CALCULUS 1260

TAKE HOME TEST 1  due Friday 09/02/2016

Your Name (PRINT IN BLOCK LETTERS):

INSTRUCTIONS

Work all problems. SHOW YOUR WORK. Circle your answers. Each problem is worth 10 points max.

YOU CAN GET HELP ON CONCEPTS INVOLVED BUT NOT ON THE PROBLEMS THEMSELVES.

1. (a) Write a parametric equation of a line through the points (2, 3, 4) and (1, 2, 3).

(b) Where does this line cross the $yz$-plane?

\[
(x, y, z) = (1-t)(2, 3, 4) + t(1, 2, 3)
\]

\[
\begin{align*}
  x &= 2 - 2t + t \\
  y &= 3 - 3t + 2t \\
  z &= 4 - 4t + 3t
\end{align*}
\]

\[
\begin{align*}
  x &= 2-t \\
  y &= 3-t \\
  z &= 4-t
\end{align*}
\]

(b) The line intersects the $xz$-plane when $x = 0$, i.e., $t = 2$. Thus, the point of intersection is $0, 1, 2$. Thus, the point of intersection is $0, 1, 2$. Thus, the point of intersection is $0, 1, 2$.
2. (a) Find the distance from the point \( Q = (2, -1, 2) \) to the plane \( 2x - y + z = 5 \).

(b) Find the point on this plane which is closest to \( Q \).

(a) In standard form the equation of the plane is:

\[
2x - y + z = 5 
\]

Thus the distance is:

\[
\frac{|2(2) - (-1) + 2 - 5|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2}{\sqrt{6}} \]

or:

\[
\frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3} \]

(b) The desired point \( Q \) is the intersection of the line and the plane.

The line is:

\[
\begin{align*}
x &= 2 + 2t \\
y &= -1 - t \\
z &= 2 + t
\end{align*}
\]

Plugging into the plane:

\[
2(2 + 2t) - (-1 - t) + 2 + t = 5
\]

\[
4 + 4t + 1 + 2 + t = 5
\]

\[
7 + 5t = 5 \quad \rightarrow \quad t = -\frac{1}{3}
\]

So the point is \( \left( \frac{4}{3}, -\frac{2}{3}, \frac{5}{3} \right) \).
3. (a) Find the distance from the point $Q = (0, 1, 2)$ to the line

$$x = 2t + 1, y = 2t, z = -t$$

(b) Find the point on this line which is closest to $Q$.

(a)

$$d = \text{distance} = \frac{||\vec{u} \times \vec{v}||}{||\vec{u}||}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
2 & 2 & -1 \\
-1 & 1 & 2
\end{vmatrix} = \langle 5, -3, 4 \rangle$$

$$||\vec{u} \times \vec{v}|| = 5\sqrt{2}, \quad ||\vec{u}|| = 3 \quad \text{Thus} \quad d = \frac{5\sqrt{2}}{3}$$

(b)

Let $x = 2t + 1, y = 2t, z = -t$ be an arbitrary point on the line.

To get the point $P$ closest to $Q$, we want $\vec{w} \perp \vec{u}$, i.e., $\vec{w} \cdot \vec{u} = 0$

$$\vec{w} = \langle 2t + 1, 2t - 1, -t - 2 \rangle$$

$$\vec{w} \cdot \vec{u} = \langle 2t + 1, 2t - 1, -t - 2 \rangle \cdot \langle 2, 2, -1 \rangle = 4t + 2 + 4t - 2 + t + 2 = 9t + 2 = 0 \rightarrow t = \frac{-2}{9}$$

Hence

$$P = \left(-\frac{y}{9} + 1, -\frac{y}{9} + \frac{2}{3} \right) = \left(\frac{5}{9}, -\frac{4}{9}, \frac{2}{9} \right)$$
4. (a) Find the area of the triangle whose vertices are \((-1, -1, -1), (-1, 0, 1)\) and \((1, 0, -1)\).

(b) Find an equation of a plane containing these three points.

![Diagram of vectors and points](image)

\[ \vec{v} = \langle 0, 1, 2 \rangle \]

\[ \vec{u} = \langle 2, 1, 0 \rangle \]

\[ \vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \langle 2, -4, 2 \rangle \]

\[ \| \vec{u} \times \vec{v} \| = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6} \]

\[ \text{Area of the triangle} = \frac{1}{2} \| \vec{u} \times \vec{v} \| = \sqrt{6} \]

(b) Equation of the plane

\[ 2x - 4y + 2z = \text{const} \]

Plug \((-1, -1, -1)\)

\[-2 + 4 - 2 = \text{const} \]

\[\implies \text{equation is} \]

\[ 2x - 4y + 2z = 0 \]

\[ x - 2y + z = 0 \]
5. Find the volume of tetrahedron spanned by vectors $(1, 1, 0), (1, -1, 0)$ and $(1, 0, 1)$.

\[
\begin{vmatrix}
1 & 1 & 0 \\
1 & -1 & 0 \\
1 & 0 & 1 \\
\end{vmatrix}
\]

expand along the 3rd column

Thus the volume = \( \frac{1}{6} \left| -2 \right| = \frac{1}{3} \)
6. Show that intervals joining vertices of tetrahedron with centers of gravity of opposite sites intersect at one point.

This problem has been discussed in details in class.