8.2 Confidence limits

MATHEMATICAL TECHNIQUES

♠ In the following situations, find the probability of a result as or more extreme than the actual result for the given value of the parameter.

- EXERCISE **8.2.1**

  A coin is flipped 5 times and comes up heads every time. Does the value $p = 0.5$ for the probability of heads lie within the 95% confidence limits?

- EXERCISE **8.2.2**

  A coin is flipped 7 times and comes up heads 6 out of 7 times. Does the value $p = 0.5$ for the probability of heads lie within the 95% confidence limits?

- EXERCISE **8.2.3**

  A person wins the lottery the second time he plays. Does the value $q = 0.001$ for the probability of success lie within the 99% confidence limits?

- EXERCISE **8.2.4**

  A person wins the lottery the fifth time he plays. Does the value $q = 0.001$ for the probability of success lie within the 99% confidence limits?

- EXERCISE **8.2.5**

  One cosmic ray hits a detector in one year. Does the value $\lambda = 5$ for the rate at which rays hit lie within the 98% confidence limits?

- EXERCISE **8.2.6**

  Three cosmic rays hit a larger detector in one year. Does the value $\lambda = 10$ for the rate at which rays hit lie within the 98% confidence limits?

♠ Find the exact confidence limits in the following cases.

- EXERCISE **8.2.7**

  A coin is flipped 5 times and comes up heads every time. Find the upper and lower 95% confidence limits. Show that your result is consistent with exercise 8.2.1.

- EXERCISE **8.2.8**

  A coin is flipped 7 times and comes up heads 6 out of 7 times. Does the value $p = 0.5$ for the probability of heads lie within the 95% confidence limits? Show that your result is consistent with exercise 8.2.2.

- EXERCISE **8.2.9**

  A person wins the lottery the second time he plays. Find the 99% confidence limits for the probability q of winning. Show that your result is consistent with exercise 8.2.3.

- EXERCISE **8.2.10**

  One cosmic ray hits a detector in one year. Find the 98% confidence limits for the rate λ at which rays hit. Show that your result is consistent with exercise 8.2.5.

♠ Find the approximate 95% confidence limits using the method of support in the following cases.

- EXERCISE **8.2.11**

  A coin is flipped 5 times and comes up heads every time. Find the approximate upper and lower 95% confidence limits and compare with exercise 8.2.7.

- EXERCISE **8.2.12**

  A coin is flipped 7 times and comes up heads 6 out of 7 times. Find the approximate upper and lower 95% confidence limits and compare with exercise 8.2.8.

- EXERCISE **8.2.13**

  A person wins the lottery the second time he plays. Find the approximate upper and lower 95% confidence limits and compare with exercise 8.2.9 (recall that the earlier exercise found 99% confidence limits).

- EXERCISE **8.2.14**

  One cosmic ray hits a detector in one year. Find the 98% confidence limits for the rate λ at which rays hit. Show that your result is consistent with exercise 8.2.5.

♠ Explain how you would use the Monte Carlo method to estimate confidence limits in the following cases.

• EXERCISE **8.2.15**

A coin is flipped 5 times and comes up heads every time. Describe how the Monte Carlo method could be used to find the approximate upper and lower 95% confidence limits.

• EXERCISE **8.2.16**

A coin is flipped 7 times and comes up heads 6 out of 7 times. Describe how the Monte Carlo method could be used to find the approximate upper and lower 95% confidence limits.

• EXERCISE **8.2.17**

A person wins the lottery the second time he plays. Describe how the Monte Carlo method could be used to find the approximate upper and lower 99% confidence limits.

• EXERCISE **8.2.18**

One cosmic ray hits a detector in one year. Describe how the Monte Carlo method could be used to find the approximate upper and lower 98% confidence limits.

♠ Check whether the expected value of p (1/2 for a fair coin and 1/6 for a fair die) lies within the approximate 95% confidence limits given by the method of support.

• EXERCISE **8.2.19**

Flipping 2 out of 4 heads with a fair coin (as in exercise 8.1.7).

• EXERCISE **8.2.20**

Rolling 2 out of 4 6's with a fair die (as in exercise 8.1.8).

• EXERCISE **8.2.21**

Flipping 2 out of 12 heads with a fair coin (as in exercise 8.1.9).

• EXERCISE **8.2.22**

Rolling 2 out of 12 6's with a fair die (as in exercise 8.1.10).

♠ Check whether the given value of Λ lies within the approximate 95% confidence limits given by the method of support.

• EXERCISE **8.2.23**

20 events occur in one minute, with a given value of $\Lambda = 10.0$ (as in exercise 8.1.15).

• EXERCISE **8.2.24**

10 high energy cosmic rays hit in an expensive detector over the course of 1 year, with a given value of $\Lambda = 8.0$ (as in exercise 8.1.16).

♠ Prove that the support takes on its maximum where the likelihood does.

• EXERCISE **8.2.25**

Using the likelihood function in exercise 8.1.7.

• EXERCISE **8.2.26**

Using the likelihood function in exercise 8.1.15.

• EXERCISE **8.2.27**

Using the likelihood function in exercise 8.1.17.

• EXERCISE **8.2.28**

For a general likelihood function.

APPLICATIONS

♠ Consider a tiny data set where one out of two people is found with a gene.

• EXERCISE **8.2.29**

Find the exact 95% confidence limits.

• EXERCISE **8.2.30**

Find the exact 99% confidence limits.

• EXERCISE **8.2.31**

How would you use the Monte Carlo method to estimate the 99% confidence limits?

• EXERCISE **8.2.32**

Use the method of support to estimate 95% confidence limits in and compare your results.

♠ Consider a data set where three out of three people are found with a gene.

• EXERCISE **8.2.33**
Find the exact 95% confidence limits.
• EXERCISE **8.2.34**
Find the exact 99% confidence limits.
• EXERCISE **8.2.35**
Why is the upper confidence limit strange?
• EXERCISE **8.2.36**
Use the method of support to estimate 95% confidence limits in and compare your results.

♠ A person places an advertisement to sell his car in the newspaper and settles down to await calls, which he expects will arrive with an exponential distribution. The first call arrives in 20 minutes.

• EXERCISE **8.2.37**
Find the maximum likelihood estimator of the rate.
• EXERCISE **8.2.38**
Find the 95% confidence limits.
• EXERCISE **8.2.39**
Find the 98% confidence limits.
• EXERCISE **8.2.40**
How many calls might he expect to miss if he went out for a two hour hike?

♠ 14 mutations are counted in one million base pairs.

• EXERCISE **8.2.41**
Write the equations for the 95% confidence limits, and solve them if you have a computer.
• EXERCISE **8.2.42**
Write the equations for the approximate 95% confidence limits using the method of support, and solve them if you have a computer.

♠ Recall the couple that has 7 boys before having a girl.

• EXERCISE **8.2.43**
Find 95% confidence limits around the maximum likelihood estimate of q. How do you interpret these results?
• EXERCISE **8.2.44**
Find 99% confidence limits around the maximum likelihood estimate of q. How do you interpret these results?

Chapter 9

Answers

8.2.1. The probability of a result as or more extreme is the probability of 5 heads (a more extreme result is not possible), or $b(5;5,0.5) = 0.5^5 = 0.03125$. This is greater than 0.025, so $p = 0.5$ lies within the 95% confidence limits.

8.2.3. The probability of a result as or more extreme is the probability of winning the first or second time, or $g_1 + g_2 = q + q(1-q) = 0.0002$. This is less than 0.005, so lies outside the 99% confidence limits.

8.2.5. The probability of a result as or more extreme is the probability of zero or one cosmic rays. With the rate $\lambda = 5$ the number of rays that hits follows a Poisson distribution with $\Lambda = 5$. The probability is $p(0;5) + p(1;5) = e^{-5} + 5e^{-5} = 0.04$. This is greater than 0.01, so lies within the 98% confidence limits.

8.2.7. Because all were successes, the upper confidence limit is $p_h = 1$. With values of p less than 1, the probability of a result as or more extreme is the probability of 5 heads (a more extreme result is not possible), or $b(5;5,p)$ To find the lower confidence limit, we must solve $b(5;5,p_l) = p_l^5 = 0.025$. This has solution $p_l = 0.478$. As we found early, the value $p = 0.5$ does lie within the 95% confidence limits.

8.2.9. The maximum likelihood estimator of q is 0.5. For values of q less than 0.5, the probability of a result as or more extreme is the probability of winning the first or second time, or $g_1 + g_2 = q + q(1-q)$. This is equal to 0.005 when $q + q(1-q) = 0.005$ which has solution $q = 0.0025$. For values of q greater than 0.5, the probability of a result as a more extreme is winning after the first time, or $1 - q$, which is equal to 0.005 when $q = 0.995$. The value $q = 0.001$ is less than 0.0025, and thus lies outside the 99% confidence limits.

8.2.11. The likelihood function is $L(p) = p^5$, so the support is $S(p) = 5\ln(p)$. The maximum occurs at $p = 1$, so the maximum support is 0. The support takes on a value 2 less than this when $S(p) = -2$, or $5\ln(p) = -2$, or 0.67. This value is rather larger than the lower confidence limit found in exercise 8.2.7.

8.2.13. The maximum likelihood estimator of q is 0.5, with likelihood function $L(q) = q(1-q)$ and support $S(q) = \ln(q(1-q))$. At $q = 0.5$, the support is -1.386. The support takes on a value 2 less than this when $S(p) = -3.386$, or $q(1-q) = 0.0338$. Solving with the quadratic formula gives confidence limits of 0.035 and 0.965. These are narrower than the 99% confidence limits found earlier.

8.2.15. I would test computer coins with different values of the parameter p to find the smallest value of p that produced 5 heads in five tries in more than 2.5% of my trials. For example, I could run a series of tests where I repeated flipping 5 computer coins 1000 times, with $p = 1.0$, $p = 0.95$, $p = 0.9$, and so forth, and waited for the last value of p that produced more than 25 trials out of 1000 with all heads.

8.2.17. I would test computer coins with different values of the parameter q to find the smallest value of q that produced a win on the first or second try with probability greater than 0.005 (to find the lower confidence limit) and the largest value of q that produced that produced a win on the second try or after with probability greater than 0.005 (to find the upper confidence limit).

8.2.19. The maximum likelihood estimator of p is 0.5. This has to lie within any confidence limits. We could also check that $L(p) = b(2;4,p) = 6p^2(1-p)^2$, $S(p) = \ln(6p^2(1-p)^2)$, so $S(0.5) = -0.98$, and note that the support for the value $p = 0.5$ is certainly within 2 of this maximum.

8.2.21. $L(p) = b(2;12,p) = 66p^2(1-p)^{10}$, with maximum at $p = 1/6$. The support is $S(p) = \ln(66p^2(1-p)^{10})$. Then $S(1/6) = -1.217$ and $S(0.5) = -4.128$, which is more than 2 less. Thus $p = 0.5$ lies outside the approximate 95% confidence limits.

8.2.23. We found that $L(20.0) = 0.089$ and $L(10.0) = 0.0058$. Then $S(20.0) = \ln(0.089) = -2.419$ and $S(10.0) = \ln(0.0058) = -5.149$. These differ by more than 2, so the value $\Lambda = 10$ lies outside the approximate 95% confidence limits.
8.2.25. $L(p) = 6p^2(1-p)^2$, so $S(p) = \ln(6) + 2\ln(p) + 2\ln(1-p)$ (using various laws of logs). Then $S'(p) = \frac{2}{p} - \frac{2}{1-p}$ which is equal to 0 at $p = 1/2$. Furthermore, $S''(p) = \frac{-2}{p^2} - \frac{2}{(1-p)^2} < 0$, hence this is a maximum. This matches the location of the maximum of the likelihood function itself.
8.2.27. $L(\Lambda) = \frac{e^{-3\Lambda}\Lambda^{57}}{20!16!21!}$, so $S(\Lambda) = 57\ln(\Lambda) - 3\Lambda - \ln(20!16!21!)$ (using various laws of logs). Then $S'(\Lambda) = \frac{57}{\Lambda} - 3$ which is equal to 0 at $\Lambda = 19$. Furthermore, $S''(\Lambda) = \frac{-57}{\Lambda^2} < 0$, hence this is a maximum. This matches the location of the maximum of the likelihood function itself.
8.2.29. The probability of one or more out of two is equal to $2p(1-p) + p^2$. The lower confidence limit satisfies

$$2p(1-p) + p^2 = 0.025,$$

which has solution $p_l = 0.0126$ (by the quadratic formula). Similarly,

$$\text{the probability of one or fewer out of two } = 2p(1-p) + (1-p)^2.$$

The upper confidence limit satisfies

$$2p(1-p) + (1-p)^2 = 0.025,$$

which has solution $p_h = 0.987$.
8.2.31. To find p_l, I would try the experiment 1000 times with small values of p, seeking one so small that a result of one or more out of two occurred only ten times. To find p_h, I would try the experiment ten times with large values of p, seeking one so large that a result of one or fewer out of two occurred only ten times.
8.2.33. The probability of three or more out of three $= p^3$. The lower confidence limit solves $p^3 = 0.025$, which has solution $p_l = 0.292$. Furthermore,

$$\text{the probability of three or fewer out of three } = 1.$$

This is always true, so the upper confidence limit is $p_h = 1$.
8.2.35. The maximum likelihood estimator is the largest possible value.
8.2.37. The likelihood function is $L(\lambda) = \lambda e^{-20\lambda}$, which has maximum at $\lambda = 0.05$.
8.2.39. The lower confidence limit λ_l satisfies

$$\begin{aligned} \Pr(\text{wait is less than or equal to 20 if } \lambda = \lambda_l) &= 0.01 \\ 1 - e^{-20\lambda_l} &= 0.01 \\ \lambda_l &= \frac{\ln(0.99)}{-20} = 0.0005. \end{aligned}$$

The upper confidence limit λ_h satisfies

$$\begin{aligned} \Pr(\text{wait is greater than or equal to 20 if } \lambda = \lambda_h) &= 0.01 \\ e^{-20\lambda_h} &= 0.01 \\ \lambda_h &= \frac{\ln(0.01)}{-20} = 0.230. \end{aligned}$$

8.2.41. The confidence limits Λ_l and Λ_h satisfy

$$\begin{aligned} \Pr(\text{14 or more}) &= \sum_{k=14}^{\infty} p(k; \Lambda_l) = 0.025 \\ \Pr(\text{14 or fewer}) &= \sum_{k=0}^{14} p(k; \Lambda_h) = 0.025 \end{aligned}$$

By substituting various values into the computer, we find $\Lambda_l = 7.65$ and $\Lambda_h = 23.5$.

8.2.43. If q is less than $q = 0.125$, the wait for a girl would be greater than 8 on average. The lower confidence limit q_l satisfies

$$\begin{aligned} \text{Pr(wait is less than or equal to 8 if } q = q_l) &= 0.025 \\ 1 - (1 - q_l)^7 &= 0.025 \\ q_l &= 1 - 0.975^{1/7} = 0.0036. \end{aligned}$$

The upper confidence limit q_h satisfies

$$\begin{aligned} \text{Pr(wait is greater than or equal to 8 if } q = q_h) &= 0.025 \\ (1 - q_h)^7 &= 0.025 \\ q_h &= 1 - 0.025^{1/7} = 0.410. \end{aligned}$$

The normal probability $q = 0.5$ seems to lie outside these confidence limits. Perhaps the father does have some genetic abnormality.