12) $y''' + y' = 2 - \sin x$

y_H: $r^3 + r = 0$

$r(r^2+1) = 0$

so both 2 & $\sin x$ are y_H's.

so, for

$L(y_p) = 2$

try $y_p = Ax$

$y_p' = A$

$y_p'' = y_p''' = 0$

$L(y_p) = A = 2$

$\boxed{y_p = 2x}$

$L(y_p) = -\sin x$

try $y_p = x(A\cos x + B\sin x)$

get $y_p''' + y_p' = -2A\cos x - 2B\sin x = -\sin x$

(after a lot of work or MAPLE)

$A = 0$

$B = \frac{1}{2}$

$\boxed{y_p = \frac{1}{2} x \sin x}$

by superposition, for #12 $\boxed{y_p = 2x + \frac{1}{2} x \sin x}$

19) $y^{(5)} + 2y^{(3)} + 2y'' = 3x^2 - 1$

y_H: $r^5 + 2r^3 + 2r^2 = 0$

$r^2(r^3 + 2r + 2) = 0$

so 1, x are y_H's

so try

$y_p = x^2[Ax^2 + Bx + C]$

$= Ax^4 + Bx^3 + Cx^2$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx$

$2(\ y_p'' = 12Ax^2 + 6Bx + 2C$

$2(\ y_p''' = 24Ax + 6B$

$0(\ y_p^{(4)} = 24A$

$+1(\ y_p^{(5)} = 0$

$L(y_p) = 24Ax^2 + (12B + 48A)x + (4C + 12B)1 = 3x^2 - 1$

$24A = 3$; $A = 1/8$

$12B + 48A = 0$; $12B + 6 = 0$, $B = -\frac{1}{2}$

$4C + 12B = -1$; $4C - 6 = -1$ $4C = 5$ $C = 5/4$

$\boxed{y_p = \frac{1}{8}x^4 - \frac{1}{2}x^3 + \frac{5}{4}x^2}$

34) $\begin{cases} y'' + y = \cos x \\ y(0) = 1 \\ y'(0) = -1 \end{cases}$

$y_H(x) = c_1 \cos x + c_2 \sin x$

$1(\ y_p = x(A\cos x + B\sin x)$

$0(\ y_p' = x(-A\sin x + B\cos x) + A\cos x + B\sin x$

$+1(\ y_p'' = x(-A\cos x - B\sin x) + 2(-A\sin x + B\cos x)$

$L(y_p) = -2A\sin x + 2B\cos x = \cos x$

$A = 0$, $B = \frac{1}{2}$

$y_p = \frac{1}{2} x \sin x$

$y = \frac{1}{2} x \sin x + c_1 \cos x + c_2 \sin x$

$y(0) = 1 = c_1$

$y'(0) = -1 = c_2$

$\boxed{y = \frac{1}{2} x \sin x + \cos x - \sin x}$

1)

$y^{(3)} - 2y'' + y' = 1 + xe^x$

$\begin{cases} y(0) = y'(0) = 0 \\ y''(0) = 1 \end{cases}$

: $r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r(r-1)^2 = 0.$

$(x) = C_1 + C_2 e^x + C_3 x e^x$

ipe for y_p: to get $L(y_p) = 1$, $s = 1$ try

$y_p = x \cdot (A) = Ax$

$y_p' = A$

$y_p'' = 0 \ldots$

$L(y_p) = A = 1$; $\boxed{y_p = x}$

et $L(y_p) = xe^x$, $s = 2$, try $y_p = x^2(Ax + B)e^x$

$= (Ax^3 + Bx^2)e^x$

$y_p' = e^x[Ax^3 + Bx^2 + 3Ax^2 + 2Bx]$

$= e^x[A(x^3 + 3x^2) + B(x^2 + 2x)]$

$y_p'' = e^x[A(x^3 + 3x^2 + 3x^2 + 6x) + B(x^2 + 2x + 2x + 2)]$

$= e^x[A(x^3 + 6x^2 + 6x) + B(x^2 + 4x + 2)]$

$y_p''' = e^x[A(x^3 + 6x^2 + 6x + 3x^2 + 12x + 6) + B(x^2 + 4x + 2 + 2x + 4)]$

$= e^x[A(x^3 + 9x^2 + 18x + 6) + B(x^2 + 6x + 6)]$

$Ly_p = e^x[x^3(A - 2A + A) + x^2(B + 9A - 2B - 12A + B + \ldots$

over

for $L(y_p) = xe^x$:

$L(y_p) = e^x \begin{bmatrix} x^3(0) \\ + x^2(\cancel{B} + \cancel{9A} - \cancel{2B} - \cancel{12A} + \cancel{B} + \cancel{3A}) \\ + x(\cancel{6B} + 18A - \cancel{8B} - 12A + \cancel{2B}) \\ + 1(6B + 6A - 4B) \end{bmatrix}$

$= xe^x$

deduce:

x: $6A = 1$; $A = 1/6$

1: $2B + 6A = 0$; $2B = -1$, $B = -\frac{1}{2}$

$y_p = \frac{1}{6}x^3e^x + \frac{1}{2}x^2e^x$

so for $L(y) = 1 + xe^x$ use superposition to deduce

$\boxed{y_p = x + \frac{1}{6}x^3e^x - \frac{1}{2}x^2e^x}$

general sol'n.

$y = x + \frac{1}{6}x^3e^x - \frac{1}{2}x^2e^x + c_1 + c_2e^x + c_3xe^x$

$y(0) = 0 = c_1 + c_2$

$y'(0) = 0 = 1 + c_2 + c_3$

$y''(0) = 1 = -1 + c_2 + 2c_3$

xe^x

$e^x(x+1)$

$e^x(x+2)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 2 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$c_3 = 3$

$c_2 = -4$

$c_1 = 4$

$\boxed{y(x) = x + \frac{1}{6}x^3e^x + \frac{1}{2}x^2e^x + 4 - 4e^x + 3xe^x}$!

5

43 a) $\cos 3x + i\sin 3x = e^{i3x} = (e^{ix})^3 = (\cos x + i\sin x)^3$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ (Pascal expansion)

so $(\cos x + i\sin x)^3 = \cos^3 x + 3\cos^2 x(i\sin x) + 3\cos x(i\sin x)^2 + (i\sin x)^3$

$\cos 3x + i\sin 3x = (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x)$

so $\cos 3x = \cos^3 x - 3\cos x\sin^2 x = \cos^3 x - 3\cos x(1 - \cos^2 x)$

$= 4\cos^3 x - 3\cos x$

$\sin 3x = 3\cos^2 x\sin x - \sin^3 x$

$= 3(1 - \sin^2 x)\sin x - \sin^3 x$

$= -4\sin^3 x + 3\sin x$

solving for $\cos^3 x$ & $\sin^3 x$ gives

$\cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$, $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

43b) $y'' + 4y = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$

$y_p = A\cos 3x$

$y_p'' = -9A\cos 3x$

$L(y_p) = A(-9 + 4)\cos 3x$

$= -5A\cos 3x$

$-5A = \frac{1}{4}$

$A = -\frac{1}{20}$

$y_p = A\cos x$

$y_p'' = -A\cos x$

$L(y_p) = A(-1 + 4)\cos x$

$= \frac{3}{4}\cos x$

$3A = \frac{3}{4}$

$A = \frac{1}{12}$

$y_p = -\frac{1}{20}\cos 3x + \frac{1}{4}\cos x$

$\boxed{y = -\frac{1}{20}\cos 3x + \frac{1}{4}\cos x + c_1\cos 2x + c_2\sin 2x}$

y_H: $r^2 + 4 = 0$

$r = \pm 2i$

$y_H = c_1\cos 2x + c_2\sin 2x$

49) $y'' - 4y' + 4y = 2e^{2x}$

y_H: $r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$y_H = \underbrace{c_1 e^{2x}}_{y_1} + \underbrace{c_2 x e^{2x}}_{y_2}$

$y_p = u_1(x) y_1 + u_2(x) y_2$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2x} \end{bmatrix}$$

$$\begin{bmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2x} \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & xe^{2x} \\ 2e^{2x} & e^{2x}(1+2x) \end{vmatrix}}{e^{4x}[1+2x-2x]} = \frac{-2xe^{4x}}{e^{4x}} = -2x$$

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{2x} \end{vmatrix}}{e^{4x}} = \frac{2e^{4x}}{e^{4x}} = 2$$

$u_1 = -x^2$, $u_2 = 2x$

$y = -x^2(e^{2x}) + 2x(xe^{2x}) = \boxed{x^2 e^{2x}}$

51) $y'' + 4y = \cos 3x$

$y_H = c_1 \cos 2x + c_2 \sin 2x$

$\uparrow y_1 \quad \uparrow y_2$

(oops, wanted # 52)

$y_p = u_1 y_1 + u_2 y_2$

$$\begin{bmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \cos 3x \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin 2x \\ \cos 3x & 2\cos 2x \end{vmatrix}}{2} = \frac{-\sin 2x \cos 3x}{2}; \quad u_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \cos 3x \end{vmatrix}}{2} = \frac{\cos 2x \cos 3x}{2}$$

$u_1 = \int \frac{-\sin 2x \cos 3x}{2} = \frac{1}{20}\cos 5x - \frac{1}{4}\cos(x)$ (Maple)

$u_2 = \int \frac{\cos 2x \cos 3x}{2} = \frac{1}{4}\sin x + \frac{1}{20}\sin(5x)$ (Maple)

$y_p = \left[\frac{1}{20}\cos 5x - \frac{1}{4}\cos x\right]\cos 2x + \left[\frac{1}{4}\sin x + \frac{1}{20}\sin 5x\right]\sin 2x$

$= \frac{1}{20}\left[\underbrace{\cos 5x \cos 2x + \sin 5x \sin 2x}_{\cos(5x-2x)}\right] + \frac{1}{4}\left[\underbrace{-\cos 2x \cos x + \sin 2x \sin x}_{-\cos(2x+x)}\right.$

$y_p = \frac{1}{20}\cos 3x - \frac{1}{4}\cos 3x$

$\boxed{y_p = -\frac{1}{5}\cos 3x}$

Math 2250-4
HW due Monday Nov 5
§5.6 (3, 5, 7, 14, 16, 21, 22)

3. $\begin{cases} x'' + 100x = 15\cos 5t + 20\sin 5t \\ x(0) = 25 \\ x'(0) = 0. \end{cases}$

x_H: $c_1\cos 10t + c_2 \sin 10t \qquad \left(\omega_0 = \sqrt{\tfrac{k}{m}} = 10\right)$

$$
\begin{aligned}
100\,(\; & x_p = A\cos 5t + B\sin 5t \\
+0\,(\; & x_p' = -5A\sin 5t + 5B\cos 5t \\
+1\,(\; & x_p'' = -25A\cos 5t - 25B\sin 5t
\end{aligned}
$$

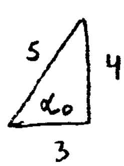
$$
\begin{aligned}
Lx_p &= \cos 5t\,(100A - 25A) &&= \cos 5t\,(15) \qquad & 75A = 15;\ A = \tfrac{1}{5} \\
&\quad + \sin 5t\,(100B - 25B) && \quad + \sin 5t\,(20) & 75B = 20;\ B = \tfrac{4}{15}
\end{aligned}
$$

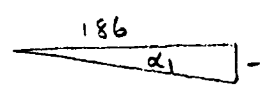
$x(t) = \frac{1}{5}\cos 5t + \frac{4}{15}\sin 5t + c_1\cos 10t + c_2 \sin 10t$

$x(0) = 25 = \frac{1}{5} + c_1 \qquad c_1 = 24\tfrac{4}{5} = \frac{124}{5}$

$x'(0) = 0 = \frac{20}{15} + 10c_2 \qquad c_2 = -\frac{2}{15}$

$$
\begin{aligned}
x(t) &= \tfrac{1}{5}\cos 5t + \tfrac{4}{15}\sin 5t + \tfrac{124}{5}\cos 10t - \tfrac{2}{15}\sin 10t \\
&= \tfrac{1}{15}\left(3\cos 5t + 4\sin 5t\right) + \tfrac{2}{15}\left(186\cos(10t) - \sin 10t\right)
\end{aligned}
$$





$$
\boxed{
\begin{aligned}
x(t) &= \tfrac{1}{3}\cos(5t - \alpha_0) + \tfrac{2}{15}\sqrt{(186)^2 + 1}\,\cos(10t - \alpha_1) \\
\alpha_0 &= \tan^{-1}\left(\tfrac{4}{3}\right) \approx .927 \text{ rad} \qquad \alpha_1 = \tan^{-1}\left(-\tfrac{1}{186}\right) \approx -.0054 \text{ rad}
\end{aligned}
}
$$

5. $\begin{cases} mx'' + kx = F_0\cos\omega t \qquad \omega \neq \omega_0;\ \omega_0 = \sqrt{k/m} \\ x(0) = x_0 \\ x'(0) = 0 \end{cases}$

We (and book p.340), found $x_p = \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t$,

so $x = x_p + x_H$

$x = \dfrac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t$

$x(0) = x_0 = \dfrac{F_0/m}{\omega_0^2 - \omega^2} + c_1 \ ; \ c_1 = x_0 + \dfrac{F_0/m}{\omega^2 - \omega_0^2}$

$x'(0) = 0 = \omega c$ ...

so

$$
\boxed{x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2}\cos\omega t + \left(x_0 + \frac{F_0/m}{\omega^2 - \omega_0^2}\right)\cos\omega_0 t}
$$

7. $x'' + 4x' + 4x = 10\cos 3t$

$x_{sp} = A\cos 3t + B\sin 3t$;

we could use the results of derivation p.346, but it is good practice to work it out:

$$
\begin{aligned}
4(\ \ & x_p = A\cos 3t + B\sin 3t \\
+4(\ \ & x_p' = -3A\sin 3t + 3B\cos 3t \\
+1(\ \ & x_p'' = -9A\cos 3t - 9B\sin 3t
\end{aligned}
$$

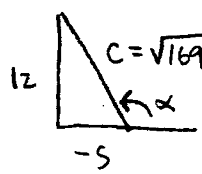
$$
\begin{aligned}
L(x_p) &= \cos 3t\,[4A + 12B - 9A] &= \cos 3t\,[10] \\
&+ \sin 3t\,[4B - 12A - 9B] &+ \sin 3t\,[0]
\end{aligned}
$$

$$
\begin{aligned}
-5A + 12B &= 10 \\
-12A - 5B &= 0
\end{aligned}
$$

$$\begin{bmatrix} -5 & 12 \\ -12 & -5 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{25+144}\begin{bmatrix} -5 & -12 \\ 12 & -5 \end{bmatrix}\begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-50}{169} \\ \frac{120}{169} \end{bmatrix} = \frac{10}{169}\begin{bmatrix} -5 \\ 12 \end{bmatrix}$$

$$x_{sp}(t) = \frac{10}{169}\left(-5\cos 3t + 12\sin 3t\right)$$



$$= \frac{10}{169}\, 13\cos(3t - \alpha) \qquad \alpha = \cos^{-1}\left(-5/13\right)$$

$$\boxed{x_{sp}(t) = \frac{10}{13}\cos(3t - \alpha)\ ,\quad \alpha = \cos^{-1}(-5/13)}$$

$$\boxed{\alpha \approx 1.966 \text{ rad}}$$

14. $\begin{cases} x'' + 8x' + 25x = 5\cos t + 13\sin t \\ x(0) = 5 \\ x'(0) = 0 \end{cases}$

$$
\begin{aligned}
25(\ \ & x_p = A\cos t + B\sin t \\
+8(\ \ & x_p' = -A\sin t + B\cos t \\
1(\ \ & x_p'' = -A\cos t - B\sin t
\end{aligned}
$$

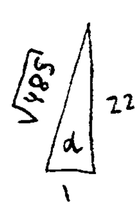
$$
\begin{aligned}
L(x_p) &= \cos t\,[25A + 8B - A] &= \cos t\,[5] \\
&+ \sin t\,[25B - 8A - B] &+ \sin t\,[13]
\end{aligned}
$$

$$
\begin{aligned}
24A + 8B &= 5 \\
-8A + 24B &= 13
\end{aligned}
$$

$$\begin{bmatrix} 24 & 8 \\ -8 & 24 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{4\cdot 144 + 64}\begin{bmatrix} 24 & -8 \\ 8 & 24 \end{bmatrix}\begin{bmatrix} 5 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{640}\begin{bmatrix} 16 \\ 352 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 1 \\ 22 \end{bmatrix}$$



$$x_{sp}(t) = \frac{1}{40}\left(\cos t + 22\sin t\right)$$

$$x_{sp} = \frac{\sqrt{485}}{40}\cos(t - \alpha)\ ; \quad \alpha = \tan^{-1}(22) \approx 1.525 \text{ rad}$$

14 cont'd.

$$x_H:\ e^{rt}\ :\ p(r) = r^2 + 8r + 25 = 0$$
$$= (r+4)^2 + 9$$
$$= (r+4+3i)(r+4-3i)$$
$$r = -4 \pm 3i$$

$$x_H(t) = e^{-4t}\left(c_1 \cos 3t + c_2 \sin 3t\right)$$

$$\boxed{x(t) = \tfrac{1}{40}(\cos t + 22\sin t) + e^{-4t}\left(c_1 \cos 3t + c_2 \sin 3t\right)}$$

you could get this far very easily in Maple:

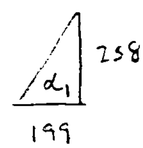
```
> deqtn := diff(x(t),t,t) + 8*diff(x(t),t)
        + 25*x(t) =
        5*cos(t) + 13*sin(t);
> ics := x(0)=5, D(x)(0)=0;
> dsolve({deqtn,ics}, x(t),
        method=laplace);
```

$$x(0) = 5 = \tfrac{1}{40} + c_1\ ; \quad c_1 = 4\tfrac{39}{40} = \boxed{\tfrac{199}{40}}$$

$$x'(0) = 0 = \tfrac{22}{40} - 4c_1 + 3c_2\ ; \quad 3c_2 = 4c_1 - \tfrac{22}{40} = \frac{796-22}{40} = \frac{774}{40}\ ; \quad \boxed{c_2 = \tfrac{258}{40}}$$

so

$$x(t) = \frac{\sqrt{485}}{40}\cos(t-\alpha_0) + \tfrac{1}{40}e^{-4t}\left[199\cos 3t + 258\sin 3t\right]$$



$$= \frac{\sqrt{485}}{40}\cos(t-\alpha_0) + \frac{\sqrt{106165}}{40}e^{-4t}\cos(3t-\alpha_1) \qquad \alpha_1 = \arctan\left(\tfrac{258}{199}\right)$$

$$\boxed{x(t) \approx .5506\cos(t-1.525) + 8.1457e^{-4t}\cos(3t-.9138)}$$

16. $mx'' + cx' + kx = 10\cos\omega t$

$m=1,\ c=4,\ k=5,\ F_0=10$

eqtn (21) p. 346:

$$C(\omega) = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}} = \frac{10}{\sqrt{(5-\omega^2)^2+16\omega^2}}$$

practical resonance looks unlikely, since denom doesn't get very small.

any local max? $C'(\omega)=0$

$$= \underbrace{(10)\left(-\tfrac{1}{2}\right)}_{\neq 0}\underbrace{\left((5-\omega^2)^2+16\omega^2\right)^{-3/2}}_{\neq 0}\underbrace{\left(2(5-\omega^2)(-2\omega)+32\omega\right)}_{=0\,?}$$

$\omega\left[-4(5-\omega^2)+32\right]$

$\omega\left[12+4\omega^2\right]$ nope

shows $C'(\omega)<0$ for $\omega>0$

no practical resonance.

Let x = horiz. displacement (to right)

21. $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 + mgh$

$x = L\sin\theta$
$v = L\theta'$

$h = L(1-\cos\theta)$

$E = \frac{1}{2}kL^2\sin^2\theta + \frac{1}{2}m(L\theta')^2 + mgL(1-\cos\theta)$

$0 \equiv E'(t) = kL^2\sin\theta\cos\theta + mL^2\theta'\theta'' + mgL\sin\theta\,\theta'$

$= L\theta'\left[kL\sin\theta\cos\theta + mL\theta'' + mg\sin\theta\right]$

$\cong$ (linearize)

$L\theta'\left[kL\theta + mL\theta'' + mg\theta\right]$

deduce $mL\theta'' + (mg+kL)\theta = 0$

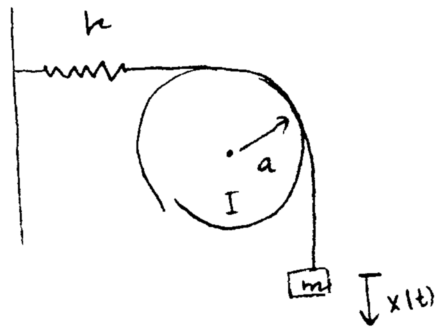
$\theta'' + \left(\frac{mg+kL}{mL}\right)\theta = 0$

$\theta'' + \left(\frac{g}{L}+\frac{k}{m}\right)\theta = 0$

$$\boxed{\omega_0 = \sqrt{\frac{g}{L}+\frac{k}{m}}}$$

(5)

22.



$$E = \tfrac{1}{2}kx^2 + mgh + \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

$$x' = v = a\omega$$
$$h = -x$$

$$E = \tfrac{1}{2}kx^2 - mgx + \tfrac{1}{2}m(x')^2 + \tfrac{1}{2}I\left(\tfrac{x'}{a}\right)^2$$
$$0 \equiv E' = kxx' - mgx' + mx'x'' + I/a^2\, x'x''$$
$$= x'\left[kx - mg + mx'' + I/a^2\, x''\right]$$

deduce $(m + I/a^2)\,x'' + kx = mg$

So, for x_H, $\boxed{\omega_0^2 = \dfrac{k}{m + I/a^2}}$