Math 2250-3
HW due Wed 10/8

4.1 1, 7 9 10 15 16 19 21 25, 26 31 33 37, 38
4.2 5, 6 9 15 16 24, 25, 27 29
4.3 1 3 6 9 10

7. $\vec{u}=\begin{bmatrix}2\\2\end{bmatrix}, \vec{v}=\begin{bmatrix}2\\-2\end{bmatrix}$ are <u>independent</u> since they are not scalar multiples [THIS TEST ONLY WORKS IN TESTING INDEPENDENCE FOR <u>TWO</u> VECTORS]

in terms of the precise def. of independence,
if $c_1\vec{u}+c_2\vec{v}=\vec{0}$ we need to show $c_1=c_2=0$ is only soltn:

$$\left[\begin{array}{cc|c} 2 & 2 & 0\\ 2 & -2 & 0\end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 1 & 0\\ 1 & -1 & 0\end{array}\right]$$
$$\begin{array}{l} \\ -R_1+R_2\end{array}\left[\begin{array}{cc|c} 1 & 1 & 0\\ 0 & -2 & 0\end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 1 & 0\end{array}\right] \Rightarrow c_1=c_2=0$$

10. $c_1\begin{bmatrix}3\\4\end{bmatrix}+c_2\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}$

$$\left[\begin{array}{cc|c} 3 & 2 & 0\\ 4 & 3 & -1\end{array}\right]$$
$$\begin{array}{l} R_2-R_1\\ R_2\end{array}\left[\begin{array}{cc|c} 1 & 1 & -1\\ 4 & 3 & -1\end{array}\right]$$
$$\begin{array}{l} \\ -4R_1+R_2\end{array}\left[\begin{array}{cc|c} 1 & 1 & -1\\ 0 & -1 & 3\end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & 2\\ 0 & 1 & -3\end{array}\right]$$

$c_1=2$
$c_2=-3$

check:
$$2\begin{bmatrix}3\\4\end{bmatrix}-3\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix} \checkmark$$

or, we could have used Cramer's rule:

$$c_1=\frac{\begin{vmatrix}0 & 2\\ -1 & 3\end{vmatrix}}{\begin{vmatrix}3 & 2\\ 4 & 3\end{vmatrix}}=\frac{2}{1}=2 \checkmark$$

$$c_2=\frac{\begin{vmatrix}3 & 0\\ 4 & -1\end{vmatrix}}{1}=-3$$

16. Are there any solutions to
$$c_1\vec{u}+c_2\vec{v}+c_3\vec{w}=\vec{0}$$
besides $c_1=c_2=c_3=0$. ("trivial soltn")
This is the matrix eqtn
$$\underbrace{\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}}_{A}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
has only "trivial solution" iff A^{-1} exists iff $\det(A)\neq 0$:

$$\begin{vmatrix}5 & 2 & 4\\ -2 & -3 & 5\\ 4 & 5 & -7\end{vmatrix}=\begin{vmatrix}1 & -3 & 11\\ -2 & -3 & 5\\ 4 & 5 & -7\end{vmatrix}\ -R_3+R_1$$
$$=\begin{vmatrix}1 & -3 & 11\\ 0 & -9 & 27\\ 0 & 17 & -51\end{vmatrix}\begin{array}{l} \\ 2R_1+R_2\\ -4R_1+R_3\end{array}$$
$$=-9\begin{vmatrix}1 & -3 & 11\\ 0 & 1 & -3\\ 0 & 17 & -51\end{vmatrix}$$
$$=(-9)(17)\begin{vmatrix}1 & -3 & 11\\ 0 & 1 & -3\\ 0 & 1 & -3\end{vmatrix}=0!$$

thus there <u>are</u> non-trivial solutions
so $\vec{u},\vec{v},\vec{w}$ are <u>dependent</u>

[in fact $\mathrm{rref}(A)=\left[\begin{array}{ccc|c}1 & 0 & 2 & 0\\ 0 & 1 & -3 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$

So homog soltns are $c_3=t,\ c_2=3t,\ c_1=-2t$
e.g. $(t=1)$ $c_1=-2$, $c_2=3$, $c_3=1$
$$-2\begin{bmatrix}5\\-2\\4\end{bmatrix}+3\begin{bmatrix}2\\-3\\5\end{bmatrix}+1\begin{bmatrix}4\\5\\-7\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$]

19. $c_1\begin{bmatrix}2\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-3\\1\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

rref: $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$c_3 = t$
$c_2 = 2t$
$c_1 = 3t$

e.g. $t=1$ $\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}3\\2\\1\end{bmatrix}$

dependent : $3\begin{bmatrix}2\\0\\1\end{bmatrix} + 2\begin{bmatrix}-3\\1\\-1\end{bmatrix} + 1\begin{bmatrix}0\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

21. $c_1\begin{bmatrix}1\\1\\-2\end{bmatrix} + c_2\begin{bmatrix}-2\\-1\\6\end{bmatrix} + c_3\begin{bmatrix}3\\7\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ 2

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 1 & -1 & 7 & 0 \\ -2 & 6 & 2 & 0 \end{array}\right]$$

rref: $\left[\begin{array}{ccc|c} 1 & 0 & 11 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$c_3 = t$
$c_2 = -4t$
$c_1 = -11t$

e.g. $t=1$ $\begin{bmatrix}-11\\-4\\1\end{bmatrix}$

dependent $-11\begin{bmatrix}1\\1\\-2\end{bmatrix} - 4\begin{bmatrix}-2\\-1\\6\end{bmatrix} + \begin{bmatrix}3\\7\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

26. $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{t}$

$\left[\vec{u}\,\middle|\,\vec{v}\,\middle|\,\vec{w}\right]\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \vec{t}$

$$\left[\begin{array}{ccc|c} 5 & 1 & 5 & 5 \\ 2 & 5 & -3 & 30 \\ -2 & -3 & 4 & -21 \end{array}\right]$$

rref: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$ (technology!)

$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\5\\-1\end{bmatrix}$

$\begin{bmatrix}5\\2\\-2\end{bmatrix} + 5\begin{bmatrix}1\\5\\-3\end{bmatrix} - \begin{bmatrix}5\\-3\\4\end{bmatrix} = \begin{bmatrix}5\\30\\-21\end{bmatrix}$ ✓

31. $\{(x,y,z) \text{ s.t. } 2x=3y\} = V$

a) V closed under +:

Let $\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}, \begin{bmatrix}x_2\\y_2\\z_2\end{bmatrix}$ in V

so $2x_1 = 3y_1$
$2x_2 = 3y_2$

thus $2(x_1+x_2) = 3(y_1+y_2)$

so $\begin{bmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\end{bmatrix}$ is in V

b) V closed under scalar multiplication

if $2x_1 = 3y_1$
then $2kx_1 = 3ky_1$

so $\begin{bmatrix}kx_1\\ky_1\\kz_1\end{bmatrix}$ is in V when $\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix}$ is.

so V is a subspace

33. V is not a subspace since, for example, it is not closed under addition. [or scalar mult.]

e.g. $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is in V

but $\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$ is not in V

38. V is a subspace so V is closed under + & scalar mult. Let $\vec{u}, \vec{v}$ in V then $a\vec{u}, b\vec{v}$ are in V [by closure wrt scalar mult]

so $a\vec{u} + b\vec{v}$ is too [by closure wrt addition].

§4.2 #6) $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 = 3x_3 \text{ and } x_2 = 4x_4 \right\}$ is a solution subspace:

$$x_1 - 3x_3 = 0$$
$$x_2 - 4x_4 = 0$$

Thm 2.

using Thm 1: if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in W$

a) closure under +

then $x_1 = 3x_3 \quad x_2 = 4x_4$

$y_1 = 3y_3 \quad y_2 = 4y_4$

so $(x_1+y_1) = 3(x_3+y_3) \quad (x_2+y_2) = 4(x_4+y_4)$

so $\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \\ x_4+y_4 \end{bmatrix} \in W$

b) closure under scalar mult

Let $\vec{x} \in W$ as above then $kx_1 = 3kx_3$, $kx_2 = 3kx_4$

so also $\begin{bmatrix} kx_1 \\ kx_2 \\ kx_3 \\ kx_4 \end{bmatrix} \in W$.

so W is a subspace

9) Not a subspace. e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ satisfy $x_1^2 + x_2^2 = 1$

but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not

15) $\left[\begin{array}{cccc|c} 1 & -4 & 1 & -4 & 0 \\ 1 & 2 & 1 & 8 & 0 \\ 1 & 1 & 1 & 6 & 0 \end{array}\right]$

rref: (technology!)

$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_4 = t$
$x_3 = s$
$x_2 = -2t$
$x_1 = -s - 4t$

$$\vec{x} = s \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + t \underbrace{\begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

16) $\left[\begin{array}{cccc|c} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{array}\right]$

rref: $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_4 = t$
$x_3 = s$
$x_2 = -s - 3t$
$x_1 = -s - 5t$

$$\vec{x} = s \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + t \underbrace{\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

27) We showed in class that the span of <u>any</u> finite collection of vectors <u>is</u> a subspace.
In this case... $\vec{u}, \vec{v}$ are fixed,

$W := \{ a\vec{u} + b\vec{v} \text{ s.t. } a, b \in \mathbb{R} \}$

a) if $a\vec{u} + b\vec{v}$, $c\vec{u} + d\vec{v} \in W$ then their sum is $(a+c)\vec{u} + (b+d)\vec{v} \in W$
so W is closed under addition

b) if $a\vec{u} + b\vec{v} \in W$ then so is $k(a\vec{u} + b\vec{v}) = ka\vec{u} + kb\vec{v}$
so W is closed under scalar mult

so W is a subspace ∎

§4.2 #29) We did this in class several weeks ago.:

If $A\vec{x}_0 = \vec{b}$ and $A\vec{x} = \vec{b}$ then
$$A(\vec{x}-\vec{x}_0) = A\vec{x} - A\vec{x}_0 = \vec{b}-\vec{b} = \vec{0}$$
so $A(\vec{x}-\vec{x}_0) = \vec{0}$

so $\vec{y} = \vec{x}-\vec{x}_0$ solves the homogeneous system.

conversely, if $\vec{y}$ solves $A\vec{y}=\vec{0}$

then $\vec{x} := \vec{x}_0 + \vec{y}$

satisfies $A\vec{x} = A(\vec{x}_0+\vec{y}) = A\vec{x}_0 + A\vec{y} = \vec{b}+\vec{0} = \vec{b}$

i.e. all soltns to $A\vec{x} = \vec{b}$ are of the form $\vec{x} = \vec{x}_0 + \vec{y}$ where $A\vec{y} = \vec{0}$.

§4.3 1. dependent; $\vec{v}_2 = \frac{3}{2}\vec{v}_1$

3. dependent: more than 2 vectors in $\mathbb{R}^3$ are always dependent
($\vec{v}_1, \vec{v}_2$ span $\mathbb{R}^2$ in this case, so $\vec{v}_3$ must be a linear combo of them)

6. independent! $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = |\vec{v}_1 \vdots \vec{v}_2 \vdots \vec{v}_3| = 1 \neq 0$. Thm 2

9. $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{w}$:

$$c_1\begin{bmatrix}5\\3\\4\end{bmatrix} + c_2\begin{bmatrix}3\\2\\5\end{bmatrix} = \begin{bmatrix}1\\0\\-7\end{bmatrix}$$

$$\left[\begin{array}{cc|c} 5 & 3 & 1 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{array}\right]$$

$-R_3+R_1$
$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 3 & 2 & 0 \\ 4 & 5 & -7 \end{array}\right]$$

$-3R_1+R_2$, $-4R_1+R_3$
$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 8 & -24 \\ 0 & 13 & -39 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$2R_2+R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 2$
$c_2 = -3$

$$2\begin{bmatrix}5\\3\\4\end{bmatrix} - 3\begin{bmatrix}3\\2\\5\end{bmatrix} = \begin{bmatrix}1\\0\\-7\end{bmatrix} \checkmark$$

10. $c_1\begin{bmatrix}-3\\1\\-2\end{bmatrix} + c_2\begin{bmatrix}6\\-2\\3\end{bmatrix} = \begin{bmatrix}3\\-1\\-2\end{bmatrix}$

$$\left[\begin{array}{cc|c} -3 & 6 & 3 \\ 1 & -2 & -1 \\ -2 & 3 & -2 \end{array}\right]$$

$\downarrow$ rref

$$\left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

$$7\begin{bmatrix}-3\\1\\-2\end{bmatrix} + 4\begin{bmatrix}6\\-2\\3\end{bmatrix} = \begin{bmatrix}3\\-1\\-2\end{bmatrix} \checkmark$$