Math 2250-3

HW due Wed 10/19

4.3 16 17 18 21 22 23 25
4.4 1-4, 8-10, 21, 26
4.5 1, 2, 3 4, 5 6 7 8 9, 10, 12, 13, 15, 19, 21, 23, 25

4.3 #16) the linear combination question $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$ is the matrix eqtn $\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \left[\vec{w}\right]$ which we solve by reducing $A \,\vdots\, \vec{w}$, i.e.

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 1 & 2 & 3 & 11 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_1 = 6$, $c_2 = -2$, $c_3 = 3$

$$\begin{bmatrix} 7 \\ 7 \\ 9 \\ 11 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 1 \\ 3 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \\ -1 \\ 3 \end{bmatrix} \quad \checkmark$$

18) the linear indepence test eqtn $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ is the matrix eqtn $\left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$ which we can solve by row reducing $A = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right]$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 0 & -5 & 1 & 0 \\ -3 & -6 & 3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -3/5 & 0 \\ 0 & 1 & -1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

has non-trivial soltns!

$c_3 = t$, $c_2 = \frac{1}{5}t$, $c_1 = \frac{3}{5}t$

Dependent! e.g. $t=1$: $\frac{3}{5}\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 4 \\ -5 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = \vec{0}$

[in fact, since dependencies correspond to homog soltns they stay the same thru row ops, so we see by observation $\vec{v}_3 = -\frac{3}{5}\vec{v}_2 - \frac{1}{5}\vec{v}_1$;

$$\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = -\frac{3}{5}\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 4 \\ -5 \\ -6 \end{bmatrix} \quad \checkmark$$

22) Proceeding as in 18,

$$\begin{bmatrix} 3 & 3 & 4 \\ 9 & 0 & 7 \\ 0 & 9 & 5 \\ 5 & -7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 7/9 \\ 0 & 1 & 5/9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>DEPENDENT</u>

$\vec{v}_3 = \frac{7}{9}\vec{v}_1 + \frac{5}{9}\vec{v}_2$, or $9\vec{v}_3 = 7\vec{v}_1 + 5\vec{v}_2$

$$9\begin{bmatrix} 4 \\ 7 \\ 5 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 3 \\ 9 \\ 0 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 0 \\ 9 \\ -7 \end{bmatrix} \quad \checkmark$$

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25) let $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = 0$

then $c_1(\vec{v}_1) + c_2(\vec{v}_1 + 2\vec{v}_2) + c_3(\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3) = 0$.

collect terms:

$$(c_1 + c_2 + c_3)\vec{v}_1 + (2c_2 + 2c_3)\vec{v}_2 + (3c_3)\vec{v}_3 = 0$$

since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is <u>independent</u>, the coeffs in this last eqtn must equal 0:

$c_1 + c_2 + c_3 = 0 \quad \Rightarrow c_1 = 0$

$2c_2 + 2c_3 = 0 \quad \Rightarrow c_2 = 0$

$3c_3 = 0 \Rightarrow c_3 = 0$

(back solve)

this shows $c_1 = c_2 = c_3 = 0$ so $\vec{u}_1, \vec{u}_2, \vec{u}_3$ linearly independent!

4.4 1) dependency for only 2 vectors $\vec{v}_1, \vec{v}_2$ is just the question of whether they are scalar multiples. Hence $\left\{ \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\}$ is independent. They also span (See Theorem (3 a) page 258) So they are a basis

2) $\vec{v}_2 = 2\vec{v}_1$ so vectors are dependent, not basis! (could also use rref or det tests)

3) any collection of 4 vectors in $\mathbb{R}^3$ must be dependent, so cannot be a basis (in general if the dim of V is n, more than n vectors will always be dependent)

4) You need at least 4 vectors to span $\mathbb{R}^4$, so this set cannot be a basis ($<n$ vectors in n-dim'l space cannot span)

8) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$ iff $A = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3 | \vec{v}_4]$ is non-singular (i.e. invertible)

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 6 & 5 \end{bmatrix}$$

rref(A) = I is clear (so is $|A| = 6(35-24) \neq 0$)

so A is nonsingular

so vectors are a basis

9) One way: This is the solution space for a very small system of homogeneous eqtn(s), for which the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 0 \end{array}\right]$$

backsolving this already reduced matrix:

$z = t$
$y = s$
$x = 2s - 5t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

so $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis. (Any two independent vectors in the plane would do.)

10) $y = z$

$y - z = 0$

as in #9: $\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \end{array}\right]$

$z = t$
$y = t$
$x = s$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};$$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis.

21) $$A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

$$\text{rref}(A) = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

backsolve: $x_4 = t$
$x_3 = s$
$x_2 = -s - 3t$
$x_1 = -s - 5t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix};$$ basis $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

26) $$A = \begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & -1 & 14 \end{bmatrix}$$

$$\text{rref}(A) = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & -2 & -5 & 0 \end{array}\right]$$

$x_5 = t$
$x_4 = s$
$x_3 = 2s + 5t$
$x_2 = s - 4t$
$x_1 = -2s + 3t$

$$\vec{x} = s \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -4 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$

4.5 #2) $\{A_{3\times 3}$ s.t. $a_{ij} = a_{ji} \quad \forall \begin{smallmatrix} 1\le i\le 3 \\ 1\le j\le 3 \end{smallmatrix}\} = V$

Yes, V is a subspace

a) closed under +: if A satisfies $a_{ij} = a_{ji}$ and B satisfies $b_{ij} = b_{ji}$

then $\text{entry}_{ij}(A+B) = a_{ij} + b_{ij} = a_{ji} + b_{ji} = \text{entry}_{ji}(A+B)$

so A+B is symmetric too.

b) closed under scal mult: if A satisfies $a_{ij} = a_{ji}$ then $\text{entry}_{ij}(cA)$

$= ca_{ij}$

$= ca_{ji}$

$= \text{entry}_{ji}(cA)$ ∎

3) <u>NO</u> Every subspace must contain the zero vector. The zero matrix is singular (it is not invertible!), so it is not in the collection V of nonsingular matrices

5) <u>YES</u>

a) If $f(0)=0$ and $g(0)=0$ then $(f+g)(0) = f(0)+g(0) = 0$, so V closed under +

b) If $f(0)=0$ then $(cf)(0) = c\cdot f(0) = c\cdot 0 = 0$, so V closed under scalar mult

7) <u>NO</u> Does not contain the zero fcn. (Also, another way would be to show set is not closed under +, or under ·

e.g. if $f(1)=1$ and $g(1)=1$ then $(f+g)(1)=2$)

8) <u>YES</u>

a) If $f(-x) = -f(x)$ and $g(-x) = -g(x)$ then $(f+g)(-x) = f(-x)+g(-x)$

$= -f(x) - g(x)$

$= -(f(x)+g(x))$

$= -(f+g)(x)$

b) If $f(-x) = -f(x)$ then

$(cf)(-x) = c\,f(-x)$

$= c(-f(x))$

$= -cf(x)$

$= -(cf)(x).$

10) <u>YES</u> a) If $p(x), q(x)$ are in the collection then

$p(x) = a_2x^2 + a_3x^3$

$q(x) = b_2x^2 + b_3x^3$

so $(p+q)(x) = (a_2+b_2)x^2 + (a_3+b_3)x^3$

so p+q also has no const. or linear terms, so is in our set.

b) If $p(x) = a_2x^2 + a_3x^3$

then $cp(x) = ca_2\cdot x^2 + ca_3\cdot x^3$ is in the set.

12) <u>NO</u> set is not closed under scalar multiplication. For example $p(x) = x^3$ is in the set, but $\frac{1}{2}p(x)$ is not.

13) If $c_1\sin x + c_2\cos x \equiv 0$ then $c_1\sin x = -c_2\cos x$; so $c_1\tan x = -c_2$

at $x=0$ deduce $c_2 = 0$

then at $x = \pi/4$ deduce $c_1 = 0$.

So <u>independent</u>

(or, two vectors are dependent iff they are scalar (i.e. constant) multiples of each other.

sinx, cosx are not mults of eachother since their quotient is not const)

13) cont'd. or, let $c_1 \sin x + c_2 \cos x = 0$

at $x=0$ get $c_2 = 0$
at $x = \pi/2$ get $c_1 = 0$ $\Longrightarrow$ independent

15) Let

$c_1(1+x) + c_2(1-x) + c_3(1-x^2) \equiv 0$

collect terms:

$(c_1+c_2+c_3)1 + (c_1-c_2)x - c_3 x^2 \equiv 0$

since $\{1, x, x^2\}$ are independent deduce

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - c_2 \quad &= 0 \\ -c_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \Big\downarrow \text{rref} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Longrightarrow c_1 = c_2 = c_3 = 0$

<u>INDEPENDENT</u>

19) $\dfrac{x-5}{(x-2)(x-3)} = \dfrac{A}{x-2} + \dfrac{B}{x-3} = \dfrac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

so $x - 5 = (A+B)x + (-3A-2B)1$

"identity principle" (linear independence of $\{1, x\}$ in disguise) requires that respective coeffs are equal:

$$\begin{array}{ll} x & A+B=1 \\ 1 & -3A-2B=-5 \end{array} \qquad \begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{1}\begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \quad \boxed{\begin{array}{l} A=3 \\ B=-2 \end{array}}$$

(shortcut for easy partial fracs:

$x-5 = A(x-3) + B(x-2)$

$x=3$: $-2 = 0 + B \to B = -2$
$x=2$: $-3 = -A + 0 \to A = 3$ ✓)

21) $\dfrac{8}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} = \dfrac{A(x^2+4)+(Bx+C)x}{x(x^2+4)}$

$8 = A(x^2+4) + (Bx+C)x$

x^2: $0 = A + B$ $\qquad B=-2$
x: $0 = C$ $\qquad C=0$
1: $8 = 4A$ $\qquad A=2$

$$\boxed{\begin{array}{l} A=2 \\ B=-2 \\ C=0 \end{array}}$$

23) $y''' = 0$; integrate 3 times

$y'' = c_1$
$y' = c_1 x + c_2$
$y = \frac{1}{2}c_1 x^2 + c_2 x + c_3$
$\;\; = ax^2 + bx + c$

basis
$\downarrow$
solution space $= \text{span}\{1, x, x^2\}$

25) $y'' - 5y' = 0$

Let $v = y'$. Then

$v' - 5v = 0$
$v' = 5v$

so $v(x) = ce^{5x}$
$y'(x) = ce^{5x}$

integrate:

$y(x) = \frac{c}{5}e^{5x} + d = ae^{5x} + b$

soln space $= \text{span}\{1, e^{5x}\}$.

$\uparrow$
basis