

## 6.7 Displaying Probabilities

### MATHEMATICAL TECHNIQUES

- ♠ Draw histograms describing the probabilities of the outcomes of 4 experiments to count the number of mutants in a bacterial cultural.

Number of mutants	Probability			
	Experiment a	Experiment b	Experiment c	Experiment d
0	0.1	0.6	0.3	0.1
1	0.2	0.3	0.2	0.3
2	0.3	0.1	0.2	0.1
3	0.3	0.0	0.2	0.4
4	0.1	0.0	0.1	0.1

• EXERCISE 6.7.1

Experiment a

• EXERCISE 6.7.2

Experiment b

• EXERCISE 6.7.3

Experiment c

• EXERCISE 6.7.4

Experiment d

- ♠ Sketch the cumulative distribution associated with the histogram from the earlier problem.

• EXERCISE 6.7.5

The histogram in exercise 6.7.1.

• EXERCISE 6.7.6

The histogram in exercise 6.7.2.

• EXERCISE 6.7.7

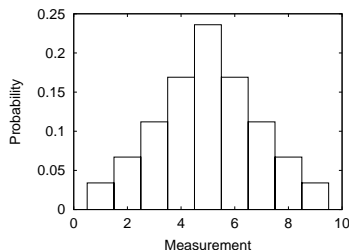
The histogram in exercise 6.7.3.

• EXERCISE 6.7.8

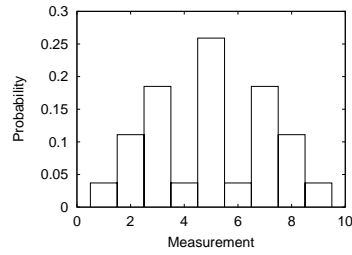
The histogram in exercise 6.7.4.

- ♠ On each histogram, find the most and least likely simple events. Is the histogram symmetric?

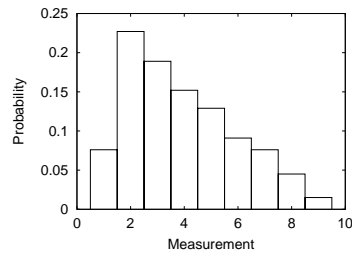
• EXERCISE 6.7.9



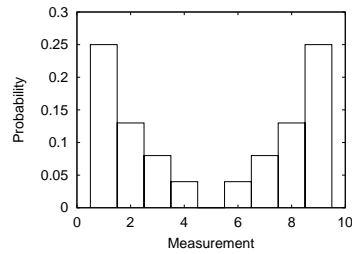
• EXERCISE 6.7.10



• EXERCISE 6.7.11



• EXERCISE 6.7.12



♠ Using the histogram indicated, estimate the probabilities of the following events.

- The measurement is equal to 7.
- The measurement is less than or equal to 4.
- The measurement is greater than 4.

• EXERCISE 6.7.13

The histogram in exercise 6.7.9.

• EXERCISE 6.7.14

The histogram in exercise 6.7.10.

• EXERCISE 6.7.15

The histogram in exercise 6.7.11.

• EXERCISE 6.7.16

The histogram in exercise 6.7.12.

♠ For each of the following p.d.f's,

- Check that the area under the curve is exactly 1,
- Sketch a graph,
- Indicate the maximum of the p.d.f., and say why you are not worried that it is sometimes greater than 1.

## • EXERCISE 6.7.17

The probability density function is  $f(x) = 2x$  for  $0 \leq x \leq 1$ .

## • EXERCISE 6.7.18

The probability density function is  $f(x) = 1 - \frac{x}{2}$  for  $0 \leq x \leq 2$ .

## • EXERCISE 6.7.19

The probability density function is  $h(t) = \frac{1}{t}$  for  $1 \leq t \leq e$ .

## • EXERCISE 6.7.20

The probability density function is  $g(t) = 6t(1 - t)$  for  $0 \leq t \leq 1$ .

♠ Find and sketch the c.d.f. associated with the given p.d.f. and check that it increases to a value of 1.

## • EXERCISE 6.7.21

The probability density function is  $f(x) = 2x$  for  $0 \leq x \leq 1$  (as in exercise 6.7.17).

## • EXERCISE 6.7.22

The probability density function is  $f(x) = 1 - \frac{x}{2}$  for  $0 \leq x \leq 2$  (as in exercise 6.7.18).

## • EXERCISE 6.7.23

The probability density function is  $h(t) = \frac{1}{t}$  for  $1 \leq t \leq e$  (as in exercise 6.7.19).

## • EXERCISE 6.7.24

The probability density function is  $g(t) = 6t(1 - t)$  for  $0 \leq t \leq 1$  (as in exercise 6.7.20).

♠ Find the probability in two ways,

a. By integrating the given p.d.f.,

b. By using the c.d.f.

and make sure that your answers match. Shade the given areas on a graph of the p.d.f.

## • EXERCISE 6.7.25

The probability density function is  $f(x) = 2x$  for  $0 \leq x \leq 1$  (as in exercises 6.7.17 and 6.7.21). Find the probability that the measurement is between 0.2 and 0.6.

## • EXERCISE 6.7.26

The probability density function is  $f(x) = 1 - \frac{x}{2}$  for  $0 \leq x \leq 2$  (as in exercises 6.7.18 and 6.7.22). Find the probability that the measurement is between 1.0 and 1.5.

## • EXERCISE 6.7.27

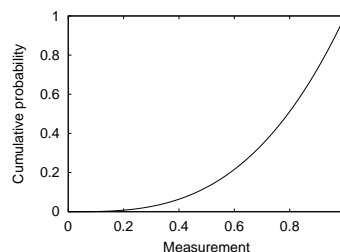
The probability density function is  $h(t) = \frac{1}{t}$  for  $1 \leq t \leq e$  (as in exercises 6.7.19 and 6.7.23). Find the probability that the measurement is between 2.0 and 2.5.

## • EXERCISE 6.7.28

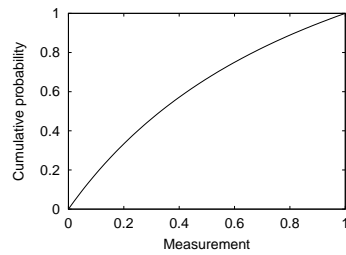
The probability density function is  $g(t) = 6t(1 - t)$  for  $0 \leq t \leq 1$  (as in exercises 6.7.20 and 6.7.24). Find the probability that the measurement is between 0.5 and 0.8.

♠ Sketch the p.d.f. associated with each of the following cumulative distribution functions.

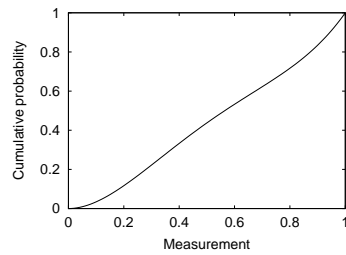
## • EXERCISE 6.7.29



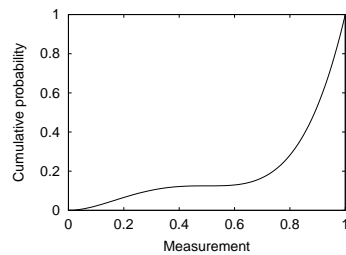
## • EXERCISE 6.7.30



• EXERCISE 6.7.31



• EXERCISE 6.7.32



## APPLICATIONS

- ♠ Draw histograms of the distributions of cell age from the assumptions in the earlier problem. Find and graph the cumulative distribution.

• EXERCISE 6.7.33

The cells in exercise 6.5.31, where

$$\Pr(\text{cell is 0 day old}) = 0.4$$

$$\Pr(\text{cell is 1 day old}) = 0.3$$

$$\Pr(\text{cell is 2 days old}) = 0.2$$

$$\Pr(\text{cell is 3 days old}) = 0.1.$$

• EXERCISE 6.7.34

The cells in exercise 6.5.32, where the cells with age greater than or equal to 3 days have been eliminated from the culture.

- ♠ 100 pairs of plants are crossed and each pair produces 10 offspring. The number of tall offspring is then counted. For the given experiment, draw a histogram of the probability of each result, and find the requested probability.

Number of tall offspring	Frequency in 100 experiments			
	Experiment a	Experiment b	Experiment c	Experiment d
0	0	0	0	5
1	0	0	2	11
2	0	1	5	35
3	0	1	10	27
4	2	6	21	13
5	8	19	27	8
6	9	20	19	1
7	31	27	8	0
8	22	16	5	0
9	22	9	3	0
10	6	1	0	0

• EXERCISE 6.7.35

Draw the histogram for experiment a, and find the probability that between 4 and 6 plants (inclusive) are tall.

• EXERCISE 6.7.36

Draw the histogram for experiment b, and find the probability that between 4 and 6 plants (inclusive) are tall.

• EXERCISE 6.7.37

Draw the histogram for experiment c, and find the probability that between 4 and 6 plants (inclusive) are tall.

• EXERCISE 6.7.38

Draw the histogram for experiment d, and find the probability that between 4 and 6 plants (inclusive) are tall.

- ♠ An experiment to see which color birds female birds prefer is repeated two times. The first time, females mate with red males with probability 0.5, with blue males with probability 0.3, and with green males with probability 0.2. The second time, females mate with red males with probability 0.4, with blue males with probability 0.35, and with green males with probability 0.25. At the end, the results of the two experiments are combined.

• EXERCISE 6.7.39

Suppose that 100 female birds were tested in each experiment. Find the number out of 200 that mated with each type of male, and convert the results into a probability distribution.

• EXERCISE 6.7.40

Suppose that 100 female birds were tested in the first experiment and 200 females in the second. Find the number out of 300 that mated with each type of male, and convert the results into a probability distribution.

• EXERCISE 6.7.41

Suppose that an equal number of female birds were used in each experiment. Use the law of total probability to find the probability distribution in the combined experiment.

• EXERCISE 6.7.42

Suppose that three times as many females birds were used in the first experiment. Use the law of total probability to find the probability distribution in the combined experiment.

- ♠ A “random number generator” on a computer is supposed to use the **uniform density** with p.d.f.  $f(x) = 1$  for  $0 \leq x \leq 1$ . The idea is that all values between 0 and 1 are equally likely to be chosen. Compare the results of using a random number generator with this form (which is more or less impossible to achieve on a computer) with two discrete versions. In the first, the values 0.0, 0.1, 0.2, ..., 1.0 are all equally likely to be chosen. In the second, the values 0.05, 0.15, 0.25, ..., 0.95 are equally likely to be chosen.

• EXERCISE 6.7.43

Attempting to simulate a fair coin by assigning “heads” when the value is greater than or equal to 0.5, and “tails” when the value is less than 0.5.

• EXERCISE 6.7.44

Attempting to simulate a fair die by assigning a 1 when the value is less than 1/6, 2 when the the value is greater than or equal to 1/6 but less than 2/6, and so forth.

- ♠ We will see that the p.d.f for the waiting time  $X$  until an event occurs often follows the **exponential distribution**, with the form  $g(x) = \alpha e^{-\alpha x}$  for some positive value of  $\alpha$ , defined for  $x \geq 0$ . For each of the following values of  $\alpha$ ,

- a. Find the c.d.f.

- b. Plot the p.d.f. and c.d.f.
- c. Check that the p.d.f. is the derivative of the c.d.f.
- d. Find  $\Pr(X \leq 1)$ . Indicate this on both of your graphs.
- e. Find  $\Pr(1 \leq X \leq 3)$ .
- f. Find  $\Pr(1 \leq X \leq 1.01)$  and show that it is approximately  $g(1) \cdot 0.01$ .

• EXERCISE 6.7.45

$$\alpha = 0.5.$$

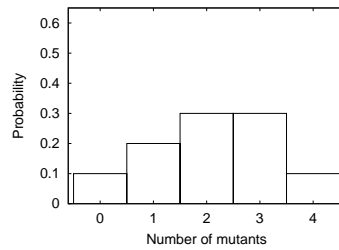
• EXERCISE 6.7.46

$$\alpha = 2.0.$$

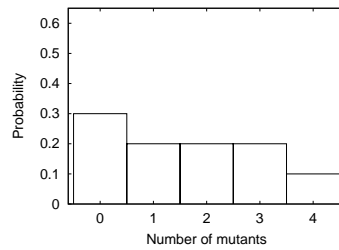
# Chapter 7

## Answers

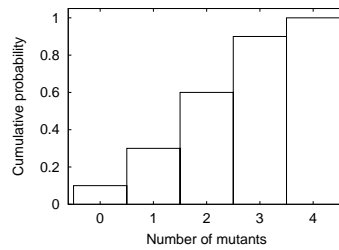
6.7.1.



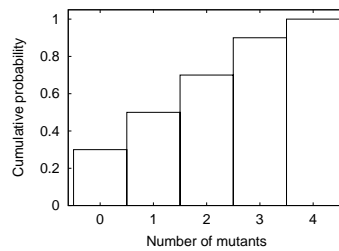
6.7.3.



6.7.5.



6.7.7.



**6.7.9.** The most likely simple event is that the measurement is 5. 1 and 9 tie for least likely. It is symmetric.

**6.7.11.** The most likely simple event is that the measurement is 2. 9 is least likely. It is not symmetric.

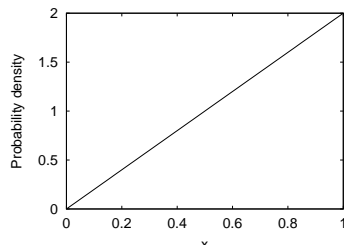
**6.7.13.** **a.** 0.11, **b.** 0.38, **c.** 0.62.

**6.7.15.** **a.** 0.08, **b.** 0.64, **c.** 0.64.

**6.7.17.** The area under the curve is

$$\int_0^1 2x dx = x^2 \Big|_0^1 = 1^2 - 0^2 = 1,$$

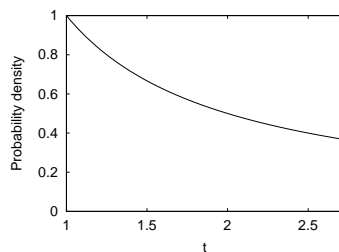
as it must be. The maximum is at  $x = 1$ , where  $f(1) = 2 > 1$ . However, the probability density can be greater than 1 because probabilities are areas under the curve, which are always less than 1.



**6.7.19.** The area under the curve is

$$\int_0^1 \frac{1}{t} dt = \ln(t) \Big|_1^e = \ln(e) - \ln(1) = 1.$$

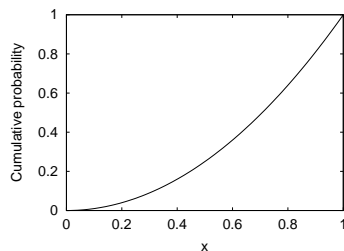
The maximum is at  $t = 1$ , where  $h(1) = 1$ . Because this is a p.d.f., this does not mean that the probability is 1, because probabilities correspond only to areas under the curve.



**6.7.21.** The cumulative distribution function is

$$F(x) = \int_0^x f(y) dy = \int_0^x 2y dy = y^2 \Big|_0^x = x^2.$$

This function is increasing because  $f(x)$  is positive, and increases from 0 at  $x = 0$  to 1 at  $x = 1$ .

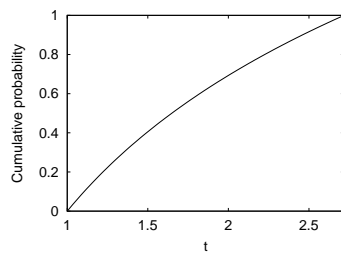


**6.7.23.** The cumulative distribution function is

$$H(t) = \int_0^t h(s) ds = \int_0^t \frac{1}{s} ds = \ln(s) \Big|_0^t = \ln(t).$$

This function is increasing because  $h(t)$  is positive, and increases from 0 at  $t = 1$  to 1 at  $t = e$ .



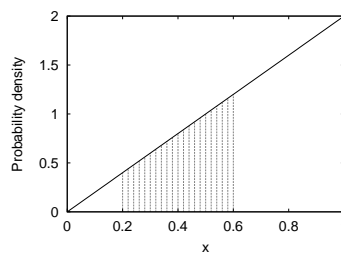


**6.7.25.** Integrating,

$$\int_0^{.2^0.6} .6f(x)dx = \int_0^{.2^0.6} .62x dx = x^2|_0^{.2^0.6} = 0.32.$$

The cumulative distribution function is  $F(x) = x^2$ , so

$$\int_0^{.2^0.6} .6f(x)dx = F(0.6) - F(0.2) = 0.6^2 - 0.2^2 = 0.32.$$

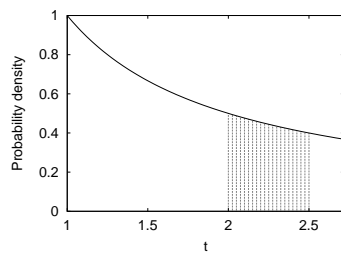


**6.7.27.** Integrating

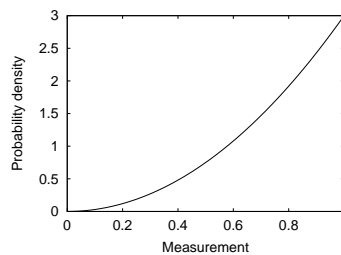
$$\int_2^{2.5} .5 \frac{1}{t} dt = \ln(t)|_2^{2.5} = \ln(2.5) - \ln(2.0) = 0.223.$$

The cumulative distribution function is  $H(t) = \ln(t)$ , so

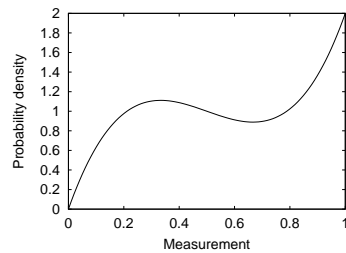
$$\int_2^{2.5} .5 \frac{1}{t} dt = H(2.5) - H(2.0) = 0.223.$$



**6.7.29.**



**6.7.31.**



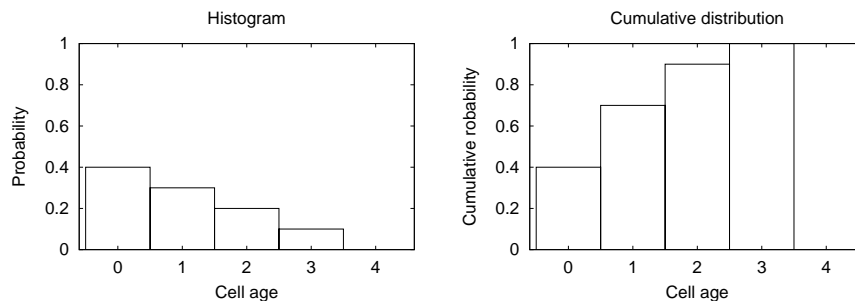
**6.7.33.** The cumulative distribution is

$$\Pr(\text{cell is less than or equal to 0 days old}) = 0.4$$

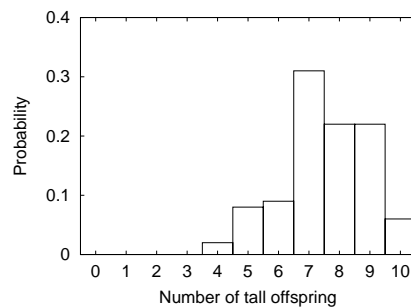
$$\Pr(\text{cell is less than or equal to 1 day old}) = 0.7$$

$$\Pr(\text{cell is less than or equal to 2 days old}) = 0.9$$

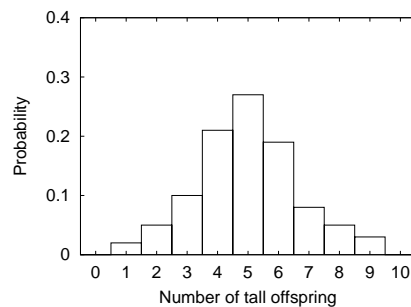
$$\Pr(\text{cell is less than or equal to 3 days old}) = 1.0$$



**6.7.35.** The probability that between 4 and 6 plants are tall is  $0.02 + 0.08 + 0.09 = 0.19$ .



**6.7.37.** The probability that between 4 and 6 plants are tall is  $0.21 + 0.27 + 0.19 = 0.67$ .



**6.7.39.** In the first experiment, 50 mated with red, 30 with blue, and 20 with green. In the second experiment, 40 mated with red, 35 with blue, and 25 with green. The total is then 90 with red, 65 with blue and 45 with green. Dividing by 200, this gives a probability of 0.45 with red, 0.325 with blue, and 0.225 with green.

**6.7.41.** Let  $R_1$  be the event of mating with a red male in the first experiment,  $R_2$  of mating with a red male in the second and  $R$  the event of mating with a red male in the combined experiment (and similarly for  $B$  and  $G$ ). Let  $p$  be the probability a female came from the first experiment and  $1 - p$  the probability she came from the second. Because the experiments are equally large,  $p = 0.5$ . Then

$$\begin{aligned}\Pr R &= \Pr(R_1) \cdot p + \Pr(R_2) \cdot (1 - p) = 0.5 \cdot 0.5 + 0.4 \cdot 0.5 = 0.45 \\ \Pr B &= \Pr(B_1) \cdot p + \Pr(B_2) \cdot (1 - p) = 0.3 \cdot 0.5 + 0.35 \cdot 0.5 = 0.325 \\ \Pr G &= \Pr(G_1) \cdot p + \Pr(G_2) \cdot (1 - p) = 0.2 \cdot 0.5 + 0.25 \cdot 0.5 = 0.225.\end{aligned}$$

**6.7.43.** With the perfect random number generator,

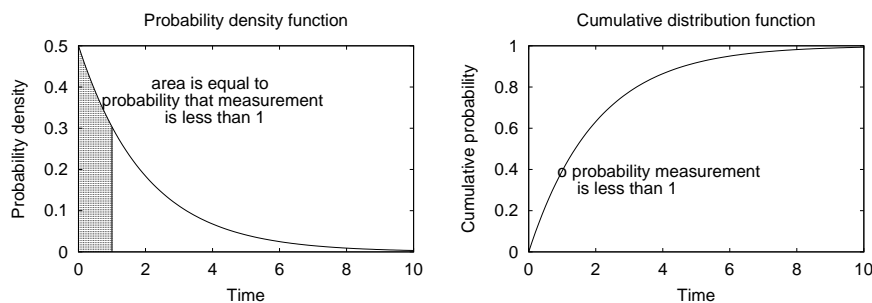
$$\Pr V < 0.5 = \int_0^{0.5} f(x)dx = \int_0^{0.5} 1dx = 0.5.$$

With the first approximate random number generator, we get “tails” if the value is 0.0, 0.1, 0.2, 0.3 or 0.4 (5 possibilities) and “heads” with each of the other 6 possibilities, with a probability of “tails” of  $5/11$ . With the second approximate random number generator, we get “tails” if the value is 0.05, 0.15, 0.25, 0.35 or 0.45 (5 possibilities) and “heads” with each of the other 5 possibilities, with a probability of “tails” of  $5/10=0.5$ .

**6.7.45.**

a. The c.d.f. is

$$G(x) = \int_0^x g(y)dy = \int_0^x 0.5e^{-0.5y}dy = -e^{-0.5y}|_0^x = -e^{-0.5x} + 1.$$



b.

c.  $G'(x) = 0 - (-0.5e^{-0.5x}) = 0.5e^{-0.5x}.$

d. This is  $G(1.0) = 0.393.$

e. This is  $G(3) - G(1) = 0.383.$

f. This is  $G(1.01) - G(1) = 0.0030$ . Dividing by 0.01 gives 0.30, very close to  $g(1.0) = 0.30$ .