CALCULUS 1250
EXAM 3 
November 14, 2017

Your Name (PRINT IN BLOCK LETTERS):

INSTRUCTIONS

Work all problems. SHOW YOUR WORK. NO CALCULATORS. Circle your answers. Each problem is worth 5 points max.

1. Let $f(x) = x^{\cos x}$. Find $f'(x)$

We use logarithmic derivative method.

$$y = x^{\cos x}$$

$$\ln y = \ln (x^{\cos x})$$

$$\ln y = \cos x \cdot \ln x$$

Differentiating

$$\frac{y'}{y} = -\sin x \ln x + \frac{\cos x}{x}$$

$$y' = y \left( \frac{\cos x}{x} - \sin x \ln x \right)$$

$$f'(x) = x^{\cos x} \left( \frac{\cos x}{x} - \sin x \ln x \right)$$
2. Integrate

\[ \int x^2 \ln x \, dx \]

Use integration by parts.

Set \( du = x^2 \, dx \) \( \Rightarrow \) \( u = \frac{x^3}{3} \)

\( v = \ln x \) \( \Rightarrow \) \( dv = \frac{1}{x} \, dx \)

\[ \int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \cdot \frac{1}{x} \, dx = \]

\[ \int \frac{x^2}{3} \, dx = \frac{x^3}{9} + C \]

\[ = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C \]
3. Integrate

\[ \int \frac{3x + 5}{(x + 1)(x + 2)} \, dx \]

\[ \frac{3x + 1}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2} \]

\[ \Rightarrow 3x + 1 = A(x + 2) + B(x + 1) \]

\[ 3x + 1 = (A + B)x + (2A + B) \]

\[ \Rightarrow A + B = 3 \]

\[ 2A + B = 1 \]

\[ \Rightarrow A = 2, \quad B = 1 \]

Hence

\[ \int \frac{3x + 5}{(x + 1)(x + 2)} \, dx = \int \frac{2}{x + 1} \, dx + \int \frac{1}{x + 2} \, dx = 2 \ln |x + 1| + \ln |x + 2| + C \]
4. Integrate

\[ \int \frac{1}{x^2 + 2x + 3} \, dx \]

\( x^2 + 2x + 3 \) does not factor with real coefficients, so we use the "completion of square" method.

\[ \frac{1}{x^2 + 2x + 3} = \frac{1}{(x^2 + 2x + 1) + 2} = \frac{1}{(x+1)^2 + 2} \]

\[ \Rightarrow \int \frac{1}{x^2 + 2x + 3} \, dx = \int \frac{1}{(x+1)^2 + 2} \, dx = \int \frac{1}{(x+1)^2 + (\sqrt{2})^2} \, dx \]

\[ = \frac{1}{\sqrt{2}} \arctan \frac{x+1}{\sqrt{2}} + C \]
5. Solve the differential equation
\[ y' + y = e^{-x} \]
with the initial condition \( y(0) = 1 \).
Note: First find the general solution.

This is a linear equation of first order.
\[ I = e^{\int 1 \, dx} = e^x \text{ is an integrating factor.} \]
\[ y' + y = e^{-x} \quad 1 \cdot e^x \]
\[ e^x y' + e^x y = 1 \]

Fold
\[ (e^x y)' = 1 \]

Integrate
\[ e^x y = x + C \]
\[ y = \frac{x + C}{e^x} \text{ general solution} \]

Plug in \( x = 0 \), \( y = 1 \)
\[ 1 = \frac{0 + C}{e^0} = C \]

Particular solution:
\[ y = \frac{x + 1}{e^x} \]
6. Solve the differential equation

\[ y' = y^4 \]

Use separation of variables.

\[ \frac{dy}{dx} = y^4 \]

\[ dy = y^4 dx \]

\[ y^{-4} dy = dx \]

\[ -\frac{1}{3} y^{-3} = x + C \]

\[ y^{-3} = -3(x + C) \]

\[ y = \left[ -3(x + C) \right]^{-\frac{1}{3}} \]

or, in radical notation

\[ y = \frac{1}{\sqrt[3]{-3(x + C)}} \]