar $a$ multiplies the length of $\mathbf{v}$ by $a$.

25. For any nonzero and nonperpendicular vectors $\mathbf{u}$ and $\mathbf{v}$ with angle $θ$ between them, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin θ$.

26. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = 0$, then $\mathbf{u}$ or $\mathbf{v}$ is 0.

27. The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. 
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28. For all vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \),
\[
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}
\]

29. If \( a \mathbf{i} + b \mathbf{j} + c \mathbf{k} \) is a vector in the plane \( bx + by + bz = 0 \), then \( a \mathbf{b}_1 + a \mathbf{b}_2 + a \mathbf{b}_3 = 0 \).

30. Any line can be represented by both parametric equations and symmetric equations.

31. When \( \mathbf{e}(t) = 0 \) for all \( t \), the path is a straight line.

32. An ellipse has its maximum curvature at points on the major axis.

33. The curvature depends on the shape of the curve and the speed with which you move along the curve.

34. The curvature of the curve determined by \( x = 3t + 4 \) and \( y = 2t - 1 \) is zero for all \( t \).

35. The curvature of the curve determined by \( x = 2 \cos t \) and \( y = 2 \sin t \) is \( 2 \) for all \( t \).

36. If \( T = T(t) \) is a unit vector tangent to a smooth curve, then \( T(t) \) and \( T'(t) \) are perpendicular.

37. If \( v = |\mathbf{v}| \) is the speed of a particle moving along a smooth curve, then \( |\mathbf{v}||\mathbf{a}| \) is the magnitude of the acceleration.

38. If \( y = f(x) \) and \( y' = 0 \) everywhere, then the curvature of this curve is zero.

39. If \( y = f(x) \) and \( y' = \text{constant} \), then the curvature of this curve is a constant.

40. If \( \mathbf{u} \cdot \mathbf{v} = 0 \), then either \( \mathbf{u} = 0 \) or \( \mathbf{v} = 0 \), or both \( \mathbf{u} \) and \( \mathbf{v} \) are \( 0 \).

41. If \( |r(t)| = 1 \) for all \( t \), then \( |r'(t)| \) is constant.

42. If \( \mathbf{v} \cdot \mathbf{v} = \text{constant} \), then \( \mathbf{v} \cdot \mathbf{v}' = 0 \).

43. For motion along a helix, \( \mathbf{N} \) always points toward the z-axis.

44. If the velocity of the motion along the curve is constant magnitude, then there can be no acceleration.

45. \( \mathbf{T}, \mathbf{N} \), and \( \mathbf{B} \) depend only on the shape of the curve and not on the speed of motion along the curve.

46. If \( \mathbf{v} \) is perpendicular to \( \mathbf{a} \), then the speed of motion along the curve must be a constant.

47. If \( \mathbf{v} \) is perpendicular to \( \mathbf{a} \), then the path of motion is a circle.

48. The only curves with constant curvature are straight lines and circles.

49. The curves given by \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k} \) and \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t' \mathbf{k} \) for \( 0 \leq t \leq 1 \) are identical.

50. The motions along the curves given by \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t \mathbf{k} \) and \( r(t) = \sin t \mathbf{i} + \cos t \mathbf{j} + t' \mathbf{k} \) for \( 0 \leq t \leq 1 \) are identical.

51. The length of a given curve is independent of the parameterization used to describe the curve.

52. If a curve lies in a plane, then the binormal vector \( \mathbf{B} \) must be a constant.

53. If \( |r(t)| = \text{constant} \), then \( r'(t) = 0 \).

54. The curve that is the intersection of the sphere \( x^2 + y^2 + z^2 = 1 \) and the plane \( ax + by + cz = 0 \) has constant curvature 1.

55. The graph of the equation \( \phi = 0 \) is the z-axis (here \( \phi \) is a spherical coordinate).

56. The graph of \( y = x^2 \) in three-space is a paraboloid.

57. If we restrict \( \rho, \theta, \) and \( \phi \) by \( \rho \geq 0, 0 \leq \theta < 2\pi, \) and \( 0 \leq \phi \leq \pi \), then each point in three-space has a unique set of spherical coordinates.

Sample Test Problems

1. Find the equation of the sphere that has \((-2, 3, 3)\) and \((4, 1, 5)\) as end points of a diameter.

2. Find the center and radius of the sphere with equation \( x^2 + y^2 + z^2 - 6x + 2y - 8z = 0 \).

3. Let \( \mathbf{a} = (2, -5), \mathbf{b} = (1, 1), \) and \( \mathbf{c} = (-6, 0) \). Find each of the following:
   (a) \( \mathbf{a} \cdot \mathbf{b} \)
   (b) \( \mathbf{a} \cdot \mathbf{b} \)
   (c) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) \)
   (d) \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) \)
   (e) \( \mathbf{i} \mathbf{c} \cdot \mathbf{b} \)
   (f) \( \mathbf{i} \mathbf{c} \cdot \mathbf{b} \)

4. Find the cosine of the angle between \( \mathbf{a} \) and \( \mathbf{b} \) and make a sketch.
   (b) \( \mathbf{a} = 3\mathbf{i} + 2\mathbf{j}, \mathbf{b} = -\mathbf{i} + 4\mathbf{j} \)
   (c) \( \mathbf{a} = -3\mathbf{i} - 2\mathbf{j}, \mathbf{b} = 2\mathbf{i} - 2\mathbf{j} \)
   (d) \( \mathbf{a} = (7, 0), \mathbf{b} = (5, 1) \)

5. Let \( \mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{b} = \mathbf{j} - 2\mathbf{k}, \) and \( \mathbf{c} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k} \). Find each of the following if they are defined:
   (a) \( \mathbf{a} + \mathbf{b} + \mathbf{c} \)
   (b) \( \mathbf{b} \cdot \mathbf{c} \)
   (c) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)
   (d) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \)
   (e) \( \mathbf{a} \cdot \mathbf{b} \)
   (f) \( \mathbf{b} \cdot \mathbf{c} \)

6. Find the angle between each pair of vectors.
   (a) \( \mathbf{a} = (1, 5, -1), \mathbf{b} = (0, 1, 3) \)
   (b) \( \mathbf{a} = -\mathbf{i} + 2\mathbf{k}, \mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} \)

7. Sketch the two position vectors \( \mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \) and \( \mathbf{b} = 3\mathbf{i} + \mathbf{j} - 3\mathbf{k} \). Then find each of the following:
   (a) their lengths
   (b) their direction cosines
   (c) the unit vector with the same direction as \( \mathbf{a} \)
   (d) the angle \( \theta \) between \( \mathbf{a} \) and \( \mathbf{b} \)

8. Let \( \mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \mathbf{b} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \) and \( \mathbf{c} = i + 2\mathbf{j} - \mathbf{k} \). Find each of the following:
   (a) \( \mathbf{a} \times \mathbf{b} \)
   (b) \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) \)
   (c) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)
   (d) \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \)

9. Find all vectors that are perpendicular to both of the vectors \( 3\mathbf{i} + 3\mathbf{j} - \mathbf{k} \) and \(-\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} \).

10. Find the unit vectors that are perpendicular to the plane determined by the three points \((-3, -6, 4), (2, 1, 1), \) and \((5, 0, -2) \).

11. Write the equation of the plane through the point \((-5, 7, -2) \) that satisfies each condition.
   (a) Parallel to the \( xy \)-plane
   (b) Perpendicular to the \( z \)-axis
   (c) Parallel to both the \( x \)- and \( y \)-axes
   (d) Parallel to the plane \( 3x - 4y + z = 7 \)

12. A plane through the point \((2, -4, -5) \) is perpendicular to the line joining the points \((-1, 5, -7) \) and \((4, 1, 1) \).
   (a) Write a vector equation of the plane.
(b) Find a Cartesian equation of the plane.
(c) Sketch the plane by drawing its traces.

13. Find the value of C if the plane $x + 5y + Cz + 6 = 0$ is perpendicular to the plane $4x - y + z - 17 = 0$. 

14. Find a Cartesian equation of the plane through the three points $(2, 3, -1), (-1, 5, 2),$ and $(-4, -2, 2)$. 

15. Find parametric equations for the line through $(-2, 1, 5)$ and $(6, 2, -3)$. 

16. Find the points where the line of intersection of the planes $x - 2y + 4z - 14 = 0$ and $-x + 2y - 3z + 30 = 0$ pierces the $yz$- and $xz$-planes. 

17. Write the equation of the line in Problem 16 in parametric form. 

18. Find symmetric equations of the line through $(4, 5, 8)$ and perpendicular to the plane $3x + 5y + 2z = 30$. Sketch the plane and the line. 

19. Write a vector equation of the line through $(2, -2, 1)$ and $(-3, 2, 4)$. 

20. Sketch the curve whose vector equation is $\mathbf{r}(t) = t\hat{i} + \frac{1}{2}t^2\hat{j} + \frac{3}{4}t^3\hat{k}$, $-2 \leq t \leq 3$. 

21. Find the symmetric equations for the tangent line to the curve of Problem 20 at the point where $t = 2$. Also find the equation of the normal plane at this point. 

22. Find $\mathbf{r}(\pi/2), \mathbf{T}(\pi/2)$, and $\mathbf{r}'(\pi/2)$ if $\mathbf{r}(t) = (t \cos t, t \sin t, 2t)$. 

23. Find the length of the curve $\mathbf{r}(t) = t^2 \hat{i} + t^3 \hat{j} + 3t \hat{k}, 1 \leq t \leq 5$. 

24. Two forces $\mathbf{F}_1 = 2\hat{i} - 3\hat{j}$ and $\mathbf{F}_2 = 3\hat{i} + 12\hat{j}$ are applied at a point. What force $\mathbf{F}$ must be applied at the point to counteract the resultant of these two forces? 

25. What heading and airspeed are required for an airplane to fly 450 miles per hour due north if a wind of 100 miles per hour is blowing in the direction N 60° E? 

26. If $\mathbf{r}(t) = \langle t^2, 2t, t^3 \rangle$ find each of the following:
   (a) $\lim_{t \to -\infty} \mathbf{r}(t)$
   (b) $\lim_{h \to 0} \frac{\mathbf{r}(0 + h) - \mathbf{r}(0)}{h}$
   (c) $\int_0^1 \mathbf{r}(t) \, dt$
   (d) $D_t[\mathbf{r}(t)]$
   (e) $D_t[\mathbf{r}(3t + 10)]$
   (f) $D_t[\mathbf{r}(t) \cdot \mathbf{r}'(t)]$

27. Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$ for each of the following:
   (a) $\mathbf{r}(t) = (\ln t)\hat{i} - 3t^2\hat{j}$
   (b) $\mathbf{r}(t) = \sin t\hat{i} + \cos 2t\hat{j}$
   (c) $\mathbf{r}(t) = \tan t\hat{i} - t^2\hat{j}$

28. Suppose that an object is moving so that its position vector at time $t$ is $\mathbf{r}(t) = \cos t\hat{i} + \sin t\hat{j} + 2t\hat{k}$. Find $\mathbf{v}(t), \mathbf{a}(t)$, and $\mathbf{u}(t)$ at $t = \ln 2$.

29. If $\mathbf{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ is the position vector for a moving particle at time $t$, find the tangential and normal components, $\alpha(T)$ and $\alpha(N)$, of the acceleration vector at $t = 1$.

For each equation in Problems 30-38, name and sketch the graph in three-space.

30. $x^2 + y^2 = 81$
31. $x^2 + y^2 = 81$
32. $z^2 = 4y$
33. $x^2 + z^2 = 4y$
34. $3y - 6z - 12 = 0$
35. $3x + 3y - 6z - 12 = 0$
36. $x^2 + y^2 - z^2 - 1 = 0$
37. $3x^2 + 4y^2 + 9z^2 - 36 = 0$
38. $3x^2 + 4y^2 + 9z^2 - 36 = 0$

39. Write the following Cartesian equations in cylindrical coordinate form.
   (a) $x^2 + y^2 = 9$
   (b) $x^2 + 4y^2 = 16$
   (c) $x^2 + y^2 = 9z$
   (d) $x^2 + y^2 + 4z^2 = 10$

40. Find the Cartesian equation corresponding to each of the following cylindrical coordinate equations.
   (a) $r^2 + z^2 = 9$
   (b) $r^2 \cos^2 \theta + z^2 = 4$
   (c) $r^2 \cos 2\theta + z^2 = 1$

41. Write the following equations in spherical coordinate form.
   (a) $x^2 + y^2 + z^2 = 4$
   (b) $2x^2 + 2y^2 - 2z^2 = 0$
   (c) $x^2 - y^2 - z^2 = -1$
   (d) $x^2 + y^2 = z$

42. Find the (straight-line) distance between the points whose spherical coordinates are $(8, \pi/4, 5\pi/6)$ and $(4, \pi/3, 3\pi/4)$.

43. Find the distance between the parallel planes $2x - 3y + \sqrt{3}z = -4$ and $2x - 3y + \sqrt{3}z = 9$.

44. Find the acute angle between the planes $2x - 4y + z = 7$ and $3x + 2y - 5z = 9$.

45. Show that if the speed of a moving particle is constant then its velocity and acceleration vectors are orthogonal.
In Problems 1–4, sketch a graph of the cylinder or quadric surface.

1. \( x^2 + y^2 + z^2 = 64 \)  
2. \( x^2 + z^2 = 4 \)  
3. \( z = x^2 + 4y^2 \)  
4. \( z = x^2 - y^2 \)

In Problems 5–8, find the indicated derivative.

5. (a) \( \frac{d}{dx} 2x^3 \)  
(b) \( \frac{d}{dx} \sin x \)  
(c) \( \frac{d}{dx} k x^3 \)  
(d) \( \frac{d}{dx} ax^3 \)

6. (a) \( \frac{d}{dx} \sin 2x \)  
(b) \( \frac{d}{dt} \sin 17t \)  
(c) \( \frac{d}{dt} \sin at \)  
(d) \( \frac{d}{dt} \sin bt \)

7. (a) \( \frac{d}{du} \sin 2u \)  
(b) \( \frac{d}{du} \sin 17u \)  
(c) \( \frac{d}{du} \sin tu \)  
(d) \( \frac{d}{du} \sin su \)

8. (a) \( \frac{d}{dx} e^{x+1} \)  
(b) \( \frac{d}{dx} e^{-x+4} \)  
(c) \( \frac{d}{dx} e^{2x+b} \)  
(d) \( \frac{d}{dx} e^{x+c} \)

In Problems 9–12, say whether the function is continuous and whether it is differentiable at the given point.

9. \( f(x) = \frac{1}{x^2 - 1} \) at \( x = 2 \)

10. \( f(x) = \tan x \) at \( x = \pi/2 \)

11. \( f(x) = |x - 4| \) at \( x = 4 \)

12. \( f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \) at \( x = 0 \)

In Problems 13–14, find the maximum and minimum value of the function on the given interval. Use the Second Derivative Test to determine whether each stationary point is a maximum or a minimum.

13. \( f(x) = 3x - (x - 1)^3 \) on \([0, 4]\)

14. \( f(x) = x^4 - 18x^3 + 113x^2 - 288x + 252 \) on \([2, 6]\)

15. A storage can is to be made in the shape of a right circular cylinder of height \( h \) and radius \( r \). Find the surface area of the container (including the circular top and bottom) as a function of \( r \) only, if the volume is to be 8 cubic feet.

16. A three-dimensional box without a lid is to be made of a material that costs $1 per square foot for the sides and $3 per square foot for the bottom. The box is to contain 27 cubic feet. Let \( l, w, \) and \( h \) denote, respectively, the length, width, and height of the box. Since the box must contain 27 cubic feet, we must have \( lwh = 27 \), or, equivalently, \( h = 27/(lw) \). Use this expression for \( h \) to find a formula for the cost of a box having length \( l \) and width \( w \).