Math 2250-3
HW due 9/10

2.1 1, 3, 6, 8, 9, 11, 12, 19
2.2 5, 7, 9, 12, 14, 15, 21
2.3 2, 3, 9, 10, 17, 18

2.1 3) $\begin{cases} \frac{dx}{dt} = 4x(7-x) \\ x(0) = 11 \end{cases}$

$$\frac{dx}{x(7-x)} = 4\,dt$$

I can see by adjusted guessing that $\frac{1}{7}\left[\frac{1}{x} + \frac{1}{7-x}\right] = \frac{1}{x(7-x)}$

or, if you prefer, partial fractions:

$$\frac{1}{x(7-x)} = \frac{A}{x} + \frac{B}{7-x} = \frac{A(7-x)+Bx}{x(7-x)}$$

so $1 = A(7-x) + Bx$

$x=0 \Rightarrow 1 = 7A$; $A = 1/7$
$x=7 \Rightarrow 1 = 7B$; $B = 1/7$

$$\frac{1}{7}\int \frac{1}{x} + \frac{1}{7-x}\,dx = \int 4\,dt$$

$$\frac{1}{7}\ln\left|\frac{x}{7-x}\right| = 4t + C$$

$$\ln\left|\frac{x}{7-x}\right| = 28t + C$$

$$\left|\frac{x}{7-x}\right| = e^C e^{28t} = \tilde{C} e^{28t}$$

$x(0) = 11$; $\tilde{C} = \left|\frac{11}{7-11}\right| = \frac{11}{4}$

$$\frac{x}{7-x} = \pm \frac{11}{4} e^{28t} = -\frac{11}{4} e^{28t} \quad [\text{try } x=0!]$$

$$x = (7-x)\left(-\frac{11}{4}\right)e^{28t}$$

$$x\left[1 - \frac{11}{4}e^{28t}\right] = -\frac{77}{4}e^{28t}$$

$$x = \frac{-\frac{77}{4}e^{28t}}{1-\frac{11}{4}e^{28t}}\left[\frac{-4e^{-28t}}{-4e^{-28t}}\right] = \boxed{\frac{77}{-4e^{-28t}+11} = x(t)}$$

← check $x(0) = \frac{77}{7} = 11$ ✓
$x(t) \to 7$ as $t \to \infty$ ✓

8) $\begin{cases} \frac{dP}{dt} = kP^2 \\ P(0) = 1 \text{ dozen (1988 is } t=0). \\ P(1) = 2 \text{ dozen} \end{cases}$

[I chose to measure years in decades and population in dozens since this makes the numbers simpler]

$\frac{dP}{P^2} = k\,dt \rightarrow -\frac{1}{P} = kt + C$
at $t=0$: $-1 = C$

$-\frac{1}{P} = kt - 1$
at $t=1$
$-\frac{1}{2} = k - 1$; $k = \frac{1}{2}$

$$-\frac{1}{P} = \frac{1}{2}t - 1 = \frac{t-2}{2}$$

so $\boxed{P = \frac{2}{2-t}}$

doomsday occurs at $t = 2$ decades, i.e. 2008, since as $t \nearrow 2$ $P(t) \nearrow \infty$.

2.1 9) a) $\beta = k_1 P \quad \delta = k_2 P$

$$\frac{\frac{dP}{dt}}{P} = \beta - \delta = (k_1 - k_2) P \Rightarrow \frac{dP}{dt} = kP^2 \qquad k = k_1 - k_2$$

$$\frac{dP}{P^2} = k\,dt$$

$$-\frac{1}{P} = kt + C \; ; \qquad -\frac{1}{P_0} = C \quad [\text{plug in } t=0].$$

$$-\frac{1}{P} = kt - \frac{1}{P_0}$$

$$-\frac{1}{P} = \frac{ktP_0 - 1}{P_0}, \quad \text{so} \quad P = \frac{-P_0}{ktP_0 - 1} = \frac{P_0}{1 - ktP_0} \checkmark$$

b) $P_0 = 6$, so $P(t) = \dfrac{6}{1 - 6kt}$

$P(10) = 9$

$$9 = \frac{6}{1 - 60k} \; ; \quad 9 - 540k = 6$$

$$3 = 540k$$

$$\frac{1}{180} = k$$

$\to$ so $P(t) = \dfrac{6}{1 - \frac{1}{30}t}$

hence doomsday occurs at $t = 30$ months [when denom $= 0$]

12) $\dfrac{dP}{dt} = aP - bP^2$ where $B = aP$ = birthrate, $D = -bP^2$ = death rate.

at $t = 0$: $P_0 = 120$

$$aP_0 = 8 \Rightarrow 120a = 8 \; ; \quad a = \frac{8}{120} = \frac{1}{15}$$

$$bP_0^2 = 6 \Rightarrow 120^2 b = 6 \Rightarrow b = \frac{6}{(120)^2} = \frac{1}{2400}$$

so

$$\frac{dP}{dt} = \frac{1}{15}P - \frac{1}{2400}P^2$$

$$\begin{cases} \dfrac{dP}{dt} = \dfrac{1}{2400} P \left[160 - P\right] \\ P(0) = 120 \end{cases}$$

from formula (4) p.77 deduce

$M = 160$
$P_0 = 120$
$k = \frac{1}{2400}$
$kM = \frac{1}{15}$

$$P(t) = \frac{(120)(160)}{120 + 40e^{-t/15}}$$

$$= \frac{480}{3 + e^{-t/15}}$$

set $P(t) = .95 P_0 = \frac{19}{20} 160 = 152$

$$152 = \frac{480}{3 + e^{-t/15}} \; ; \quad 152(3 + e^{-t/15}) = 480$$

$$e^{-t/15} = \frac{480}{152} - 3 \cong .158$$

$$-t/15 \cong \ln(.158) \cong -1.845$$

$$\boxed{t \approx 27.7 \text{ months}}$$

19) $\dfrac{dx}{dt} = .8x - .004x^2$

$$\frac{dx}{dt} = .004x(200 - x)$$

(a) = limiting salt concentration "M" = 200 g.

(b) $x_0 = 50$
when does $x(t) = 100$?

from p.77 (4)

$k = .004$
$x_0 = 50$
$kM = .8$
$M = 200$

$$x(t) = \frac{200(50)}{50 + 150e^{-.8t}} = \frac{200}{1 + 3e^{-.8t}}$$

set $x(t) = 100$: $\quad 100 = \dfrac{200}{1 + 3e^{-.8t}} \; ; \quad 1 + 3e^{-.8t} = 2$

$$3e^{-.8t} = 1 \qquad e^{-.8t} = 1/3$$

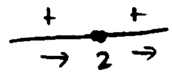
$$t = \frac{\ln(1/3)}{-.8} \cong 1.37 \text{ sec.}$$

$$(= \tfrac{5}{4} \ln 3)$$

2.2 7) $\frac{dx}{dt} = (x-2)^2$

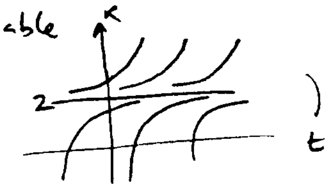
slope $=(x-2)^2$

$x=2$ only equilibrium



so $x=2$ is unstable

(or via slope field



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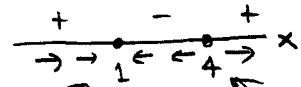
9) $\frac{dx}{dt} = x^2 - 5x + 4$
$= (x-4)(x-1)$

equilibria: $x=1$, $x=4$

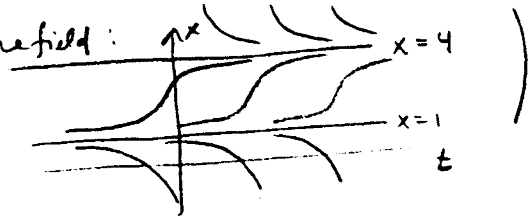


slope $= (x-4)(x-1)$

$x=4$ unstable

$x=1$ stable

(or via slopefield:



)

14) $\frac{dx}{dt} = kx(M-x) - hx$

$\frac{dx}{dt} = kx\left[\left(M-\frac{h}{k}\right) - x\right]$

$= kx\left[M' - x\right]$, where $M' = M - h/k$

(a) this is logistic, with $M' = M - h/k$, as long as $M' > 0$
i.e. $M - h/k > 0$

$\boxed{kM > h}$

(b) If $h = kM$ then $M' = 0$

and $\frac{dx}{dt} = -kx^2 \Rightarrow \frac{dx}{-x^2} = k\,dt \Rightarrow \frac{1}{x} = kt + C = kt + x_0$

$\Rightarrow x = \frac{1}{x_0 + kt} \to 0$ as $t \to \infty$

If $h > kM$ then $M' < 0$ but the method of sol'n on page 77 still holds,

so $P(t) = \frac{M'P_0}{P_0 + (M'-P_0)e^{-kM't}}$

eqn 4 p.77!

note, if $P_0 > 0$ both num & denom are negative!

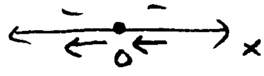
since $M' < 0$ the denom exponential $e^{-kM't} \to \infty$ as $t \to \infty$

so $P(t) \searrow 0$

one could also argue in (b) using slope fields or phase diagrams:

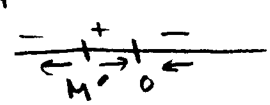
$h = kM$

$\frac{dx}{dt} = -kx^2$



$h > kM$
$M' < 0$

$\frac{dx}{dt} = kx(M'-x)$



15) $\frac{dx}{dt} = kx(x-M)$

$$\frac{dx}{x(x-M)} = k\,dt$$

$$\frac{1}{M}\left[-\frac{1}{x} + \frac{1}{x-M}\right]dx = k\,dt$$

$$\left[-\frac{1}{x} + \frac{1}{x-M}\right]dx = Mk\,dt$$

$$\ln\left|\frac{x-M}{x}\right| = Mkt + C$$

$$\left|\frac{x-M}{x}\right| = e^C e^{Mkt} \;; \quad \frac{x-M}{x} = \tilde{c}\,e^{Mkt}$$

$$\frac{x-M}{x} = \frac{x_0-M}{x_0}\,e^{Mkt} \quad \text{(check at } t=0\text{).}$$

$$x - M = \left(1-\frac{M}{x_0}\right)e^{Mkt}\,x$$

$$-M = x\left[\left(1-\frac{M}{x_0}\right)e^{Mkt} - 1\right]$$

$$M = x\left[1 - \left(1-\frac{M}{x_0}\right)e^{Mkt}\right]$$

$$\frac{M}{1-\left(1-\frac{M}{x_0}\right)e^{Mkt}} = x$$

$$x = \frac{Mx_0}{x_0 - (x_0-M)e^{Mkt}} = \frac{Mx_0}{x_0 + (M-x_0)e^{Mkt}} \quad \checkmark$$

or ∂ part fracs:

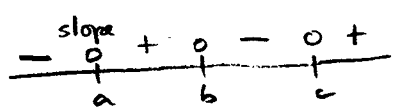
$$\frac{1}{x(x-M)} = \frac{A}{x} + \frac{B}{x-M} = \frac{A(x-M)+Bx}{x(x-M)}$$

$$1 = A(x-M) + Bx$$

$x=0 \Rightarrow 1 = -AM$; $\quad A = -\frac{1}{M}$

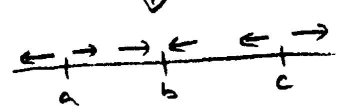
$x=M \Rightarrow 1 = MB$; $\quad B = \frac{1}{M}$

21) $\frac{dx}{dt} = (x-a)(x-b)(x-c)$ $\qquad a<b<c$



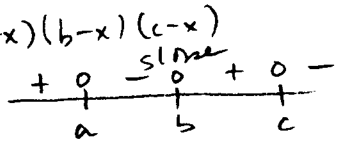
⟹ phase diagram

stable

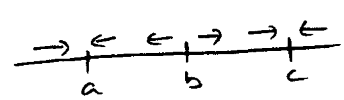


only b is stable

$\frac{dx}{dt} = (a-x)(b-x)(c-x)$

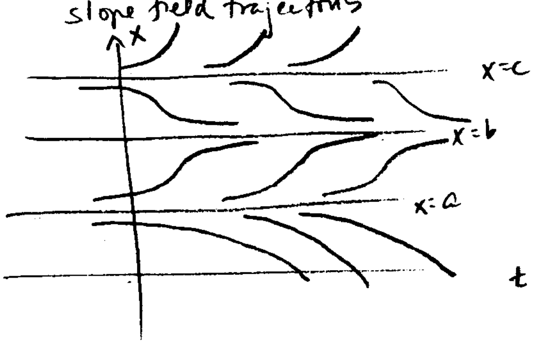


⟹ phase diagram

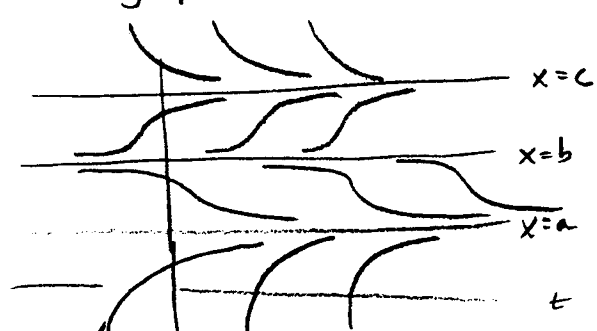


a & c are stable

slope field trajectories



graphs



2.3 2) $\frac{dv}{dt} = -kv$

(a) $\frac{dv}{v} = -k\,dt$

$\ln|v| = -kt + C$

$|v| = \tilde{C}e^{-kt}$

$v = v_0 e^{-kt}$ this should look familiar!

$\frac{dx}{dt} = v_0 e^{-kt}$

$x = \int v_0 e^{-kt}\,dt = -\frac{v_0}{k} e^{-kt} + C$

$x(0) = x_0 = -\frac{v_0}{k} + C; \quad C = x_0 + \frac{v_0}{k}$

$x(t) = -\frac{v_0}{k} e^{-kt} + x_0 + \frac{v_0}{k}$

$= x_0 + \frac{v_0}{k}(1 - e^{-kt})$ ✓

(b) $x(t) \nearrow$ as $t \nearrow$ and $\lim_{t\to\infty} x(t) = x_0 + \frac{v_0}{k}(1-0) = x_0 + \frac{v_0}{k}$

So object traveled $\frac{v_0}{k}$

10) $v(t) = (v_0 + \frac{g}{\rho})e^{-\rho t} - \frac{g}{\rho}$

$y(t) = y_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t})$

initially $y_0 = 10^4$, $v_0 = 0$

$\rho = .15$

$v_\tau = -\frac{g}{\rho} = \frac{-32}{.15} = -213\frac{1}{3}$ ft/sec

so $y(20) = 10^4 - \frac{32}{.15}(20) + \frac{1}{.15}\left(\frac{32}{.15}\right)(1 - e^{-(.15)(20)})$

$v(20)$ ... go to Maple!

Math 2250-4
homework calculations
Wednesday Sept 12

Section 2.3 10)

```
> restart:with(DEtools):with(plots):
Warning, the name changecoords has been redefined
> v:=t->(v0+g/rho)*exp(-rho*t) -g/rho;
  y:=t->y0+vtau*t+(1/rho)*(v0-vtau)*(1-exp(-rho*t));
    #formulas from pages 96-97
```

$$v := t \to \left(v0 + \frac{g}{\rho}\right) e^{(-\rho t)} - \frac{g}{\rho}$$

$$y := t \to y0 + vtau\, t + \frac{(v0 - vtau)\,(1 - e^{(-\rho t)})}{\rho}$$

```
> g:=32.0:
  y0:=10000:  #data during initial fall
  v0:=0:
  rho:=.15:
  vtau:=-g/rho:
> v(t);
  y(t);  #just checking
```

$$213.3333333\, e^{(-.15\, t)} - 213.3333333$$

$$11422.22222 - 213.3333333\, t - 1422.222222\, e^{(-.15\, t)}$$

```
> y(20);
  v(20);  #position and velocity after 20 seconds
```

$$7084.747281$$

$$-202.7120920$$

```
> y0:=7084.747281:
  v0:=-202.7120920:
  rho:=1.5:
  vtau:=-g/rho:  #data after parachute opens
> y(t);
  v(t);  #just checking
```

$$6963.828108 - 21.33333333\, t + 120.9191725\, e^{(-1.5\, t)}$$

$$-181.3787587\, e^{(-1.5\, t)} - 21.33333333$$

```
> solve(y(t)=0,t); #find landing time
```

$$326.4294426$$

```
> 326.4294426+20;  #total time until touchdown,
    #accounting for before/after parachute opens
```

$$346.4294426$$

```
> %/60;
```

$$5.773824043$$

```
> .773824043*60;
```

$$46.42944258$$

So the woman is airborn for about **5 minutes and 46 seconds.**

17)

```
> restart:with(DEtools):with(plots):
Warning, the name changecoords has been redefined
> v:=t->sqrt(g/rho)*tan(C1-t*sqrt(rho*g));
  C1:=arctan(v0*sqrt(rho/g));
  y:=t->y0+(1/rho)*ln(abs(cos(C1-t*sqrt(rho*g))/cos(C1)));
    #equations from page 98-99
```

$$v := t \to \sqrt{\frac{g}{\rho}}\,\tan(C1 - t\sqrt{\rho\, g})$$

$$C1 := \arctan\left(v0\sqrt{\frac{\rho}{g}}\right)$$

$$y := t \to y0 + \frac{\ln\left(\left|\frac{\cos(C1 - t\sqrt{\rho\, g})}{\cos(C1)}\right|\right)}{\rho}$$

```
> g:=9.8:
  v0:=49:
  y0:=0:
  rho:=.0011:  #data from problem
> v(t);
  y(t);  #just checking
```

$$-94.38798074\,\tan(-.4788372920 + .1038267788\,t)$$

$$909.0909091\,\ln(1.126720906\,|\cos(-.4788372920 + .1038267788\,t)|)$$

```
> solve(v(t)=0,t);  #find time at ymax
```

$$4.611886235$$

```
> y(4.611886235);  #find ymax
```

$$108.4650555$$

18)

```
> restart:
>
  v:=t->sqrt(g/rho)*tanh(C2-t*sqrt(rho*g));
  y:=t->y0-(1/rho)*ln(abs(cosh(C2-t*sqrt(rho*g)))/cosh(C2));
    #formulas from text
```

$$v := t \to \sqrt{\frac{g}{\rho}}\,\tanh(C2 - t\sqrt{\rho\, g})$$

$$y := t \to y0 - \frac{\ln\left(\frac{|\cosh(C2 - t\sqrt{\rho\, g})|}{\cosh(C2)}\right)}{\rho}$$

```
> y0:=108.47:   #data from 18
```

```
  rho:=.0011:
  v0:=0:
  g:=9.8:
  C2:=arctanh(v0*sqrt(rho/g)):
> solve(y(t)=0,t);  #find when ground is hit - second
    #root is complex:
```

$$4.799067204,\ 4.799067204 + 30.25801908\,I$$

```
> v(4.799067204); #speed when ground is hit
```

$$-43.48990830$$

```
>
```