

1.4 3, 9, 12, 14, 19, 20, 22, 37, 45, 54, 55, 60

1.5 1, 7, 8, 13, 20, 33, 36, 38, 41

1.4 9. $(1-x^2) \frac{dy}{dx} = 2y$

$$\frac{dy}{y} = \frac{2}{1-x^2} dx$$

$$\ln|y| = 2 \cdot \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right| + C \text{ (Integral table \#18)}$$

exponentiate:

$$|y| = e^C \left| \frac{x+1}{x-1} \right| \Rightarrow \boxed{y = \tilde{C} \left(\frac{x+1}{x-1} \right)}$$

20. $\begin{cases} \frac{dy}{y^2+1} = 3x^2 dx \\ y(0)=1 \end{cases}$

$$\tan^{-1} y = x^3 + C$$

tangent (?!):

$$y = \tan(x^3 + C)$$

$$y(0)=1 = \tan(C); C = \pi/4 \text{ works}$$

$$\boxed{y = \tan(x^3 + \pi/4)}$$

22. $\begin{cases} \frac{dy}{dx} = y(4x^3-1) \\ y(1)=-3 \end{cases}$

$$\frac{dy}{y} = (4x^3-1) dx$$

$$\ln|y| = x^4 - x + C$$

$$|y| = e^C e^{x^4-x}$$

$$y = \tilde{C} e^{x^4-x}$$

$$y(1) = -3 = \tilde{C}$$

$$\boxed{y = -3e^{x^4-x}}$$

37. $x(t) = \# \text{ of } ^{238}\text{U atoms}$

$$x(0) = x_0$$

$$y(t) = \# \text{ of Pb atoms}$$

from problem (conservation of atoms) deduce

$$x(t) + y(t) \equiv x_0 \text{ always. (key point)}$$

12. $y \frac{dy}{dx} = x(y^2+1)$

$$\int \frac{y dy}{y^2+1} = \int x dx$$

$$\uparrow$$

$$u = y^2+1$$

$$du = 2y dy$$

$$\frac{1}{2} \int \frac{1}{u} du = \frac{x^2}{2} + C$$

$$\ln|y^2+1| = x^2 + \tilde{C}$$

$$|y^2+1| = e^{\tilde{C}} e^{x^2}$$

$$y^2+1 = \tilde{C} e^{x^2} \Rightarrow y^2 = \tilde{C} e^{x^2} - 1$$

$$\Rightarrow \boxed{y = \pm \sqrt{\tilde{C} e^{x^2} - 1}}$$

37) cont'd.

$\tau = \text{half life} = 4.51 \times 10^9 \text{ years}$
related to k in

$$\begin{cases} \frac{dx}{dt} = -kx \\ x(0) = x_0 \end{cases} \rightarrow \text{soln } x(t) = x_0 e^{-kt}$$

$$\text{so } x(\tau) = \frac{1}{2} x_0 = x_0 e^{-k\tau}$$

$$-\ln 2 = -k\tau \quad k = \frac{\ln 2}{\tau} \quad (\text{eqn 18})$$

$$\text{so } k = \frac{\ln 2}{\tau} = 1.54 \times 10^{-10}$$

(let $T = \text{present}$.)

$$\text{given } \frac{x(T)}{y(T)} = .9 = \frac{x(T)}{x_0 - x(T)}$$

$$.9(x_0 - x(T)) = x(T)$$

$$.9x_0 = 1.9x(T)$$

$$\text{so } \boxed{x(T) = \frac{.9}{1.9} x_0}$$

⑤

36. $a = 5 \text{ mi}$
 $v_0 = 9 \text{ mi/h}$
 $v_s = 3 \text{ mi/h}$

but $v = v_0 \left(1 - \frac{x^4}{a^4}\right)$

so $\frac{dy}{dx} = \frac{v_0}{v_s} \left(1 - \frac{x^4}{a^4}\right)$

$y = \int \frac{dy}{dx} dx = \frac{v_0}{v_s} \left(x - \frac{x^5}{5a^4}\right) + C$

$y(a) = 0 = \frac{v_0}{v_s} \left(-a + \frac{a}{5}\right) + C$
 $= \frac{v_0}{v_s} \left(-\frac{4}{5}a\right) + C; \quad C = \frac{v_0}{v_s} \frac{4}{5}a$

swimmer ends up at $y(a) = \frac{v_0}{v_s} \left(a - \frac{a}{5}\right) + \frac{v_0}{v_s} \left(\frac{4}{5}a\right)$

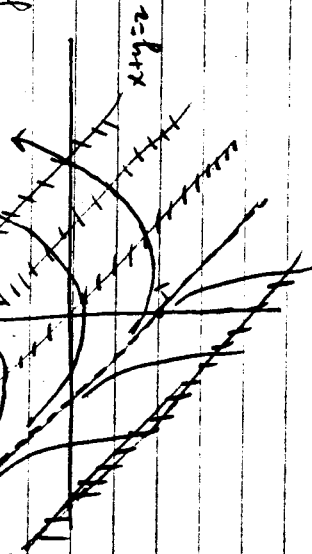
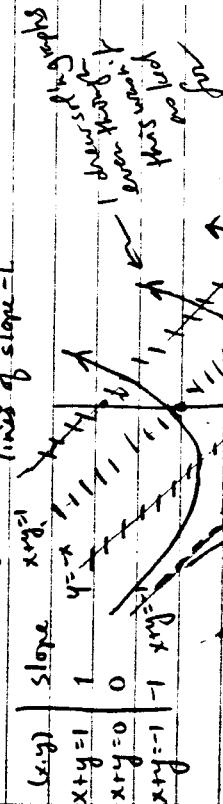
b) 3 b) → see back of book A-28

8) "

12) $\frac{dy}{dx} = x + y$

$= \frac{8}{5} (1.5)(3) = 2.4 \text{ miles downstream.}$

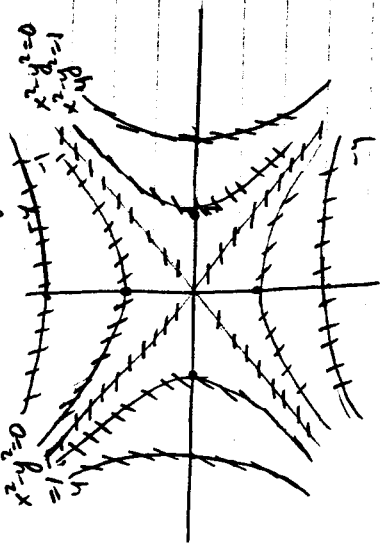
isoclines $x+y = \text{const}$; $y = -x + \text{const}$



⑥

b) 13 #16) $\frac{dy}{dx} = x^2 - y^2$

isoclines $x^2 - y^2 = \text{const}$ hyperbolas



the $x-y$ plane

21) $\left\{ \begin{aligned} \frac{dy}{dx} &= 2x^2y^2 \\ y(1) &= -1 \end{aligned} \right.$

$f(x,y)$ continuous on all of $\mathbb{R}^2$
 so soln exists on some interval containing $x=1$.

$\frac{\partial f}{\partial y} = 4x^2y$ also cont, so soln is unique

24) $\left\{ \begin{aligned} \frac{dy}{dx} &= y^{1/3} \\ y(0) &= 0 \end{aligned} \right.$

$f(x,y) = y^{1/3}$ is cont so $\exists$ soln (there exists)

$\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$ is not continuous along the x -axis

since initial cond (i.e. when $y=0$)

is $y(0)=0$ there is no rectangle with $(0,0)$ in its interior on which $\frac{\partial f}{\partial y}$ is continuous,

so uniqueness is not guaranteed

61.4 #37 cont'd.

(2)

$\ln(x(T)) = x_0 e^{-kT}$ as well, so

$$x_0 e^{-kT} = \frac{.9}{1.9} x_0$$

$$e^{-kT} = \frac{.9}{1.9} \Rightarrow \ln\left(\frac{.9}{1.9}\right) = -kT ; \quad T = -\frac{\ln\left(\frac{.9}{1.9}\right)}{k} = \frac{-\ln\left(\frac{.9}{1.9}\right)}{1.54 \times 10^{-10}} ;$$

$$T = 4.85 \times 10^9 \text{ years}$$

45. Newton's law of cooling

$$\frac{dT}{dt} = k(A - T) = -k(T - A)$$

$$T(0) = 210$$

$$A = 70$$

$$\frac{dT}{T - A} = -k dt$$

$$\ln(T - A) = -kt + C \quad (T > A)$$

$$T - A = e^C e^{-kt}$$

$$T = A + \tilde{C} e^{-kt}$$

$$T(0) = 210 = 70 + \tilde{C} \Rightarrow \tilde{C} = 140$$

$$T = 70 + 140 e^{-kt}$$

(if 1 lux hours for time) $T\left(\frac{1}{2}\right) = 140 = 70 + 140 e^{-k/2} \Rightarrow \frac{1}{2} = e^{-k/2} ; \frac{1}{4} = e^{-k}$

$$T(t) = 70 + 140 \left(\frac{1}{4}\right)^t$$

$$\text{set } T(t) = 100 = 70 + 140 \left(\frac{1}{4}\right)^t$$

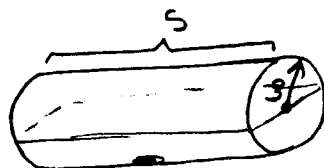
$$30 = 140 \left(\frac{1}{4}\right)^t$$

$$\frac{3}{14} = \left(\frac{1}{4}\right)^t$$

$$t = \frac{\ln(3/14)}{\ln(1/4)} \approx 1.111 \text{ hours}$$

$$\approx 66.67 \text{ minutes}$$

54.



cross-section when depth is y is rectangle of length 5 and width w . From cross-section picture & Pythagorean Thm deduce

$$\frac{1}{2}w = \sqrt{9 - (3-y)^2} = \sqrt{6y - y^2} ; \quad w = 2\sqrt{6y - y^2}$$

$$\text{so } A(y) = 10(6y - y^2)^{1/2}$$

Torricelli: (eqn 24)

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy}$$

$$\text{so } 10(6y - y^2)^{1/2} \frac{dy}{dt} = -\pi \left(\frac{1}{12}\right)^2 8y^{1/2} \quad (a = \pi \left(\frac{1}{12}\right)^2 ; g = 32 \text{ ft/sec}^2)$$

$$\cancel{y^{1/2}} (6-y)^{1/2} \frac{dy}{dt} = \frac{-\pi}{180} \cancel{y^{1/2}}$$

$$\text{so } \begin{cases} (6-y)^{1/2} \frac{dy}{dt} = \frac{-\pi}{180} \\ y(0) = 3 \end{cases}$$

#54 cont'd
sol'n

(3)

$$(6-y)^{1/2} dy = -\frac{\pi}{180} dt$$

integrate:

$$\left(-\frac{2}{3}\right)(6-y)^{3/2} = -\frac{\pi}{180} t + C$$

$$y(0)=3, \text{ so } -\frac{2}{3} 3^{3/2} = C$$

$$-\frac{2}{3} (6-y)^{3/2} = -\frac{\pi}{180} t - \frac{2}{3} 3^{3/2}$$

$$\frac{2}{3} (6-y)^{3/2} = \frac{\pi}{180} t + 2\sqrt{3}$$

now set $y=0$:

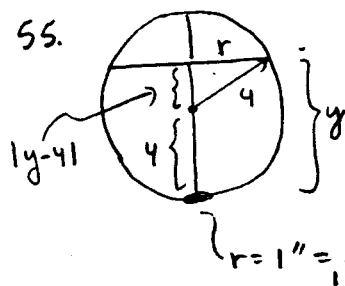
$$\frac{2}{3} (6)^{3/2} = \frac{\pi}{180} t + 2\sqrt{3}$$

$$\frac{2}{3} \cdot 6 \cdot \sqrt{2}\sqrt{3} - 2\sqrt{3} = \frac{\pi}{180} t$$

$$2\sqrt{3} (2\sqrt{2}-1) = \frac{\pi}{180} t$$

$$t = \frac{180}{\pi} (2\sqrt{3})(2\sqrt{2}-1) \approx \underline{\underline{\quad}}$$

55.



$$A(y) = \pi r^2 = \pi (16 - (y-4)^2) = \pi (-y^2 + 8y)$$

$$A(y) \frac{dy}{dt} = -a\sqrt{2gy} = -\frac{\pi}{144} 8y^{1/2} = -\frac{\pi}{18} y^{1/2}$$

$$\text{so } \begin{cases} \pi(-y^2 + 8y) \frac{dy}{dt} = -\frac{\pi}{18} y^{1/2} \\ y(0) = 8 \end{cases}$$

cancel π 's, divide by $y^{1/2}$,
mult by dt

$$-y^{3/2} + 8y^{1/2} dy = -\frac{1}{18} dt$$

$$\text{integrate: } -\frac{2}{5} y^{5/2} + \frac{16}{3} y^{3/2} = -\frac{1}{18} t + C$$

$$y(0)=8 \Rightarrow -\frac{2}{5} 64\sqrt{8} + \frac{16}{3} 8\sqrt{8} = C = (\sqrt{8})(64) \left[-\frac{2}{5} + \frac{2}{3} \right] = \frac{512}{15} \sqrt{2}$$

When $y=0$ get

$$0 = -\frac{1}{18} t + \frac{512}{15} \sqrt{2} \Rightarrow t = \left(\frac{512}{15} \sqrt{2} \right) (18) \approx 869 \text{ secs} \approx 14 \text{ mins } 29 \text{ secs}$$

60 $\frac{dVol}{dt} = \text{const}$ where $Vol(t)$ is amt of snow plowed at time t

a) $t=0$ when snow started. snow falls at const rate so $\text{depth}(t) = k, t$
some k ,

in time dt plow moves dx

$$\text{So volume removed} = dVol = (\text{area of snow cross-section}) dx = c (\text{depth}) dx \quad (c = \text{"width"})$$

$$dVol = (ck, t) dx$$

$$\text{so } \text{const} = \frac{dVol}{dt} = ck, t \frac{dx}{dt} ; \left[\text{const} = ck, t \frac{dx}{dt} \right] \Rightarrow k \frac{dx}{dt} = \frac{1}{t}, \text{ rewrite then}$$

60 cont'd.

$$b) \quad k \frac{dx}{dt} = \frac{1}{t}$$

$$k dx = \frac{1}{t} dt$$

$$kx = \ln t + C \quad (t > 0).$$

Let T correspond to 7:00.

$$x(T) = 0 \Rightarrow 0 = \ln T + C; \quad C = -\ln T.$$

$$kx = \ln t - \ln T$$

$$\boxed{kx = \ln \frac{t}{T}}$$

$$8:00: \quad x(T+1) = 2$$

$$10:00 \quad x(T+3) = 4$$

$$\left. \begin{aligned} 2k &= \ln\left(\frac{T+1}{T}\right) \\ 4k &= \ln\left(\frac{T+3}{T}\right) \end{aligned} \right\}$$

$$2 \ln\left(\frac{T+1}{T}\right) = \ln\left(\frac{T+3}{T}\right)$$

$$\left(\frac{T+1}{T}\right)^2 = \frac{T+3}{T}$$

$$(T+1)^2 = T(T+3)$$

$$T^2 + 2T + 1 = T^2 + 3T$$

$$\boxed{1 = T}$$

so snow started at 6 AM!

9 1.5

$$8. \quad 3xy' + y = 12x$$

$$y' + \frac{1}{3x}y = 4$$

$$e^{\int P(x) dx} = e^{\int \frac{1}{3x} dx} = e^{\frac{1}{3} \ln x} = x^{1/3}$$

$$x^{1/3} [y' + \frac{1}{3x}y] = 4x^{1/3}$$

$$(x^{1/3}y)' = 4x^{1/3}$$

$$x^{1/3}y = 3x^{4/3} + C$$

$$\boxed{y = 3x + Cx^{-1/3}}$$

$$13. \quad \begin{cases} y' + y = e^x \\ y(0) = 1 \end{cases}$$

$$e^{\int P(x) dx} = e^{\int 1 dx} = e^x.$$

$$e^x(y' + y) = e^{2x}$$

$$(e^x y)' = e^{2x}$$

$$e^x y = \int e^{2x} dx = \frac{e^{2x}}{2} + C$$

$$\boxed{y = \frac{e^x}{2} + Ce^{-x}}$$

$$y(0) = 1 = \frac{1}{2} + C; \quad C = \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}(e^x + e^{-x})}$$

$$20. \quad \begin{cases} y' = 1 + x + y + xy \\ y(0) = 0 \end{cases}$$

is separable:

$$y' = (1+x)(1+y)$$

$$\frac{dy}{1+y} = (1+x) dx$$

$$\ln|1+y| = x + \frac{x^2}{2} + C$$

$$y(0) = 0: \quad 0 = C.$$

$$1+y = e^{x+x^2/2}$$

$$\boxed{y = e^{x+x^2/2} - 1}$$

is also linear

$$y' - (1+x)y = 1+x$$

$$(\bar{e}^{x-x^2/2} y)' = \bar{e}^{x-x^2/2} (1+x)$$

$$\bar{e}^{x-x^2/2} y = \int (1+x) \bar{e}^{x-x^2/2} dx = \int e^{-u} du$$

$$u = x + x^2/2$$

$$du = 1+x dx$$

$$= -e^{-x-x^2/2} + C$$

$$y = -1 + Ce^{-x-x^2/2} \quad y(0) = 0 \Rightarrow C = 1;$$

same as boxed sol'n.

33. $\frac{dx}{dt} = r_i c_i - r_o c_o$

$c_i = 0$ (pure water)

$c_o = \frac{x}{1000} \text{ kg/l}$

$r_o = 5 \text{ l/s}$

$\frac{dx}{dt} = -\frac{5}{1000} x = -\frac{1}{200} x$

so $x(t) = x_0 e^{-\frac{1}{200} t}$; $x_0 = 100$

set $10 = x(t) = 100 e^{-\frac{1}{200} t}$

$\frac{1}{10} = e^{-\frac{1}{200} t}$; $t = 200 \ln(10) \approx 460.5 \text{ sec}$
 $\approx 7 \text{ min } 40.5 \text{ sec.}$

38. $\frac{dx}{dt} = r_i c_i - r_o c_o$
 $= 0 - 5 \left(\frac{x}{100} \right)$ since $c_i = 0$ for tank 1

(a) $\begin{cases} \frac{dx}{dt} = -\frac{x}{20} \\ x(0) = 50 \end{cases}$ so $x(t) = 50 e^{-t/20}$

(b) $\frac{dy}{dt} = r_i c_i - r_o c_o = 5 \frac{x}{100} - 5 \frac{y}{200}$
 $\uparrow \quad \uparrow$
 $c_i \quad c_o$

From (a),

$\frac{dy}{dt} = \frac{5}{100} 50 e^{-t/20} - \frac{y}{40}$

$y' + \frac{y}{40} = 2.5 e^{-t/20}$

$(e^{t/40} y)' = e^{t/40} 2.5 e^{-t/20} = 2.5 e^{-t/40}$

$e^{t/40} y = -100 e^{-t/40} + C$

$y = -100 e^{-t/20} + C e^{-t/40}$

$y(0) = 50 = -100 + C$; $C = 150$

$y = -100 e^{-t/20} + 150 e^{-t/40}$

note $y(t) \rightarrow 0$ as $t \rightarrow \infty$

(c) $y(t) \rightarrow 0$ as $t \rightarrow \infty$ so if y has a max at some pos. t , will have $y'(t) = 0$, $y' = -\frac{100}{20} e^{-t/20} - \frac{150}{40} e^{-t/40} = 0$

at this time

$y(t) = -100 \left(\frac{3}{4} \right)^2 + 150 \left(\frac{3}{4} \right)$
 $= \frac{225}{4} = 56.25 \text{ lb}$

$5 e^{-t/20} = \frac{15}{4} e^{-t/40}$; mult by $e^{t/40}$
divide by 5
 $e^{-t/40} = \frac{3}{4}$

41. $S(t) = 30e^{t/20}$ salary at time t (\$/year)

$A(t)$ = amt in retirement account

in time dt years

$$dA = \underbrace{.12 S(t) dt}_{\text{from salary}} + \underbrace{.06 A(t) dt}_{\text{accumulating from retirement account}} \quad (\Delta A \approx \text{the RHS})$$

so $\frac{dA}{dt} = .12 S(t) + .06 A(t)$

$$\begin{cases} A'(t) - .06 A = 3.6 e^{t/20} \\ A(0) = 0 \end{cases}$$

$$e^{-.06t} (A' - .06 A) = e^{-.06t} 3.6 e^{t/20} = 3.6 e^{-.01t}$$

$$(e^{-.06t} A)' = 3.6 e^{-.01t}$$

$$e^{-.06t} A = -360 e^{-.01t} + C$$

$$A(0) = 0 \Rightarrow C = +360$$

$$\text{so } A(t) = 360 (e^{.06t} - e^{.05t})$$

$$\begin{aligned} A(40) &= 360 (e^{2.4} - e^2) \\ &= 360 (e^2) (e^{.4} - 1) \end{aligned}$$

$$A(40) \approx \$1,308,283$$