6.6 Independence and Markov Chains

MATHEMATICAL TECHNIQUES

♠ Check whether the following events are independent by checking three equations:

$$\begin{aligned} \Pr(A) &= \Pr(A|B) && \text{A is independent of B} \\ \Pr(B) &= \Pr(B|A) && \text{B is independent of A} \\ \Pr(A \cap B) &= \Pr(A)\Pr(B) && \text{the multiplication rule.} \end{aligned}$$

Do you ever find a case where only one or two of these equations is satisfied?

• EXERCISE **6.6.1**

As in exercises 6.4.5 and 6.5.5, the sample space is S=$\{0,1,2,3,4\}$, $\Pr(\{0\}) = 0.2$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$ and $\Pr(\{4\}) = 0.0$, A=$\{0,1,2\}$ and B=$\{0,2,4\}$.

• EXERCISE **6.6.2**

As in exercises 6.4.5 and 6.5.6, the sample space is S=$\{0,1,2,3,4\}$, $\Pr(\{0\}) = 0.2$, $\Pr(\{1\}) = 0.3$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$, $\Pr(\{4\}) = 0.0$, A=$\{1,2,3\}$ and B=$\{2,3,4\}$.

• EXERCISE **6.6.3**

The sample space is S=$\{1,2,3,4\}$, $\Pr(\{1\}) = 0.48$, $\Pr(\{2\}) = 0.12$, $\Pr(\{3\}) = 0.32$, $\Pr(\{4\}) = 0.08$, A=$\{3,4\}$ and B=$\{1,3\}$.

• EXERCISE **6.6.4**

The sample space is S=$\{1,2,3,4\}$, $\Pr(\{1\}) = 0.4$, $\Pr(\{2\}) = 0.4$, $\Pr(\{3\}) = 0.1$, $\Pr(\{4\}) = 0.1$, A=$\{1,2\}$ and B=$\{1,3\}$.

♠ Consider again the three-sided die that gives scores of 1,2 or 3, each with probability 1/3 (exercises 6.5.9–6.5.12). Suppose that the results of rolls are independent. Use the multiplication rule to find the following probabilities.

• EXERCISE **6.6.5**

The probability of rolling a 1 followed by a 3.

• EXERCISE **6.6.6**

The probability of rolling three 1's in a row.

• EXERCISE **6.6.7**

The probability of rolling two odd values in a row.

• EXERCISE **6.6.8**

The probability of rolling an odd value followed by an even value.

• EXERCISE **6.6.9**

The probability of rolling the die three times and not repeating a value.

• EXERCISE **6.6.10**

The probability of rolling the die three times and having the total be odd.

♠ In each of the following problems, the sample space is S=$\{1,2,3,4\}$. From the probabilities of the given events A and B, and the assumption that A and B are independent, find $\Pr(\{1\})$, $\Pr(\{2\})$, $\Pr(\{3\})$ and $\Pr(\{4\})$.

• EXERCISE **6.6.11**

A=$\{1,2\}$, B=$\{1,3\}$, $\Pr(A) = 0.4$,$\Pr(B) = 0.6$.

• EXERCISE **6.6.12**

A=$\{1,4\}$, B=$\{1,3\}$, $\Pr(A) = 0.8$,$\Pr(B) = 0.3$.

♠ Show that the multiplication rule (theorem 6.3) does not work in the following cases.

• EXERCISE **6.6.13**

For two events A and B that are disjoint, as long as $\Pr(A) > 0$ and $\Pr(B) > 0$.

• EXERCISE **6.6.14**

For two events A and B where A is a subset of B, as long as $\Pr(A) > 0$ and $0 < \Pr(B) < 1$.

♠ Write the information from each of the following two state Markov chains in terms of conditional probability. Write an updating function for the probability, and use the method presented in equation 6.9 to find the long-term probability.

- EXERCISE **6.6.15**
  The mutants described in exercise 6.2.19, where a gene has a 1.0% chance of mutating each time a cell divides, and a 1.0% chance of correcting the mutation.
- EXERCISE **6.6.16**
  The lemmings described in exercise 6.2.20, where a lemming has a probability 0.2 of jumping off the cliff each hour and a probability 0.1 of crawling back up.
- EXERCISE **6.6.17**
  The molecules described in exercise 6.2.21, where a molecule has a probability 0.05 of binding and a probability of 0.02 of unbinding each second.
- EXERCISE **6.6.18**
  The molecules described in exercise 6.2.22, where a caterpillar has a probability 0.15 of being taken over by a parasitoid each day, and a probability 0.03 of recovering.

♠ The formula for the probability of the union of two events,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B),$$

(from exercises 6.4.13– 6.4.16) is simpler when events are independent.

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B),$$

Test the formula on the following independent events.

- EXERCISE **6.6.19**
  Using the probabilities found in exercise 6.6.11, where A=$\{1,2\}$, B=$\{1,3\}$, $\Pr(A) = 0.4$,$\Pr(B) = 0.6$.
- EXERCISE **6.6.20**
  Using the probabilities found in exercise 6.6.12, where A=$\{1,4\}$, B=$\{1,3\}$, $\Pr(A) = 0.8$,$\Pr(B) = 0.3$.

APPLICATIONS

♠ An ecologist is looking for the effects of eagle predation on the behavior of jack rabbits, as in exercises 6.5.27 and 6.5.28. Assuming that the rabbits and eagles behave independently,

a. Find the probability that the ecologist sees both a rabbit and an eagle during a particular hour of observation.

b. Draw a Venn diagram to illustrate the situation.

c. Find the probability that she saw a jack rabbit conditional on her seeing an eagle. How might you interpret this result? Compare with the overall probability of seeing a jack rabbit.

d. Find the probability that she saw an eagle conditional on her seeing a jack rabbit. How might you interpret this result?

- EXERCISE **6.6.21**
  She sees an eagle with probability 0.2 during an hour of observation and a jack rabbit with probability 0.5.
- EXERCISE **6.6.22**
  She sees an eagle with probability 0.4 during an hour of observation and a jack rabbit with probability 0.8.

♠ Someone comes up with a cut-rate "test" for a disease. This test gives a positive result with probability 0.5 whether or not the patient has the disease. In each of the following cases, find the probability of having the disease conditional on a positive test in two ways.

a. Work it out directly as in the previous chapter,

b. Use independence.

- EXERCISE **6.6.23**
  1% of people have the disease.
- EXERCISE **6.6.24**
  10% of people have the disease.

♠ We have seen that one force that can alter the ratio of heterozygotes produced by a selfing heterozygote is **meiotic drive** (exercises 6.3.35–6.3.38), where one allele, say **A**, pushes its way into more than half of the gametes (ovules or pollen). Another possibility is that the alleles in surviving offspring are not independent. Compare the fraction of heterozygotes produced in the following cases.

- EXERCISE **6.6.25**

  Compare a case of meiotic drive where 60% of both pollen and ovules carry the **A** allele independently, with a case of non-independent assortment where an offspring gets an **A** allele from the pollen with probability 0.6 when the ovule provides an **A** and gets an **A** allele from the pollen with probability 0.4 when the ovule provides an **a**. The ovule provides **A** with probability 0.5.

- EXERCISE **6.6.26**

  Compare a case of meiotic drive where 70% of the pollen and 40% of the ovules carry the **A** allele independently, with a case of non-independent assortment where an offspring gets an **A** allele from the pollen with probability 0.7 when the ovule provides an **A** and gets an **A** allele from the pollen with probability 0.3 when the ovule provides an **a**. The ovule provides **A** with probability 0.5.

♠ A species of bird comes in three colors, red, blue and green. 20% are red, 30% are blue and 50% are green. Females prefer red to blue and blue to green, and meet with the best male they find.

- EXERCISE **6.6.27**

  Females pick the better of the first two males they meet. What is the probability a female mates with a green bird? What did you have to assume about independence?

- EXERCISE **6.6.28**

  Females pick the better of the first two males they meet. What is the probability a female mates with a blue bird and the probability a female mates with a red bird?

♠ A small class has only 3 students. Each comes to class with probability 0.9. Find the probability that all the students come to class and the probability that no students come to class in the following circumstances.

- EXERCISE **6.6.29**

  The students act independently.

- EXERCISE **6.6.30**

  Student 2 comes to class with probability 1.0 if student 1 does. Student 3 ignores them.

- EXERCISE **6.6.31**

  Student 2 comes to class with probability 8/9 if student 1 does. Student 3 ignores them.

- EXERCISE **6.6.32**

  Student 3 comes to class with probability 1.0 if both the others come. Students 1 and 2 ignore each other.

♠ A popular probability problem refers to a once popular game show called "Let's make a deal." In this game, the host (named Monte Hall) hands out large prizes to contestants for no reason at all. In one situation, Monte would show the contestant three doors, named door 1, door 2 and door 3. One would hide a new car, one $500 worth of false eyelashes, and the other a goat (deemed worthless by the purveyors of the show). The contestant picks door 1. But instead of showing her the prize, Monte opens door 3 to reveal the goat.

- EXERCISE **6.6.33**

  Should the contestant switch her guess to door 2?

- EXERCISE **6.6.34**

  If she uses the right strategy, what is her probability of getting the new car?

- EXERCISE **6.6.35**

  It is later revealed that Monte does not always show what is behind one of the other doors, but does so only when the contestant guessed right in the first place (the so-called "Machiavellian Monte"). How often would a contestant who used the strategy in exercise 6.6.33 get the new car?

- EXERCISE **6.6.36**

  What is the right strategy to use for dealing with the Machiavellian Monte? How well would the contestant do?

♠ Suppose a molecule is transferred among 3 cells according to a Markov chain. Write down conditional probabilities to describe the following situations. It can help to draw a picture.

- EXERCISE **6.6.37**

  The position of the molecule in one minute is independent of the position in the previous minute.

- EXERCISE **6.6.38**
  The molecule only rarely leaves a cell. When it does so it enters each of the other cells with equal probability.
- EXERCISE **6.6.39**
  Imagine the three cells arranged in a ring. The molecule only rarely leaves a cell, and when it does so it always moves to the right.
- EXERCISE **6.6.40**
  Imagine the three cells arranged in a line. The molecule only rarely leaves a cell. If it is at the end, it moves to the middle. If it is in the middle, it enters the end cells with equal probability.

Chapter 7

Answers

6.6.1. We have found that Pr(A) = 0.9, Pr(B) = 0.6, Pr(A ∩ B) = 0.6, Pr(A|B) = 2/3 and Pr(B|A) = 1. Therefore, Pr(A) = 0.9 ≠ Pr(A ∩ B) = 0.6, Pr(B) = 0.6 ≠ Pr(B ∩ A) = 1, and Pr(A ∩ B) = 2/3 ≠ Pr(A)Pr(B) = 0.54. None of the three conditions are satisfied. The events A and B are not independent.
6.6.3. Pr(A) = 0.4, Pr(B) = 0.8, Pr(A ∩ B) = 0.32. Therefore, Pr(A|B) = 0.32/0.8 = 0.4 and Pr(B|A) = 0.32/0.4 = 0.8. In this case, Pr(A) = 0.4 = Pr(A ∩ B), Pr(B) = 0.8 = Pr(B ∩ A), and Pr(A ∩ B) = 0.32 = Pr(A)Pr(B). All three of the conditions are satisfied, and the events A and B are independent.
6.6.5. The probability of rolling a 1 is 1/3, as is the probability of rolling a 3. Because the rolls are independent, the probability of both events is the product of the probabilities, or $1/3 \cdot 1/3 = 1/9$.
6.6.7. The probability of rolling an odd value is 2/3, because it could be either a 1 or a 3. The probability of rolling two in a row is the product $2/3 \cdot 2/3 = 4/9$.
6.6.9. The first roll could be anything (probability 1). The second role must be different from the first (probability 2/3) and the last must be different from the first 2 (probability 1/3). Because these are independent, we multiply to find $2/3 \cdot 1/3 = 2/9$.
6.6.11. $A \cap B = \{1\}$. Because A and B are independent, Pr(A ∩ B) = Pr(A)Pr(B) = $0.4 \cdot 0.6 = 0.24$. Also, A=$\{1\} \cup \{2\}$, so Pr(A) = $\Pr\{1\} + \Pr\{2\}$ because the simple events $\{1\}$ and $\{2\}$ are disjoint. Therefore, $0.4 = 0.24 + \Pr\{2\}$ and $\Pr\{2\} = 0.16$. Similarly, we can find $\Pr\{3\} = 0.36$. Because all the probabilities must add to 1, $\Pr\{4\} = 0.24$.
6.6.13. Because the events are disjoint, the intersection is the null set and its probability is zero. But Pr(A ∩ B) = 0 ≠ Pr(A)Pr(B) if Pr(A) > 0 and Pr(B) > 0.
6.6.15. Let M_t denote the event "mutant at generation t" and Let N_t denote the event "non-mutant at generation t". Then

$$\begin{aligned}\Pr(M_{t+1}|M_t) &= 0.99\\ \Pr(M_{t+1}|C_t) &= 0.01\\ \Pr(C_{t+1}|M_t) &= 0.01\\ \Pr(C_{t+1}|C_t) &= 0.99.\end{aligned}$$

If $p_t = \Pr M_t$, then

$$p_{t+1} = 0.99p_t + 0.01(1 - p_t).$$

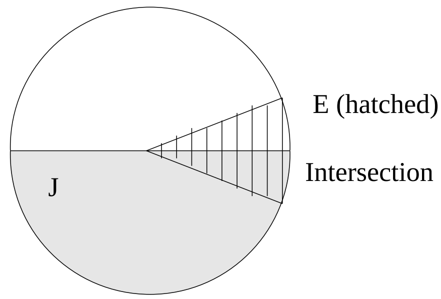
The equilibrium is where $p^* = 0.99p^* + 0.01(1 - p^*)$ which has solution $p^* = 0.5$.
6.6.17. Let B_t denote the event "bound at hour t" and Let U_t denote the event "unbound at second t". Then

$$\begin{aligned}\Pr(B_{t+1}|B_t) &= 0.98\\ \Pr(B_{t+1}|U_t) &= 0.05\\ \Pr(U_{t+1}|B_t) &= 0.02\\ \Pr(U_{t+1}|U_t) &= 0.95.\end{aligned}$$

If $p_t = \Pr B_t$, then

$$p_{t+1} = 0.98p_t + 0.05(1 - p_t).$$

The equilibrium is where $p^* = 0.98p^* + 0.05(1-p^*)$ which has solution $p^* = 0.714$.
6.6.19. $A \cup B = \{1,2,3\}$. From the probabilities found in exercise 6.6.11, $\Pr(\{1,2,3\}) = \Pr\{1\} + \Pr\{2\} + \Pr\{3\} = 0.24 + 0.16 + 0.36 = 0.76$. By the formula, $\Pr(A \cup B) = 0.4 + 0.6 - 0.24 = 0.76$. It worked.
6.6.21. The probability of both is the product of the probabilities, or 0.1.



The probability of seeing a jack rabbit conditional on seeing an eagle is just 0.5, because $\Pr(J|E) = \Pr(J)$ when events are independent. Similarly, the probability of seeing an eagle conditional on seeing a jack rabbit is 0.2, because $\Pr(E|J) = \Pr(E)$ when events are independent. It seems that rabbits and eagles ignore each other.
6.6.23.

a. First, find Pr(P) with the law of total probability,

$$\begin{aligned}\Pr(P) &= \Pr(P|D)\Pr(D) + \Pr(P|N)\Pr(N) \\ &= (0.5)(0.01) + (0.5)(0.99) = 0.5.\end{aligned}$$

Then, find the conditional probability with Bayes' Theorem,

$$\Pr(D|P) = \frac{\Pr(P|D)\Pr(D)}{\Pr(P)} = \frac{0.5 \cdot 0.01}{0.5} = 0.01.$$

The test gives no new information.

b. We have that $\Pr(P) = \Pr(P|D)$. Therefore, these two events are independent, and $\Pr(D) = \Pr(D|P) = 0.01$.

6.6.25. With meiotic drive, the probability of an **A** from the pollen is independent of the allele that came from the ovule. Let **aA** represent a plant that got an **a** from the ovule and an **A** from the pollen. By the multiplication rule, $\Pr(aA) = \Pr(A)\Pr(a) = 0.6 \cdot 0.4 = 0.24$. Similarly, $\Pr(Aa) = \Pr(A)\Pr(a) = 0.6 \cdot 0.4 = 0.24$. Therefore, the total probability of a heterozygote is $0.24 + 0.24 = 0.48$, a smaller fraction than without meiotic drive. With non-independent assortment, $\Pr(aA) = \Pr(a|A)\Pr(A) = 0.4 \cdot 0.5 = 0.2$ and $\Pr(Aa) = \Pr(A|a)\Pr(a) = 0.4 \cdot 0.5 = 0.2$. The total is 0.4.
6.6.27. Assume she runs into males independently. She will mate with a green bird only if both males are green, which occurs with probability $0.5 \cdot 0.5 = 0.25$.
6.6.29. Using the multiplication law, all come with probability 0.729, and none come with probability 0.001.
6.6.31. Let S_1 be the event student 1 comes, N_1 that student 1 does not come etc. By the law of total probability,

$$\begin{aligned}\Pr(S_2) &= \Pr(S_2|S_1)\Pr(S_1) + \Pr(S_2|N_1)\Pr(N_1) \\ 0.9 &= \frac{8}{9} \cdot 0.9 + \Pr(S_2|N_1)0.1.\end{aligned}$$

The only unknown part is $\Pr(S_2|N_1)$, which can be solved for as 1.0. The second student therefore always comes when the first does not, meaning that the class will never be empty. Both 1 and 2 come with probability

$$\Pr(\text{both 1 and 2 come}) = \Pr(S_2|S_1)\Pr(S_1) = 0.8.$$

Multiplying by 0.9 (because 3 still ignores them), the probability that all come is 0.72.
6.6.33. Yes. The probability the new car is behind door 1 is 1/3, and the probability that it is behind door 2 or door 3 is 2/3. Because Monte showed it not to be behind door 3, the probability it is behind door 2 is 2/3.

6.6.35. A contestant who switched would only do so when she was right in the first place. She would never get the new car.

6.6.37. Label the three cells **a**, **b**, and **c**, and let the events a_t, b_t, and c_t mean that it was in cell **a**, **b** or **c**, respectively, at time t. Suppose it always goes to **a** with probability 0.8, and to **b** or **c** with probability 0.1. Then

$$\begin{aligned}\Pr(\mathrm{a}_{t+1}|\mathrm{a}_t) &= 0.8, \Pr(\mathrm{a}_{t+1}|\mathrm{b}_t) = 0.8, \Pr(\mathrm{a}_{t+1}|\mathrm{c}_t) = 0.8 \\ \Pr(\mathrm{b}_{t+1}|\mathrm{a}_t) &= 0.1, \Pr(\mathrm{b}_{t+1}|\mathrm{b}_t) = 0.1, \Pr(\mathrm{b}_{t+1}|\mathrm{c}_t) = 0.1 \\ \Pr(\mathrm{c}_{t+1}|\mathrm{a}_t) &= 0.1, \Pr(\mathrm{c}_{t+1}|\mathrm{b}_t) = 0.1, \Pr(\mathrm{c}_{t+1}|\mathrm{c}_t) = 0.1.\end{aligned}$$

6.6.39. Suppose **b** is to the right of **a**, **c** to the right of **b**, and **a** to the right of **c**.

$$\begin{aligned}\Pr(\mathrm{a}_{t+1}|\mathrm{a}_t) &= 0.8, \Pr(\mathrm{a}_{t+1}|\mathrm{b}_t) = 0.0, \Pr(\mathrm{a}_{t+1}|\mathrm{c}_t) = 0.2 \\ \Pr(\mathrm{b}_{t+1}|\mathrm{a}_t) &= 0.2, \Pr(\mathrm{b}_{t+1}|\mathrm{b}_t) = 0.8, \Pr(\mathrm{b}_{t+1}|\mathrm{c}_t) = 0.0 \\ \Pr(\mathrm{c}_{t+1}|\mathrm{a}_t) &= 0.0, \Pr(\mathrm{c}_{t+1}|\mathrm{b}_t) = 0.2, \Pr(\mathrm{c}_{t+1}|\mathrm{c}_t) = 0.8.\end{aligned}$$