Math 2250-4
HW due Wed 9/19

2.4 5
2.5 5 25
2.6 5
3.1 1, 4, 11, 16, 17, 19, 23, 29, 32, 33, 34
3.2 7, 8, 9, 13, 17, 29, 30

2.4-2.6 were done with Maple - see end of sol'ns.

3.1 4) I will do these synthetically, as in class

$$\left[\begin{array}{cc|c} 5 & -6 & 1 \\ 6 & -5 & 10 \end{array}\right] \leftarrow \text{stands for } \begin{array}{l} 5x-6y=1 \\ 6x-5y=10 \end{array}$$

$$\begin{array}{l} -R_1+R_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 1 & 9 \\ 6 & -5 & 10 \end{array}\right]$$

$$\begin{array}{l} \\ -6R_1+R_2 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & -11 & -44 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/_{11} \end{array} \left[\begin{array}{cc|c} 1 & 1 & 9 \\ 0 & 1 & 4 \end{array}\right]$$

$$\begin{array}{l} -R_2+R_1 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 4 \end{array}\right] \leftarrow \text{stands for } \begin{array}{l} x=5 \\ y=4 \end{array}, \text{ so that's the sol'n } \boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}}$$

$5(5)-6(4)=1$ ✓
$6(5)-5(4)=10$ ✓

11)
$$\left[\begin{array}{ccc|c} 2 & 7 & 3 & 11 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 9 & -12 \end{array}\right]$$

$$\begin{array}{l} R_2 \\ R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 2 & 7 & 3 & 11 \\ 3 & 7 & 9 & -12 \end{array}\right]$$

$$\begin{array}{l} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 0 & -2 & 3 & -18 \end{array}\right]$$

$$\begin{array}{l} \\ \\ 2R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\begin{array}{l} -2R_3+R_1 \\ R_3+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\begin{array}{l} -3R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{array}\right] \leftarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$$

$2(1)+7(3)+3(-4) = 23-12 = 11$ ✓
$1(1)+3(3)+2(-4) = 10-8 = 2$ ✓
$3(1)+7(3)+9(-4) = 24-36 = -12$ ✓

16)
$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 1 & 4 & -1 & 4 \\ 5 & 6 & 1 & 20 \end{array}\right]$$

$$\begin{array}{l} \\ -R_1+R_2 \\ -5R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 21 & -9 & -10 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -3R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 0 & 0 & -4 \end{array}\right] \leftarrow 0x+0y+0z=-4$$

has no sol'n
so also original
system is
inconsistent

3.1 17)

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 7 \\ 3 & 2 & -2 & 3 \\ 5 & 1 & 2 & 15 \end{array}\right]$$

$-R_1+R_2$
$$\left[\begin{array}{ccc|c} 1 & 3 & -6 & -4 \\ 3 & 2 & -2 & 3 \\ 5 & 1 & 2 & 15 \end{array}\right]$$

$-3R_1+R_2$, $-5R_1+R_3$
$$\left[\begin{array}{ccc|c} 1 & 3 & -6 & -4 \\ 0 & -7 & 16 & 15 \\ 0 & -14 & 32 & 35 \end{array}\right]$$

$-2R_2+R_3$
$$\left[\begin{array}{ccc|c} 1 & 3 & -6 & -4 \\ 0 & -7 & 16 & 15 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

inconsistent

23) $y''+4y=0$ ; $y(x)=A\cos 2x + B\sin 2x$
$y'(x) = -2A\sin 2x + 2B\cos 2x$

$y(0)=3=A$ $\quad A=3$
$y'(0)=8=2B$ $\quad B=4$

$$\boxed{y = 3\cos 2x + 4\sin 2x}$$

29) $6y''-5y'+y=0$
$y(x) = Ae^{x/2} + Be^{x/3}$
$y' = \frac{1}{2}Ae^{x/2} + \frac{1}{3}Be^{x/3}$

$y(0)=7=A+B$
$y'(0)=11=\frac{1}{2}A+\frac{1}{3}B$ , or $66=3A+2B$

$A+B=7$
$3A+2B=66$ $\rightsquigarrow$
$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 3 & 2 & 66 \end{array}\right]$$

$-3R_1+R_2$
$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -1 & 45 \end{array}\right]$$

old $R_2+R_1$
$$\left[\begin{array}{cc|c} 1 & 0 & 52 \\ 0 & 1 & -45 \end{array}\right] \rightsquigarrow \begin{array}{l} A=52 \\ B=-45 \end{array}$$

$$\boxed{y(x) = 52e^{x/2} - 45e^{x/3}}$$

$52-45=7$ ✓
$26-15=11$ ✓

32) a) two planes in $\mathbb{R}^3$ are either parallel [so no points are on them], or they intersect in a line [so $\infty$'ly many points are on them] or they are identical [so $\infty$ int]

Since the solution set to
$a_1x+b_1y+c_1z=d_1$
$a_2x+b_2y+c_2z=d_2$
is exactly the set of points common to two planes, we deduce it is either empty (parallel planes) or $\infty$ (intersecting line or equal planes)

b) if $d_1=d_2=0$ then $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is in both planes, so they do intersect, hence intersection is infinite

§3.2 8) $x_1 - 10x_2 + 3x_3 - 13x_4 = 5$
$x_3 + 3x_4 = 10$

let $x_4 = t$
then $x_3 = 10-3t$
let $x_2 = s$
then $x_1 = 5 + 10s - 3(10-3t) + 13t$

$$\boxed{\text{so } \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-25\\0\\10\\0\end{bmatrix} + s\begin{bmatrix}10\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}22\\0\\-3\\1\end{bmatrix}}$$

9) $2x_1 + x_2 + x_3 + x_4 = 0$
$3x_2 - x_3 - 2x_4 = 2$
$3x_3 + 4x_4 = 9$
$x_4 = 6$

$x_4 = 6$
$3x_3 = 9-24 = -15$ ; $x_3 = -5$
$3x_2 = 2+(-5)+2(6) = 9$ ; $x_2 = 3$
$2x_1 = -3 - -5 - 6 = -4$
$x_1 = -2$

$$\boxed{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-2\\3\\-5\\6\end{bmatrix}}$$

§3.2 13)

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \\ 2 & 7 & 8 & 29 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ -2R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\begin{array}{c} \\ -R_2 \\ \text{old } R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} -3R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$
$x_2 = 3-2t$
$x_1 = 4+3t$

so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

29 a) row p is the only issue:

$R_p \to cR_p \to \frac{1}{c}cR_p = R_p$

in A    in B    back to A

b) clear

c) $R_q \to cR_p + R_q \to [cR_p+R_q] - cR_p = R_q$

d) (a)-(c) show each row op is reversible. So if we do

$Op_1$
$Op_2$
$Op_3$
⋮
$Op_k$

to A in order to get B then doing

$Op_k^{-1}$
$Op_{k-1}^{-1}$
⋮
$Op_2^{-1}$
$Op_1^{-1}$

returns us back to A.

17)

$$\left[\begin{array}{cccc|c} 1 & -4 & -3 & -3 & 2 \\ 2 & -6 & -5 & -5 & 5 \\ 3 & -1 & -4 & -5 & -7 \end{array}\right]$$

$$\begin{array}{c} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -4 & -3 & -3 & 2 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 11 & 5 & 4 & -13 \end{array}\right]$$

$$\begin{array}{c} \\ -5R_2+R_3 \\ R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -4 & -3 & -3 & 2 \\ 0 & 1 & 0 & -1 & -18 \\ 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -2R_2+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & -4 & -3 & -3 & 2 \\ 0 & 1 & 0 & -1 & -18 \\ 0 & 0 & 1 & 3 & 37 \end{array}\right]$$

(didn't do anything)

$$\left[\begin{array}{cccc|c} 1 & -4 & -3 & -3 & 2 \\ 0 & 1 & 0 & -1 & -18 \\ 0 & 0 & 1 & 3 & 37 \end{array}\right]$$

$$\begin{array}{c} 3R_3+R_1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & -4 & 0 & 6 & 113 \\ 0 & 1 & 0 & -1 & -18 \\ 0 & 0 & 1 & 3 & 37 \end{array}\right]$$

$$\begin{array}{c} 4R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 41 \\ 0 & 1 & 0 & -1 & -18 \\ 0 & 0 & 1 & 3 & 37 \end{array}\right]$$

18, 11, 18, 18, 198, −13, 185

37 × 3 = 111

18 × 4 = 72; 113 − 72 = 41

$x_4 = t$
$x_3 = 37-3t$
$x_2 = -18+t$
$x_1 = 41-2t$

$$\boxed{\text{so } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 41 \\ -18 \\ 37 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix}}$$

check: well, we can get smaller numbers by letting $t=20$:

$$\begin{bmatrix} 41-40 \\ -18+20 \\ 37-60 \\ 20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -23 \\ 20 \end{bmatrix}$$

$1-8+69-60 = 2$ ✓
$2-12+115-100 = 5$ ✓
$3-2+92-100 = -7$ ✓

yea!

Math 2250-4
Numerical Computations
Homework due Wednesday, Sept 19, 2001

This is the work for #5 in sections 2.4-2.6, and for #25 in section 2.6

## 2.5 #5:

```
> restart:with(DEtools):
> deqtn:=diff(y(x),x)=y(x)-x-1:
  dsolve({deqtn,y(0)=1},y(x)); #exact solution.
```

$$y(x)=x+2-e^x$$

**Find exact solution by hand:** You were supposed to find this solution by hand, using the fact that the DE is linear. You probably figured out that the integrating factor is exp(-x), and proceeded as follows:

$$\frac{dy}{dx}-y=-x-1$$

$$\left[\frac{d}{dx}\right]e^{(-x)}y=e^{(-x)}(-x-1)$$

$$e^{(-x)}y=e^{(-x)}x+2e^{(-x)}+C$$

$$y=x+2+C\,e^x$$

$$1=2+C$$

$$y=x+2-e^x$$

**Now we will do Euler as requested, first with h=.1, between x=0 and x=.5:**

```
> x0:=0.0: xn:=0.5: y0:=1.0: n:=5: h:=(xn-x0)/n:
   #specify initial values, number of steps, and size
> f:=(x,y)->y-x-1:    #this is the slope function f(x,y)
    #in dy/dx = f(x,y), for #5.
> x:=x0: y:=y0:   #initialize x,y for the do loop

> for i from 1 to n do
       k:= f(x,y):  #current slope, use : to suppress output
       y:= y + h*k: #new y value via Euler
       x:= x + h:   #updated x-value:
       print(x,y,x+2-exp(x));  #display current values,
                        #and compare to exact solution.
    od:                 #``od'' ends a do loop
```

.1000000000, 1.0, .994829082

.2000000000, .9900000000, .978597242

.3000000000, .9690000000, .950141192

.4000000000, .9359000000, .908175302

.5000000000, .8894900000, .851278729

**Change h to 0.05, and display results every 0.1:**



```
> x:=x0: y:=y0:n:=10:h:=0.05:
  for i from 1 to n do
       k:= f(x,y):  #current slope, use : to suppress output
       y:= y + h*k: #new y value via Euler
       x:= x + h:   #updated x-value:
         if frac(i/2)=0  #only print at multiples of 0.1
          then print(x,y,x+2-exp(x));  #display current values,
                     #and compare to exact solution.
         fi;         #end of if clause
     od:             #``od'' ends a do loop
```

.10, .9975, .994829082

.20, .98449375, .978597242

.30, .9599043594, .950141192

.40, .9225445563, .908175302

.50, .8711053733, .851278729

**2.5 #5: improved Euler with step size 0.1, same DEqtn:**

```
> x:=x0: y:=y0:n:=5:h:=0.1:
  for i from 1 to n do
       k1:= f(x,y): #left slope
       k2:=f(x+h,y+h*k1):  #right slope approx.
       k:=(k1+k2)/2:
       y:= y + h*k: #new y value via Euler
       x:= x + h:   #updated x-value:
         print(x,y,x+2-exp(x));  #display current values,
                     #and compare to exact solution.
     od:             #``od'' ends a do loop
```

.1, .9950000000, .994829082

.2, .9789750000, .978597242

.3, .9507673750, .950141192

.4, .9090979494, .908175302

.5, .8525532341, .851278729

Notice we did much better than with unimproved Euler, getting the correct answer to two decimal places.

2.6#5) Same differential equation, with Runge Kutta, two steps to 0.5:

```
> x:=x0: y:=y0:n:=2:h:=0.25:

>
   for i from 1 to n do        #stolen Runge-Kutta code
     k1:=f(x,y):               #left-hand slope
     k2:=f(x+h/2,y+h*k1/2):    #1st guess at midpoint slope
     k3:=f(x+h/2,y+h*k2/2):    #second guess at midpoint slope
     k4:=f(x+h,y+h*k3):        #guess at right-hand slope
     k:=(k1+2*k2+2*k3+k4)/6:   #Simpson's approximation for the
  integral
     x:=x+h:                   #x update
```

```
      y:=y+h*k:                    #y update
      print(x,y,x+2-exp(x));       #display current values and exact sol.
 >   od:
```

.25, .9659830729, .965974583

.50, .8513005309, .851278729

Good to almost 4 decimal places!!!

**2.5 #25) improved Euler, with n=50 subintervals:**

```
 > t0:=0: v0:=49.0: n:=50: h:=.2:
   t:=t0: v:=v0:
   f:=(t,v)->-.04*v-9.8:
   t:=t0:  v:=v0:
 > for i from 1 to n do
        k1:= f(t,v): #left slope
        k2:=f(t+h,v+h*k1):  #right slope approx.
        k:=(k1+k2)/2:
        v:= v + h*k: #new v value via Euler
        t:= t+ h:   #updated t-value:
          if frac(i/5)=0 then
            print(t,v,294*exp(-t/25)-245);
           #display current values,
           #and compare to exact solution, when time is a
           #whole second
         fi:
      od:                 #``od'' ends a do loop
```

1.0, 37.47221636, 37.4720951

2.0, 26.39643881, 26.3962058

3.0, 15.75494416, 15.7546084

4.0, 5.530704084, 5.5302740

5.0, -4.292642013, -4.2931586

6.0, -13.73081325, -13.7314088

7.0, -22.79891241, -22.7995800

8.0, -31.51145004, -31.5121831

9.0, -39.88236779, -39.8831601

10.0, -47.92506060, -47.9259065

Looks very close.

**2.5 #25) improved Euler, with n=100 subintervals:**

```
 > t0:=0: v0:=49.0: n:=100: h:=.1:
   t:=t0: v:=v0:
   f:=(t,v)->-.04*v-9.8:
   t:=t0:  v:=v0:
 > for i from 1 to n do
        k1:= f(t,v): #left slope
```



```
        k2:=f(t+h,v+h*k1):  #right slope approx.
        k:=(k1+k2)/2:
        v:= v + h*k: #new v value via Euler
        t:= t+ h:   #updated t-value:
          if frac(i/10)=0 then
            print(t,v,294*exp(-t/25)-245);
           #display current values,
           #and compare to exact solution, when time is a
           #whole second
         fi:
      od:                 #``od'' ends a do loop
```

1.0, 37.47212533, 37.4720951

2.0, 26.39626391, 26.3962058

3.0, 15.75469208, 15.7546084

4.0, 5.530381164, 5.5302740

5.0, -4.293029838, -4.2931586

6.0, -13.73126040, -13.7314088

7.0, -22.79941363, -22.7995800

8.0, -31.51200039, -31.5121831

9.0, -39.88296265, -39.8831601

10.0, -47.92569564, -47.9259065

There is not much difference between n=50 and n=100 time steps, and both approximations are very close to the exact solution.

**How to approximate time to apex and impact velocity after 9.41 seconds:**

We can narrow in on the time of zero velocity, we can see from above that it's between 4 and 5 seconds. By linear interpolation it seems to be near 4.6 seconds. I can rerun Euler between 4 and 5 seconds, using the velocity at 4 seconds as an initial velocity. I can print out every computation with time between 4.5 and 4.7 seconds:

```
 >
 > t0:=4.0: v0:=5.530381164: n:=100: h:=.01:
      #narrowing in...
   t:=t0: v:=v0:
   f:=(t,v)->-.04*v-9.8:
   t:=t0:  v:=v0:
 >
 > for i from 1 to n do
        k1:= f(t,v): #left slope
        k2:=f(t+h,v+h*k1):  #right slope approx.
        k:=(k1+k2)/2:
        v:= v + h*k: #new v value via Euler
        t:= t+ h:   #updated t-value:
          if (i >= 50) and  (i <= 70)
            then
           print(t,v,294*exp(-t/25)-245);
          #display current values,
          #and compare to exact solution, in our interval of
```

```
concern
    fi:
  od:              #``od'' ends a do loop
```

4.50, .5695473697, .5694422

4.51, .4713391963, .4712340

4.52, .3731702983, .3730652

4.53, .2750406600, .2749356

4.54, .1769502657, .1768452

4.55, .07889909974, .0787941

4.56, -.01911285359, -.0192178

4.57, -.1170856100, -.1171906

4.58, -.2150191851, -.2151241

4.59, -.3129135946, -.3130185

4.60, -.4107688542, -.4108737

4.61, -.5085849795, -.5086897

4.62, -.6063619862, -.6064667

4.63, -.7040998899, -.7042046

4.64, -.8017987063, -.8019034

4.65, -.8994584510, -.8995631

4.66, -.9970791396, -.9971837

4.67, -1.094660788, -1.0947653

4.68, -1.192203411, -1.1923079

4.69, -1.289707025, -1.2898115

4.70, -1.387171645, -1.3872761

so we see that zero velocity occurs at about t=4.56 seconds, in agreement with section 2.3.

We could use the identical method to estimate the impact velocity after 9.41 seconds - rerun Euler between 9 and 10 seconds and display results in a subinterval. (If we didn't know that the impact occured then we could find a numerical approximation by doing a numerical integration of the velocity function!, )